Chapter 1

INTRODUCTION

1.1 Background of the Study

All bodies possessing mass and elasticity are capable of vibration thus all structures experience vibration to some degree. Structural vibrations are oscillatory in nature and can be linear or non-linear. For linear systems the principle of superposition holds and their analysis is well developed.

There are two classes of vibrations, free and forced vibration. Free vibrations are vibrations that take place in the absence of any external force. They are due to the action of forces inherent in the system itself. A system under free vibration will vibrate at one or more of its natural frequencies. The natural frequencies of a system depend on the distribution of its mass and stiffness and hence are a property of the dynamical system.

Forced vibrations take place under the excitation of external forces. If the excitation is oscillatory, the system is forced to vibrate at the same excitation frequency. These excitations can undermine the safety of the structure when large amplitudes develop. If the frequency of excitation coincides with any of the natural frequencies of the structural system, a phenomenon known as resonance occurs resulting in large oscillations in the system. This can lead to the failure of bridges, buildings, airplane wings and other structures subject to dynamic loads.

Resonance is to be avoided in most systems. The effect can be reduced with dampers and absorbers. It is important to note that all vibrating systems are subjected to some degrees of damping. This is because energy is dissipated by friction, air resistance and other resistances. When damping is small, it has very little influence on the natural frequency of vibration hence calculations for natural frequencies are done on the basis of zero damping. Damping is very important in limiting the amplitude of oscillations at the natural frequencies of the system.

When a system is excited by a suddenly applied non-periodic force, its response is transient and the oscillations take place at the natural frequencies of the system. The amplitude of vibration will vary in a manner dependent on the type of excitation.

The number of independent coordinates required to describe the motion of a system is known as the degrees of freedom of the system. A continuous structure will have an infinite number of degrees of freedom and hence an infinite number of coordinates to analyze. However certain idealizations are made and a continuous system may be treated as one having a finite number of degrees of freedom. A structure with N degrees of freedom will have N natural frequencies and each of the natural frequencies will have a displacement configuration known as normal modes. The normal modes of a system depend on the distribution of the mass and stiffness of the system. When vibrating at one of these modes all points in the system undergo simple harmonic motion that passes through their equilibrium positions simultaneously. To produce a normal mode vibration, the system has to be given an initial displacement corresponding to the normal mode.

For systems with few degrees of freedom, it is possible to formulate the equations of motion by an application of the Newton's laws of motion. The method however becomes complicated for systems with a high degree of freedom and the energy methods provide a convenient alternative. One of the noTable products of the energy method is the Lagrange's equations. The Lagrange's equation enables the analysis of structural elements as discrete masses connected together by mass-less elements. With a proper selection of representative masses the results can be very close to the exact response. When the discretization is increased by the use of more number of lump masses, the accuracy of the response improves.

In order to obtain the exact response of structural systems it is necessary to analyse them as elements with continuously distributed masses. A continuous structure has infinite degrees of freedom and normal modes but generally the first few modes are of most importance. The amplitudes of vibration associated with higher modes are very small and can be ignored. Hamilton's principle is an energy method for the analysis of continuous structures. It is an extension of the principle of minimum potential energy for the analysis of dynamic systems.

For structural systems with complex geometries, it is generally very difficult to obtain the exact dynamic response. This arises from difficulty in formulating the partial differential equations of motion for the system or difficulty in solving the equations formulated. In such situations approximate solutions are obtained by lumping the masses. This in effect reduces the system's infinite degrees of freedom to a finite value. Another method with similar consequences is the Rayleigh-Ritz method. It is actually an extension of the Rayleigh method. It involves the use of a series of shape functions multiplied by constant coefficient to depict the mode shapes. This method is known to give very good predictions of lower modes. The success of the Rayleigh-Ritz method depends on the choice of the shape functions, the number of shape functions and how well they define the mode shapes and satisfy the kinematic conditions of the structural system.

The advent of fast digital computers has made the analysis of large simultaneous equations easy. This can be put to use in the Finite Element Method. Just like in the Rayleigh-Ritz method, there is need to select a shape function. The accuracy of finite element method can be improved upon by the careful selection of better shape functions (p-version) and also by the introduction of more joints/nodes and hence more elements (h-version). The latter has the implication of increasing the size of the resulting equations and hence the computational cost.

The finite element method and the Rayleigh-Ritz method are numerical methods for obtaining approximate solutions. Their results can only be exact when the chosen shape function depicts perfectly the structure's displacement configuration and end conditions. Once these conditions are not met, their results remain approximate. In the Rayleigh-ritz method, one way of improving its accuracy is by increasing the number of terms in the shape function. This has the implication of increasing the flexibility of the shape function so as to mimic better the deformation or displacement configuration of the real structures. If the number of terms in the shape function is increased to infinity, then the resulting shape function will be able to depict any curve or deformation perfectly and hence will give an exact solution. The resulting analysis however will be too complex hence in the application of the Rayleigh ritz method a compromise is reached between the accuracy desired, number of terms in the chosen shape function and the mathematical analyses to be carried out.

The finite element method became very popular because in addition to being able to improve on its chosen shape function, it is also possible to break the structure into finite elements, enabling the chosen shape functions to march better the displacement configuration of the elements under consideration and so improve the results. The smaller the elements the more likely the chosen shape function would depict its deflection more accurately. The introduction of more nodes (increase in number of elements) however leads to an increase in the size of computation to be carried out.

On the other hand, the Hamilton's principle and Lagrange equations are energy formulations designed to give accurate or exact results. The Hamilton's principle can be used for the analysis of structures with a continuous distribution of mass (continuous structures). Since all structural elements have a continuous distribution of mass it is a powerful tool for the accurate analysis of structural elements. The lagrange equation was formulated for the analysis of structures consisting of discrete masses connected together by mass-less elements. For such a structure the results from the use of lagrange equations are accurate. The question is, are there mass-less structural elements? If a continuous mass is treated as mass-less then, the result from the use of Lagrange equation becomes approximate. One such approximate application is carried out when a beam supports a mass and the mass of the beam is small when compared to the mass of the body it is supporting. In such a case the supported mass is treated as a discrete mass and the supporting beam as mass-less. This leads to an approximate result. Another common approximate application of the lagrange equation is when a continuous mass is treated as discrete masses connected by mass-less elements. This is done primarily to ease the calculation and reduce a structure of infinite DOF to a one of finite DOF. The implication is that since the real structure does not have discrete masses but a continuous stream of masses, analyzing it as discrete masses results in approximate solutions. The discretization error is reduced by having masses lumped at uniform distance (this helps to keep the centre of gravity of the lump-massed beam same as that of the continuous structure) and by increasing the number of lumped masses (having smaller lumped masses and hence smaller discretization). Either way as long as there is lumping of continuous masses, the results remains approximate, but will continue tending to the exact solution with an increase in the number of lumped masses.

1.2 Statement of the Problem

Real structures, because of their continuous distribution of mass have infinite degrees of freedom. The degrees of freedom of a structure can be reduced to a finite value by the lumping of the continuous masses at some selected nodes. This enables an easy application of the Lagrange's equation for the dynamic analysis of the structure. However results obtained this way are not exact. This is so because the lumping of the continuous mass at selected nodes of the structure alters the mass distribution of the structure and hence introduces an error in the results of the dynamic analysis. There is need to determine an effective way of minimizing or eliminating the error introduced by the lumping of continuous mass and hence improve the results of the dynamic analysis.

1.3 Aim of Study

The aim of the study is to dynamically analyse beams by modifying the structure's stiffness distribution.

1.4 Objectives of Study

i) To formulate the force equilibrium equations for beams of different end conditions under free vibration and determine the equations for the inherent forces in a vibrating element of a beam.

ii) To determine the required modification in stiffness distribution required for the lumped mass element to possess the same natural frequency as the continuous one.

iii) Evaluate the improvement in the accuracy of the fundamental frequencies obtained by the modification of a beam’s stiffness distribution and extend the principle to an approximate analysis of frames.

iv) Validate previous works done on determination of natural frequencies of frames

The use of lumped mass comes with some benefits

i) The visual appeal: The lumping of masses can be easily appreciated in schematic diagrams. This enables easy comprehension and easy formulation of the structure stiffness and inertia matrices.

ii) Just like in the direct stiffness or finite element method of structural analysis, it enables the structure to be divided into elements connected at selected points (nodes) that are being investigated. This is done by lumping the masses at these nodes.

These have made the use of lumped masses a core part of the introductory subjects in structural dynamics.

1.5 Justification of Study

This work is necessary because the use of lumped masses to depict continuous systems is a commonly used form of idealization in structural dynamics. It is found on almost every introductory text on structural dynamics (Donaldson (2006); Graig and Kurdila (2006); Williams (2016)). This is largely due to the simplification and clarity it offers. But its use is limited by the fact that it gives approximate results. This limitation has reduced its use as a tool for research and investigation making it tenable only in works where approximate results are adequate. The results are improved by using evenly spaced lumps, but this places a constraint on the free selection of possible node points for lumping. If the use of lumped masses can be improved upon so as to give exact solutions, then it would be deployed in more advanced studies on the vibration of beams and frames.

1.6 Scope of Work

The project was limited to 2-dimensional structural elements. The loads (inertia) are assumed to act only within the elastic limits of the structural elements.

Prismatic elements (elements with a constant cross-section and which are long in comparism to their cross-section) will be considered. Connecting members are jointed at the centroids of their respective cross-sections such that forces and moments are transferred from one member to another without the generation of additional loading due to eccentricity.

In calculating the displacement of nodal points, deformation due to bending alone was considered. Longitudinal and lateral vibrations were considered in beams of different end conditions. Both single degree of freedom (SDOF) and multi-degree of freedom (MDOF) systems will be considered. The effect of damping was for some reasons ignored. This is because the effect of damping on the natural frequency of frames is very minimal. The damping ratio ζ for reinforced concrete frames range from 2% to

5% (Hesameddin et al 2015). The relationship between the damped natural frequency and the undamped natural frequency is (Thomson and Dahleh 1998)

$$w_d = w_n\sqrt{1-\zeta^2} \tag{1.1}$$

If the value of the damping is taken as being 0.02 to 0.05 then from equation (1.1) damping reduces the natural frequency of reinforced concrete frames by 0.02% to 0.125% which are very minimal and can be neglected.

Beams of different end conditions and simple frames were treated.

Chapter 2

LITERATURE REVIEW

2.1 Brief History of Structural Analysis

The foundation of modern structural analysis can be said to have been laid in the 17th century by the works of Galileo Galilei, Robert Hooke and Isaac Newton (Wikipedia 2014). In 1638 Galileo first established a scientific approach to structural engineering in his article ‘Dialogue Relating to Two New Sciences’. It contained the first attempt to develop the theory for beams and is regarded as the beginning of structural analysis (Gere *et al*, 2012; Wells 2010). Thirty-two years later Robert Hooke in his statement of Hooke’s law provided a scientific understanding of elasticity of materials and their behaviour under load (Britannica 2014).

In 1687, Sir Isaac Newton set out his laws of motion providing for the first time an understanding of the fundamental laws governing structures. He also developed the fundamental theorem of calculus which is an indispensible tool in modern structural analysis.

By the turn of the 18th century, Leonhard Euler with Daniel Bernoulli developed the Euler-Bernoulli beam equation. Euler in 1757 derived the Euler buckling formula, greatly advancing the analysis of slender members under compression. During this period, Daniel Bernoulli with Johann Bernoulli formulated the theory of virtual work which uses the equilibrium of forces and compatibility of geometry to solve structural problems.

The 19th and 20th century witnessed a rapid development in material science and structural analysis. In 1826 Claude Louis Navier formulated the general theory of elasticity in a mathematical form. He also established the elastic modulus giving insight into the structural behaviour of materials (Britannica 2014). In 1873 Carlo Alberto Castigliano presented his two theorems for computing displacement as a partial derivative of the strain energy (Timoshenko 1953). From 1875 to 1920 very little progress was made in the development of theory and analytical techniques, this is largely due to practical limitations on the solvability of algebraic equations with more than few unknowns (McGuire *et al* 2000).

The idea behind the matrix structural analysis concepts were formulated by Maney and Ostenford in their analysis of trusses and framework based on displacement parameters as unknowns. The early formulations of discrete dynamical systems were also done in the 1930s (Duncan and Collar 1934, Duncan and Collar 1935; Frazer *et al* 1938). But these efforts were highly stultified by the severe limitations on the size of unknowns that can be managed by a manual implementation of either the force or displacement method. In 1932 Hardy Cross provided a respite with the introduction of the method of 'moment distribution'. This made possible the solution of complex structural problems by an iterative process. The method of moment distribution became a sTable for structural frame analysis until the birth of digital computers in the early 1950s. This saw the codification of well established framework analysis in the format best suited for the computer, the matrix format. The matrix force method was deployed to the aircraft industry by Levy (1947). This was followed by contributions from other scholars (Rand 1951, Langefors 1952, Wehle and Lansing 1952, Denke 1954). Argyris and Kelsey (1955) systemized the concept of assembling structural system equations from elemental components. Turner in 1959 introduced the direct stiffness method which was to pave way to the finite element method. Before 1970 the direct stiffness method has become the dominant method used in the implementation of production –level structural analysis programs (Fellipa 2000)

Today there has been tremendous development in the use of computers for structural analysis (Samuelsson and Zienkiewi, 2006). The formulation of the finite element method has enabled the analysis of more sophisticated frameworks, plates and shells (Case *et al*, 1999). Advances had also been made on non-linear and plastic analysis of structures.

2.2 Analysis Type

1) <u>Linear Static Analysis</u>

   This is the most common type of analysis, under this type of analysis, loads applied to a body deform the body and the body generates internal forces and reactions at the support to balance the applied external loads. Displacements, stresses and strains under the effect of loads are calculated based on some assumptions. They include the following among others:

   a) All loads are static ie they are applied slowly and gradually until their full magnitude is reached (Gere *et al*, 2012). After reaching the full magnitude, the load remains constant i.e does not vary with time.

   b) The relationship between loads and resulting responses is linear. If you double the magnitude of loads, the response of the structure (displacement, stresses and strains) will also double (Saikat , 2001).

The linearity assumption holds when all materials of the structure obeys Hooke's law ie stress is directly proportional to strain; the induced displacements are small enough to ignore the changes in stiffness caused by loads and loads are constant in magnitude, direction and distribution (Ghali *et al* 2003).

When the above conditions are not valid, then a non-linear analysis has to be carried out.

2) Dynamic Analysis

This kind of analysis is performed on a structure with dynamic loads ie loads that vary with time. The most common case is the dynamic response of a building due to earthquake acceleration at its base. When a structure is exerted by a dynamic load with a frequency that coincides with one of the structure's natural frequencies the structure undergoes large displacements. This is a phenomenon called resonance. The natural frequencies and their corresponding mode shapes depend on the structure's geometry, material properties, its support conditions and static loads.

3) Buckling Analysis

Experience shows that structures may fail in some cases not on account of high stresses surpassing the strength of the material, but due to insufficient stability of slender or thin-walled members (Timoshenko, 1961). This occurs when the stored axial energy is converted into bending energy with no change in the externally applied loads. Mathematically when buckling occurs, the total stiffness matrix becomes singular (Ghali *et al* 2003).

Buckling always involves compression. Buckling analysis involves the calculation of the smallest (critical) loading required to buckle a model. Buckling loads are associated with buckling modes. In engineering we are usually interested in the lowest mode because it is associated with the lowest critical load, but in very slender structures where buckling is a critical factor, calculating multiple buckling modes helps in locating weak areas in the structure.

4) Thermal Analysis

The three basic mechanisms for heat transfer are conduction, convection and radiation. Thermal analysis is the analysis of the changes in stresses and internal forces in structures due to a variation in temperature.

In statically determinate frames, no stresses are generated when the temperature variation (across member sections) is linear. In this case the thermal expansion occurs freely without restraint. For composite members, when the temperature variation is non-linear, each fiber may not be free to undergo the full expansion and thus induces stresses.

In statically indeterminate structures, the increase in length or rotations at members' ends may be restrained giving rise to changes in reaction and internal forces which can be determined by a thermal analysis. The reactions produced by temperature variation must represent a set of forces in equilibrium.

2.3 Analysis by Computer

Before the advent of the computer, structural engineers had of necessity avoided structural analysis formulations which led to the solution of more than three or four simultaneous equations (Jenkins, 1990). The introduction of the method of Moment Distribution circumvented this difficulty and so was extensively developed to solve rotational equilibrium equations at the joints by an iterative process (Leet *et al*, 2005; Hibbeler 2006).

By the early 1950, the computer came on stream, their potential for carrying out the more laborious parts of analysis were appreciated and there began a steady development of methods designed to utilize their speed and numerical processing capacity. The formulation of methods of analysis using matrix algebra was found to be very suitable for the digital computer (McGuire *et al*, 2000). They could generate the required matrices and also perform the matrix operations of addition, multiplication and inversion to provide values of displacements and stress resultants. The concepts of flexibility and stiffness were developed and these new methods were able to analyse skeletal structures and later through the method of finite element plates and shells using the enormous computational power of the digital computer.

Computer aided structural analysis is the method of solving structural analysis problems with the help of a computer software (Saikat, 2001). Virtually all general purpose programs contain essentially four components (see Figure 2.1).

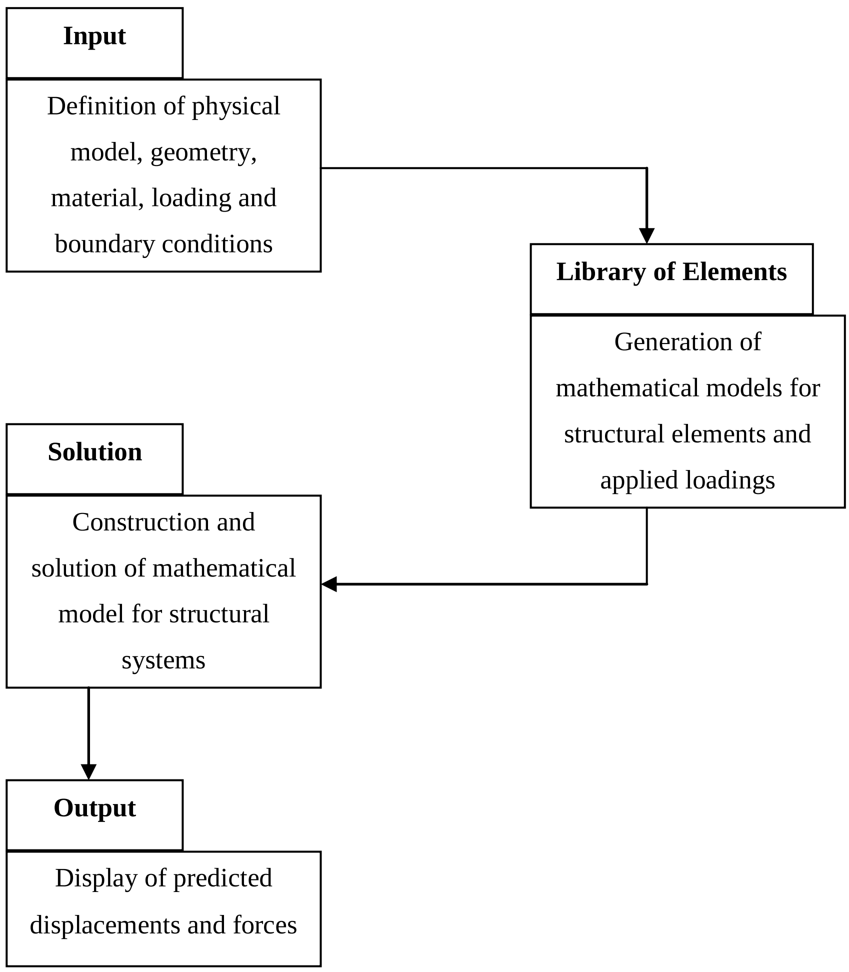


Figure 2.1 Structural Analysis Computational flow (McGuire *et al*, 2000)

i) Input: - The input phase requires information about the material, geometric description and loading of the element. More sophisticated programs produce graphical idealisations of the element enabling some errors in input to be detected early.

ii) Element Library: - The library phase contains codes necessary for the generation of element algebraic relationships for connection to the neighbouring elements as well as connection process itself. The final products are algebraic equations that characterize the response of the structure (the structure stiffness or flexibility matrix).

iii) Solution: - In this phase the generated algebraic equations are solved to find the nodal displacements or resultant stresses. In the case of a linear static analysis program this may mean no more than a simple solution.

iv) Output: - This phase presents the numerical or graphical records of the solution upon which the engineer can base decisions regarding the proportioning of the structure and other design processes.

2.4 Basic Concepts in the Analysis of structures

2.4.1 Principle of Superposition

If deformations are assumed to be small so that they do not significantly affect the geometry of the structure or alter the forces in the members, stresses, strains and displacements due to different actions (loads) can be added together by the Principle of Superposition (Vitor, 2006). The principle of superposition states that the stresses, strains or displacements due to a number of forces acting simultaneously on a system is equal to the sum of the stresses, strains or displacements due to each force acting separately. In structural and mechanical engineering superposition is used to solve for beam and structure deflections due to combined loads when the effects are linear and each load does not significantly alter the geometry of the structural system (Shigley *et al,* 2004). In structural dynamics the superposition of the mode shapes for different natural frequencies is used to characterize the dynamic response of a linear structure (Bathe, 2006).

2.4.2 Equations of Equilibrium

A structure is in a state of static equilibrium if the resultants of all forces and moments at all joints or node points are equal to zero (Wilson, 1995). The three dimensional equilibrium equations for an infinitesimal element is given as

$$\frac{\partial \sigma_1}{\partial x_1}+\frac{\partial \tau_{12}}{\partial x_2}+\frac{\partial \tau_{13}}{\partial x_3}+\beta_1=0$$

$$\frac{\partial \tau_{21}}{\partial x_1}+\frac{\partial \sigma_2}{\partial x_2}+\frac{\partial \tau_{23}}{\partial x_3}+\beta_2=0$$

$$\frac{\partial \tau_{31}}{\partial x_1}+\frac{\partial \tau_{32}}{\partial x_2}+\frac{\partial \sigma_3}{\partial x_3}+\beta_3=0 \qquad (2.1)$$

Where β_i is the body force per unit volume in the i-direction (Boresi and Schmidt, 2003).

For a body subjected to forces in three dimensions, six equations of static equilibrium can be written (Harrison and Nettleton, 1994).

They are:

$$\left.\begin{array}{l}\sum F_x=0,\ \sum F_y=0,\ \sum F_z=0\\ \sum M_x=0,\ \sum M_y=0,\ \sum M_z=0\end{array}\right\} \qquad (2.2)$$

For coplanar forces ie when all the forces acting on the body are in one plane, only three equations are required to establish the static equilibrium of the body. When forces act in the x-y plane, these equations are

$$\sum F_x=0,\ \sum F_y=0,\ \sum M_z=0 \qquad (2.3)$$

When a stucture is in equilibrium every part of it is in the same state and the above equations of statics must be satisfied.

2.4.3 Statical Determinacy

The analysis of a structure is usually carried out to determine the reactions at the supports and the internal stress resultants. If these can be determined entirely from the equations of statics (equations 2.2) above then the structure is said to be statically determinate. When there are more unknown reactive elements than the equations of static equilibrium then the structure is statically indeterminate (Beer *et al*, 1996). The reactive elements in indeterminate structures cannot be obtained without reference to the structures compatibility conditions (Barber, 1992).

The degree of statical indeterminacy of a structure can be determined by inspection or from the number of releases necessary to render the structure statically determinate. But for large complex structures with many members such an approach is difficult and a more mathematical approach is prefered.

For a plane truss the degree of indeterminacy n is given by the formula below

$$n = (m + r) - 2j \qquad (2.4)$$

Where n is the degree of statical indeterminacy. It is zero for determinate structures.

m is the number of members

r is the number of reaction components

j is the number of joints

In the case of a pin-jointed space frame the degree of statical indeteminacy is given by (Adekola, 1988)

$$n = (3m + r) - 3j \qquad (2.5)$$

While for a rigid jointed space frame it is

$$n = (6m + r) - 6j \qquad (2.6)$$

For simple plane frames and continuous beams the degree of statical indeterminacy can be determined using the formula

$$n = r - e - s \qquad (2.7)$$

Where n is the degree of statical indeterminacy.

r is the number of reaction components

s is the number of special conditions e.g hinges. s is the number of releases introduced by that special condition (Onyeyili, 2003; Meriam and Kraige, 2002).

2.4.4 Kinematic Determinacy

A structure is kinematically determinate if it is possible to obtain the nodal displacements from compatibility conditions without reference to equilibrium conditions for example a fixed end beam is kinematically determinate since the end displacements are known from the support conditions. This of course is zero.

The degree of kinematic indeterminacy of a structure is the number of independent joint displacements (rotations and translations) in a structure. This is also known as the degree of freedom of the structure. A system of joint displacements is said to be independent if each displacement can be varied arbitrarily and independently of all the others. In a plane truss, each joint other than a support has two degrees of freedom ie the translation in two orthogonal directions. In a space truss, it is three ie the translation in three mutually perpendicular axis. For plane frames and space frames each joint other than a support has 3 and 6 degrees of freedom respectively.

In modern structural analysis, the displacement method is almost entirely used and in this method the relevant indeterminacy is the kinematic one (Megson, 2005).

2.4.5 **Strain Energy**

Strain energy is defined as the energy absorbed by a bar (member) during the loading process (Gere *et al*, 2012; Nash and Potter, 2014). When the structure is gradually unloaded this elastic strain energy will return the structure to its undeformed state provided the material's elastic limit is not exceeded. From the principle of conservation of energy, strain energy is equal to the work done by the load provided no energy is added or subtracted in the form of heat.

The strain energy dU of an infinitesimal element is given by

$$dU = \frac{1}{2}\sigma\epsilon dv \qquad (2.8)$$

Where ε is the general symbol for strain, σ is the general symbol for stress and *dv* is the volume of the element.

The strain energy per unit volume of element is known as the strain energy density and is equal to the area under a stress-strain curve. For linear elastic elements the stress-strain curve is a straight line.

The total strain energy is the strain energy due to the four internal stresses (Ghali *et al* 2003)

$$U = \frac{1}{2}\int \frac{N^2}{EA}dx + \frac{1}{2}\int \frac{M^2}{EI}dx + \frac{1}{2}\aleph\int \frac{V^2}{GA}dx + \frac{1}{2}\int \frac{T^2}{GJ}dx \qquad (2.9)$$

where N and A are the normal force and cross-sectional area respectively. E is the young's modulus for the material. I is the second moment of area of the member cross section. T is the torsional moment, G is the modulus of elasticity in shear and J is a torsional constant. For members with circular cross-sections J is equal to the polar moment of inertia of the cross-section. $\aleph$ is a shape factor which depends on the shape of the member's cross-section.

2.4.6 Betti's and Maxwell's theorem

Betti's theorem states that the sum of the products of the forces of a system (say F system) and the displacements at the corresponding coordinates caused by another system of forces (say Q system) is equal to the sum of the products of the forces of the Q system and the displacements at the corresponding coordinates caused by the F system.

$$\sum F_i D_{iQ} = \sum Q_i D_{iF} \tag{2.10}$$

This theorem is valid for only linear elastic structures.

For a special case where there is only one force $F_i = 1$ in the F system and one force $Q_i = 1$ in the Q system. Equation (2.10) reduces to $D_{iQ} = D_{iF}$ and is better known as

$$f_{ij} = f_{ji} \tag{2.11}$$

Where f_{ij} is the displacement at coordinate i due to a unit load at coordinate j. f_{ji} is the displacement at coordinate j when a unit load is applied at coordinate i. Equation (2.10) is known as the Maxwell's reciprocal theorem and it states that in linear elastic structures the displacement at coordinate i due to a unit force at coordinate j is equal to the displacement at j due to a unit force acting at coordinate i. This theorem is responsible for the symmetrical nature of the flexibility and stiffness matrices used in structural analysis under the force and stiffness methods.

2.5 Energy principles in the analysis of structures

The response of structures to static and dynamic loads may be accessed in terms of energy (Segal and Val, 2006). Energy principles are formulated based on the principle of conservation of energy. They are fundamental in the study of structural dynamics and structural stability. They include the principles of virtual work and complementary virtual work, principle of minimum potential energy and minimum complementary energy and the Castigliano theorems.

2.5.1 Principle of Virtual Work

This principle was formulated by Bernoulli. The principle states that, if a structure is in equilibrium and remains in equilibrium while it is subjected to a virtual distortion, the external virtual work W_E done by the external forces acting on the structure is equal to the internal work U done by the internal stresses.

$$\delta W_E = \delta U = \int_V \sigma_{ij}\, \delta e_{ij}\, dV \quad (2.12)$$

This implies that the product of the actual displacement and the corresponding virtual forces is equal to the product of the actual internal displacements and the corresponding virtual internal forces (Ghali *et al*, 2003).

$$\sum_{i=1}^{n} (Virtual\ force\ at\ i)(Actual\ displacement\ at\ i)$$

$$= \sum_{i=1}^{n} (Virtual\ internal\ forces\ at\ i)(Actual\ Internal\ displacement\ at\ i)$$

(2.13)

This principle is used to determine the external displacement at any coordinate from the strains due to known actual internal displacements. For convenience the virtual loads are chosen in such a way that the right-hand side of the above equation directly gives the deformation. The virtual force is therefore taken as P = 1, this is known as the unit load theorem and can be applied to both linear elastic and non-linear elastic structures.

Instead of displacement, the principle of virtual work can also be used to determine the force at a coordinate if the real internal forces are known, then

$$\sum_{i=1}^{n}(Virtual\ displacement\ at\ i)(Actual\ force\ at\ i)$$

$$= \sum_{i=1}^{n}(Virtual\ internal\ displacement\ at\ i)(Actual\ Internal\ force\ at\ i)$$

(2.14)

This is known as the unit displacement theorem (Ghali *et al* 2003), it is valid for only linear elastic structures owing to the inherent assumption in its definition that strain at any point on a structure is proportional to the force at that point. Like in the unit load theorem, unit displacements are used for convenience hence its name unit displacement theorem. The principle of virtual work was extended to dynamics by D'Alembert in 1743 when he introduced the concept of inertial forces. It has been used in the formulation of symmetric stiffness matrices of structures (Pinto and Prato 2006), thin-walled curved beam equations (Rajasekaran and Padmanabham 1989) and in the dynamic analysis of beams (Boutros 2000).

2.5.2 Principle of Complementary Virtual Work

It states that the complementary virtual work δW_E^* done by an external virtual force system under the actual deformation of a structure is equal to the complementary work δU^* done by the virtual stresses under the actual strains.

$$\delta W_E^* = \delta U^* = \int_V \delta\sigma_{ij}\, e_{ij}\, dV \quad (2.15)$$

Both the principle of virtual work and the principle of complementary virtual work apply to situations involving small deformation.

2.5.3 Principle of Minimum Potential Energy

It states that of all displacement fields which satisfy the prescribed constraint condition, the correct state is that which makes the total potential energy of the structure a minimum.

$$\delta\pi = \delta(U + V_E) = 0 \tag{2.16}$$

$$\pi = U + V_E \tag{2.16a}$$

Where U is the structure's strain energy and V_E is the external potential or work done by external forces.

2.5.4 Principle of Minimum Complementary Energy

The principle states that of all states of stress which satisfy the equations of equilibrium, the correct state is that which makes the total complementary energy of the structure a minimum.

$$\delta\Pi^* = \delta(U^* + V_E^*) = 0 \tag{2.17}$$

$$\Pi^* = U^* + V_E^* \tag{2.17a}$$

For a case of an isotropic, linearly elastic structure subjected to discrete generalized forces, the complementary strain energy U* is equal to the strain energy U.

2.5.5 Castigliano's Theorems

These were published by Castigliano and Italian Engineer in 1873. The two theorems can be derived from the principles of minimum potential energy and minimum complementary energy.

Castigliano's first theorem states that if the strain energy U stored in an elastic structure is expressed as a function of the generalized displacements q_i, then the partial derivatives of U with respect to any one of the generalized displacements qi is equal to the corresponding generalized force Qi.

$$Q_i = \frac{\partial U}{\partial q_i}; \quad i = 1,2, \dots n \tag{2.18}$$

Castigliano's second theorem on the other hand states that if the strain energy U in a linearly elastic structure is expressed as a function of the generalized forces Qi, then the partial derivative of U with respect to any one of the generalized forces Qi is equal to the corresponding displacement qi.

$$q_i = \frac{\partial U}{\partial Q_i}; \quad i = 1,2, \dots n \tag{2.19}$$

This theorem is applicable to only linearly elastic structures. If the strain energy is replaced with the complementary strain energy we obtain

$$q_i = \frac{\partial U^*}{\partial Q_i}; \quad i = 1,2, \dots n \tag{2.20}$$

Equation (2.20) is known as the Engesser's theorem and applies to both linearly and non-linearly elastic structures.

2.6 Matrix Analysis of Structures

The advent and subsequent development of digital computers has made it possible to analyse complicated structures using matrix methods (Rajasekaran and Sankarasubramaniam 2001). Recall that indeterminate structures are those structures that cannot be solved using only the equations of equilibrium. To analyse an indeterminate structure there is need to generate the structure's deformation equations and the equilibrium, compatibility and force-displacement requirements of the structure must be satisfied. Equilibrium is satisfied when the reactive forces hold the structure at rest. Compatibility is satisfied when the various segments of the structure fit together without breaks or overlaps. The force displacement requirement depends on the way the material of the structure responds to loads.

There are basically two matrix methods of analyzing structures; they are the flexibility method and the displacement method.

2.6.1 The Flexibility Matrix

This method was developed by James Clerk Maxwell in 1864. It was later refined by Ottor Mohr and Heinrich Muller-Breslau (Boyajian 2005). In the force/flexibility method a system of arbitrary generalized forces which satisfy equilibrium conditions are taken as unknowns. The correct state of stress is obtained using the principle of minimum complementary energy. The structure compatibility equations are written as (Jenkins 1990)

$$\{f_o\} + [f]\{Q\} = \{q\} \tag{2.21}$$

Where $\{f_o\}$ is a vector of displacements in the generalized coordinates of the reduced /basic structure, {Q} is the vector of the generalized forces which is sought for. {q} is a vector of generalized displacement of the structure (with all redundant forces present). When there is no transversely loaded external forces on the elements of the structure $\{f_o\} = 0$.

[f] is a square matrix called the flexibility matrix. The elements of [f] are the flexibility influence coefficients f_{ij}, f_{ij} represents the displacement at point i in the direction of Q_i produced by a unit force Q_j. The flexibility influence coefficient can be obtained from the Castigliano's theorem as

$$f_{ij} = \frac{\partial^2 U}{\partial Q_i \partial Q_j} \tag{2.22}$$

Equation (2.22) applies to linearly elastic structures. These influence coefficients can also be obtained using the principle of virtual work.

In structural dynamics it may sometimes be necessary to find the flexibility matrix of a structure about a set of coordinates. If Q_i (i = 1,2,3,…n) represents a set of loads acting in coordinate 1, 2, 3,…,n and Y_i (i=1,2,3,…,n) represent the redundant forces on the structure. Then the flexibility matrix relating the displacements q_i caused by forces Q_i must satisfy the relationship

$$\{q\} = [f]\{Q\} \tag{2.23}$$

It can be shown that the flexibility matrix can be expressed as (Tauchert 1974)

$$[f] = [B_{QQ}] + [B_{QY}][D] \tag{2.24}$$

Where $[D] = -[B_{YY}]^{-1}[B_{QY}]$ (2.24a)

$$[B_{QQ}] = [A_Q]^T[C][A_Q] \tag{2.24b}$$

$$[B_{QY}] = [A_Q]^T[C][A_Y] \tag{2.24c}$$

$$[B_{YQ}] = [A_Y]^T[C][A_Q] = [B_{QY}]^T \tag{2.24d}$$

$$[B_{YY}] = [A_Y]^T[C][A_Y] \tag{2.24e}$$

[C] is the flexibility matrix of the unconstrained structure.

$[A_Q]$ is a matrix of member forces due to unit external loads Q such that an element of $[A_Q,]$, A_{Qij} represents the force in coordinate i due to a unit applied load Q_j.

$[A_Y]$ is a matrix of member forces due to unit redundant force. Such that an element of $[A_Y]$, A_{Yij} is the magnitude of member force at I due to a unit value of redundant force Y_j.

The corresponding stiffness matrix can be obtained by evaluating the inverse of the flexibility matrix.

The force method in the pre-computer era was the popular analysis tool for civil, mechanical and aerospace engineering structures (Raju and Nagabhushanam, 2000). The automation of force method on computers met a stiff challenge. The redundant analysis was the major problem. This arose from the fact that an indeterminate structure can be reduced to many different basic structures (Robinson and Haggenmacher 1971, Topou 1979, Kaneko *et al* 1983). A new method known as the integrated force method was developed to overcome this obstacle (Patnaik 1986). Unlike the basic force method, the integrated force method is independent of redundant and has successfully been used for the static analysis of frames (Patnaik and Yadagiri, 1989), plates (Kaljevic *et al*, 1996) and structural dynamics and optimization (Patnaik and Yadagiri, 1976, Patnaik *et al* 1996). Despite all these efforts at improving the force method, the displacement method still remained the preferred choice for automated structural analysis.

2.6.2 The Stiffness Matrix

In the displacement method of analysis, the unknown quantities are a set of arbitrary generalized displacements which satisfy the kinematic conditions of the structure. If q_i (i=1,2,…,n) represent a set of generalized displacements proportional to the generalized forces Q_i then

$$\{Q\} = [k]\{q\} \tag{2.25}$$

Where [k] is a square matrix known as the stiffness matrix. The coefficient k_{ij} of the stiffness matrix represent the force at i due to a unit displacement at coordinate j when all other displacements are equal to zero. In the displacement method of analysis, the displacement and corresponding forces can only be obtained at the specified generalized coordinates.

The stiffness matrix with respect to any set of prescribed coordinates q_i (i=1,2,…,n) can be obtained from (Tauchert 1974)

$$[k] = [b_{qq}] + [b_{qy}][d] \quad (2.26)$$

Where $[d] = -[b_{yy}]^{-1}[b_{yq}]$ (2.26a)

$$[b_{qq}] = [a_q]^T[K][a_q] \quad (2.26b)$$

$$[b_{qy}] = [a_q]^T[K][a_y] \quad (2.26c)$$

$$[b_{yq}] = [a_y]^T[K][a_q] = [b_{qy}]^T \quad (2.26d)$$

$$[b_{yy}] = [a_y]^T[K][a_y] \quad (2.26e)$$

[K] is the stiffness matrix of the unconstrained structure.

$[a_q]$ is a matrix of member displacements due to unit displacement at the prescribed coordinates ie an element in $[a_q$,], a_{qij} is the displacement in coordinate i when the structure experience a unit displacement in coordinate q_j, all other joint displacements q_i being zero.

If y_i (i=1,,2,3,…,n) represent the kinematic indeterminate displacements, $[a_y]$ is a matrix of member displacements per unit displacement in coordinate y_i ie an element of $[a_y]$, a_{yij} is the displacement at i corresponding to a unit displacement y_j. By inverting the stiffness matrix [k] we get the flexibility matrix [f]. The overall potential energy of a structure is directly related to the stiffness of the structure (Wong and Zhao 2007).

2.7 Dynamic Analysis

All bodies possessing mass and elasticity are capable of vibration (Chen and Zhou 1993; Thomson 2003; Rajasekaran 2009). When the vibration takes place in the absence of external excitation it is known as free vibration. The equation of motion for a viscously damped forced system is

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{2.27}$$

Where m is the mass of the vibrating system, k is the stiffness and c is the damping coefficient (Thomson and Dahleh 1998).

Damping in small amounts has little influence on the natural frequency of vibration and may be neglected in its calculation. When damping is neglected c = 0 and the equation of motion becomes

$$m\ddot{x} + kx = F(t) \tag{2.28}$$

Equation (2.28) is the equation of motion of a forced undamped system under vibration. If F(t) = 0 we have a free undamped vibration. Equation (2.28) is a non-homogenous second differential equation and the solution consists of two parts, the complementary function which is the solution of the homogenous equation and the particular integral. The complementary function is

$$x = A\sin\omega t + B\cos\omega t \tag{2.29}$$

Where w is the natural frequency of vibration, A and B are constants that can be evaluated from the initial conditions $x(0)$ and $\dot{x}(0)$ to obtain

$$x = \frac{\dot{x}(0)}{w}\sin\omega t + x(0)\cos\omega t \tag{2.30}$$

The particular integral can be obtained from the Duhamel's integral or the convolution integral as

$$x = \frac{1}{m\omega}\int_0^t F(\xi)\sin\omega(t-\xi)\,d\xi \tag{2.31}$$

Hence the complete solution of the equation of motion of an undamped system under forced vibration is

$$x = \frac{\dot{x}(0)}{\omega}\sin\omega t + x(0)\cos\omega t + \frac{1}{m\omega}\int_0^t F(\xi)\sin\omega(t-\xi)\,d\xi \qquad (2.32)$$

2.7.1 Lagrange's Equation

Equation (2.32) is for structures with only one degree of freedom. But every real structure has a continuous distribution of mass and hence an infinite number of degrees of freedom (DOF) (Sule 2011). The structure however can be modeled with a finite number of degrees of freedom. The number of degrees of freedom depends on the accuracy required and the mass distribution of the structure. A structure modeled with N DOF will have N natural frequencies and for each natural frequency there will be a natural state of vibration with a displacement configuration known as normal mode.

The analysis of multi-DOF systems can be carried out using the method of virtual work. Lagrange formulated a scalar equation in terms of generalized coordinates and is presented as

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \qquad (2.33)$$

$i = 1,2,\ldots n$

Where q_1, q_2, …, q_n are a set of independent generalized displacements, T is the kinetic energy of the structure and U is the strain energy of the structure and Q_i is the non-conservative or the non-potential force on the system.

Since it is easier to analyse complicated structures in matrix notation, the Lagrange's equations can be used to develop the matrix equation for the analysis of a damped n-degree of freedom discrete mass structure as

$$[m]\{\ddot{x}\} + [d]\{\dot{x}\} + [k]\{x\} = \{F\}_a \qquad (2.34)$$

Where [m] is the inertia matrix, [d] is the damping matrix and [k] is the structures stiffness matrix. If a set of generalized coordinates q_i different from the absolute displacement x is used, equation (2.34) is re-written as (Trauchert 1974)

$$[m]\{\ddot{q}\} + [d]\{\dot{q}\} + [k]\{q\} = \{Q\}_a \quad (2.34)$$

For a free undamped vibration equation (2.34) becomes

$$[m]\{\ddot{q}\} + [k]\{q\} = 0 \quad (2.35)$$

By pre-multiplying the equation with the structure's flexibility matrix [f]

$$[D]\{\ddot{q}\} + \{q\} = 0 \quad (2.36)$$

Where the dynamical matrix $[D] = [f][m]$ (2.37)

A solution of equation (2.36) is given as

$$\{q\} = \{\phi\}\sin(\omega t + \delta) \quad (2.38)$$

By substituting equation (2.38) into equation (2.36) and rearranging we obtain

$$([D] - \lambda[I])\{\phi\} = 0 \quad (2.39)$$

Where [I] is an identity matrix and $\lambda = {}^{1}/_{\omega^2}$ (2.40)

Equation (2.39) represents a system of n-homogenous, linear algebraic equation in the amplitudes $\{\phi\}$ and can be solved to get the frequencies $\omega_1, \omega_2, \ldots, \omega_n$ for an n-degree of freedom system. For each distinct frequency ω_j (or eigenvalue), there will be a set of amplitudes $\{\phi\}_j$ (or eigenvector).

A general solution to equation (2.35) becomes the sum of each solution multiplied by an arbitrary constant.

$$\{q\} = \{\phi\}_1 C_1 \sin(\omega_1 t + \delta_1) + \{\phi\}_2 C_2 \sin(\omega_2 t + \delta_2) + \cdots + \{\phi\}_n C_n \sin(\omega_n t + \delta_n) \quad (2.41)$$

Put in matrix form equation (2.41) becomes

$$\{q\} = [\phi]\{C\sin(\omega t + \delta)\} \quad (2.42)$$

Or

$$\{q\} = [\phi]\{A\cos(\omega t) + B\sin(\omega t)\} \tag{2.43}$$

Where $[\phi]$ is the modal matrix. The constants A_1, A_2, …, A_n and B_1, B_2, …, B_n are determined from the initial conditions. If at time t = 0 the displacement and velocities are of values $\{q_o\}$ and $\{\dot{q}_o\}$ respectively then

$$\{A\} = [\phi]^{-1}\{q_o\} \tag{2.44}$$

$$\{wB\} = [\phi]^{-1}\{\dot{q}_o\} \tag{2.45}$$

The eigenvectors or relative amplitudes $\{\phi\}_j$ obtained from a free vibration satisfy certain orthogonality conditions. They are summarized below (Tauchert 1974)

$$\{\phi\}_i^T[m]\{\phi\}_j = 0 \quad when\ i \neq j \tag{2.46a}$$

$$\{\phi\}_i^T[m]\{\phi\}_j = M_j \quad when\ i = j \tag{2.46b}$$

$$\{\phi\}_i^T[k]\{\phi\}_j = 0 \quad when\ i \neq j \tag{2.47a}$$

$$\{\phi\}_i^T k\{\phi\}_j = M_j\,\omega_j^2 \quad when\ i = j \tag{2.47b}$$

where M_j and w_j are the generalized mass and circular frequency for the j^{th} normal mode respectively. Equations (2.46a) – (2.47b) can written in terms of modal matrix as

$$[\phi]^T[m][\phi] = [M] \tag{2.48}$$

$$[\phi]^T[k][\phi] = [\mathrm{K}] \tag{2.49}$$

Where [M] is a generalized inertia matrix and [K] is a generalized stiffness matrix. Both are diagonal matrices. The matrix [K] has the following elements $M_1\omega_1^2$, $M_2\omega_2^2$, $M_3\omega_3^2$, etc on its main diagonal.

The differential equations of a multi-degree of freedom system are in general coupled. Mass or dynamical coupling exist if the mass matrix is non-diagonal, whereas stiffness or static coupling exist if the stiffness matrix is non-diagonal. There exist a coordinate system that has neither form of coupling, such coordinates are called principal or normal coordinates. Consider a set of coordinates r_1, r_2, …, r_n or {r} such that

$$\{q\} = [\phi]\{r\} \quad (2.50)$$

By substituting equation (2.50) into (2.35) and applying the orthogonality conditions equations (2.48) and (2.49) we obtain

$$[M]\{\ddot{r}\} + [K]\{r\} = 0 \quad (2.51)$$

Since [M] and [K] are diagonal matrices, equation (2.51) is uncoupled. This is particularly important in the analysis of forced vibration.

2.7.2 Hamilton's Principle

Structures can be analysed as systems with continuously distributed masses. This leads to exact results that are within the framework of the classical beam theory. While Lagrange's equations provide a way of analyzing multi-degree of freedom system, a similar approach for continuous structures is an energy theorem known as the Hamilton's principle.

The Hamilton's principle states that the motion of an elastic structure during the time interval $t_1 < t < t_2$ is such that the time integral of the total dynamic potential $U - T + V_E$ is an extremum.

$$\delta \int_{t_1}^{t_2} (U - T + V_E)dt = 0 \quad (2.52)$$

where U represents the strain energy of the system, T the kinetic energy and V_E the work done by the external forces.

The principle represents a generalization of the principle of minimum potential energy to include dynamic effects. When the dynamic effects are neglected, kinetic energy T = 0 and the equation reduces to the principle of minimum potential energy.

Using the Hamilton's principle the partial differential equation and boundary conditions governing the longitudinal vibration of a bar is derived as (Trauchert 1974)

$$(EAu^{'})^{'} - \mu\ddot{u} + F(x_1, t) = 0 \quad (2.53)$$

and

$$N_o = [EAu_1^{'}]_{x_1=0} \ \ or\ \delta u_1(0, t) = 0 \quad (2.53a)$$

$$N_L = [EAu_1^{'}]_{x_1=0} \ \ or\ \delta u_1(L, t) = 0 \quad (2.53b)$$

respectively.

Where $A(x_1)$ is the cross sectional area of the bar, $\mu(x_1)$ is the mass per unit length of the bar and E is the modulus of elasticity of the material of the bar. Equation (2.53) is the equation of motion for the longitudinal vibration of a bar while equations (2.53a – 2.53b) are the corresponding boundary conditions. x_1 is the position of a element of the bar and u_1 is the displacement of the element in question in the longitudinal axis of the bar.

For the free longitudinal vibration of a uniform bar $F(x_1, t) = 0$ and equation (2.53) reduces to

$$c^2 u_1^{''} = \ddot{u}_1 \quad (2.54)$$

where $c^2 = \frac{EA}{\mu}$ (2.54a)

Equation (2.54) is a one dimensional wave equation and has a general solution

$$u_1 = g(x_1 - ct) + h(x_1 + ct) \quad (2.55)$$

Where g and h are arbitrary functions satisfying initial conditions. For a normal mode vibration (where each particle of the bar vibrates harmonically at a circular frequency w)

$$u_1(x_1, t) = \phi_1(x_1)\sin(\omega t + \delta) \tag{2.56}$$

which upon substitution into equation (2.54) will give

$$\phi'' + \frac{\omega^2}{c^2}\phi = 0 \tag{2.57}$$

The general solution of equation (2.57) is

$$\phi(x_1) = C_1 \cos\frac{\omega x_1}{c} + C_2 \sin\frac{\omega x_1}{c} \tag{2.58}$$

By introducing the boundary conditions equation (2.58) results in an eigenvalue problem, the solution of which yields the natural circular frequencies ω_j and mode shapes (eigenvectors) ϕ_j. The general solution by mode superposition is

$$u_1(x_1, t) = \sum_{j=1}^{\infty} \phi_j(x_1)\left(A_j \cos\omega_j t + B_j \sin\omega_j t\right) \tag{2.59}$$

Where the constants A_j and B_j can be determined form the initial conditions.

The eigenfunctions ϕ_j satisfy certain orthogonality relationships. The orthogonality conditions for a uniform bar having any combination of clamped and free ends are summarized as

$$\mu\int_0^l \phi_i\phi_j\,dx_1 = 0 \ when\ i \neq j \tag{2.60a}$$

$$\mu\int_0^l \phi_i\phi_j\,dx_1 = M_j \ when\ i = j \tag{2.60b}$$

$$EA\int_0^l \phi_i'\phi_j'\,dx_1 = 0 \ when\ i \neq j \tag{2.60c}$$

$$EA\int_0^l \phi_i'\phi_j'\,dx_1 = \omega_j^2 M_j \ when\ i = j \tag{2.60d}$$

Where μ and A are the mass per unit length and cross-sectional area of element respectively, M_j is the generalized mass for the j^{th} mode of vibration.

From initial conditions the constants Aj and Bj can be expressed as

$$A_j = \frac{\mu}{M_j}\int_0^l u_1(x_1, 0)\phi_j\,dx_1 \tag{2.61a}$$

$$B_j = \frac{\mu}{M_j\omega_j}\int_0^l \dot{u}_1(x_1, 0)\phi_j\,dx_1 \tag{2.61b}$$

Using the Hamilton's principle the differential equations and boundary conditions governing the lateral vibration of a bar is obtained as

$$(EIu_2'')'' + \mu\ddot{u}_2 = F(x_1, t) \quad (2.62)$$

$$V_o = -[(EIu_2'')']_{x_1=0} \; or \; \delta u_2(0, t) = 0 \quad (2.63a)$$

$$V_l = -[(EIu_2'')']_{x_1=l} \; or \; \delta u_2(l, t) = 0 \quad (2.63b)$$

$$M_o = -[EIu_2'']_{x_1=0} \; or \; \delta u_2'(0, t) = 0 \quad (2.63c)$$

$$M_l = -[EIu_2'']_{x_1=0} \; or \; \delta u_2'(l, t) = 0 \quad (2.63d)$$

Where M_o and V_o represent the bending moment and shear force at $x_1 = 0$,

V_l and M_l represent the bending moment and shear force at $x_1 = l$.

Equation (2.62) is the equation of motion for flexural vibrations while equations (2.63a) – (2.63d) are the corresponding boundary conditions.

For a uniform bar undergoing free lateral vibration equation, $F(x_1,t) = 0$ and equation (2.62) reduces to

$$(EIu_2'')'' + \mu\ddot{u}_2 = 0 \quad (2.64)$$

For normal modes of vibration

$$u_2(x_1, t) = \phi(x_1)\sin(\omega t + \delta) \quad (2.65)$$

When equation (2.65) is substituted into (2.64) we obtain

$$\phi^{iv} - \beta^4\phi = 0 \quad (2.66)$$

Where $\beta^4 = \frac{\mu\omega^2}{EI}$ (2.66a)

The general solution of equation (2.66) is

$$\phi(x_1) = C_1 cosh\beta x_1 + C_2 sinh\beta x_1 + C_3 cos\beta x_1 + C_4 sin\beta x_1 \quad (2.67)$$

By introducing the necessary boundary conditions and solving the resulting eigenvalue problem, the beam's natural frequencies ω_j and mode shapes ϕ_j (j =

1,2,…∞) are obtained. The general free vibration solution is obtained by mode superposition as

$$u_2(x_1,t) = \sum_{j=1}^{\infty} \phi_j(x_1)\left(A_j \cos\omega_j t + B_j \sin\omega_j t\right) \tag{2.68}$$

where A_j and B_j are constants. Just like in the longitudinal vibration of beams, normal modes also exhibit some orthogonality relationships. The othogonality conditions for a uniform beam with any combination of clamped, free, simply supported and guided end can be summarized as (Tauchert 1974)

$$\mu \int_0^l \phi_i \phi_j \, dx_1 = 0 \; when \; i \neq j \tag{2.69a}$$

$$\mu \int_0^l \phi_i \phi_j \, dx_1 = M_j \; when \; i = j \tag{2.69b}$$

$$EI \int_0^l \phi_i'' \, \phi_j'' \, dx_1 = 0 \; when \; i \neq j \tag{2.69c}$$

$$EI \int_0^l \phi_i'' \, \phi_j'' \, dx_1 = \omega_j^2 M_j \; when \; i = j \tag{2.69d}$$

The constants A_j and B_j in equation (2.68) are determined from the initial conditions and generalized mass by using equations (2.61a – 2.61b).

2.7.3 The Finite Element Method

In the finite element method the structure is represented as an assemblage of subdivisions called finite elements. The elements are connected to each other at joints known as nodes. The variation of displacement along the length of an element is assumed to be governed by simple functions. These approximating functions are known as interpolation models and are defined in terms of displacement at the nodes. When the equilibrium equations of the structure are written, the unknowns in the equilibrium will be the displacement at the nodes. Once they are known, the displacement at any point on an element can be obtained using the integration models.

Basic steps of the finite Element Method

1. The structure is divided into elements, the number depends on the accuracy of results required.
2. Displacement models for elements are chosen. They should be simple functions that can easily be worked on, preferably polynomials. They are also required to meet certain convergence requirement.
3. Using the assumed displacement model, the structure stiffness matrix and inertia matrix are formulated.
4. The element equations are assembled to get an overall equilibrium equation. This includes the stiffness and inertia matrices.
5. The resulting structure matrix is analysed for the displacement at the nodes. In case of dynamic analysis, it is analysed for the natural frequencies and mode shapes (Rao 2005).

a) Axial Element

Axial elements support only longitudinal forces and hence act like a spring. For an axial element with the two ends displaced by u_1 and u_2, the displacement at any point $\xi = \frac{x}{l}$ is assumed to be a straight line and the displacement is a superposition of the two mode shapes. The normalized mode shapes are

$$\phi_1 = (1 - \xi) \tag{2.70}$$

$$\phi_2 = \xi \tag{2.71}$$

The displacement u at any point along the axis of the element is

$$u = (1 - \xi)u_1 + \xi u_2 \tag{2.72}$$

This displacement model is used to formulate the kinetic energy and potential energy of the system which are substituted into Lagrange's equations for generalized mass to obtain the element's inertia mass as

$$[m] = \frac{\mu l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{2.73}$$

This can also be obtained from the equation for generalized masses obtained previously as

$$m_{ij} = \int_0^l \mu\phi_i\phi_j\,dx_1 \tag{2.74}$$

b) Beam Element

The local coordinate for the beam elements are the lateral displacements and rotations at the two ends. The positive sense of these coordinates is arbitrary, but the one mostly accepted is presented in Figure 2.2

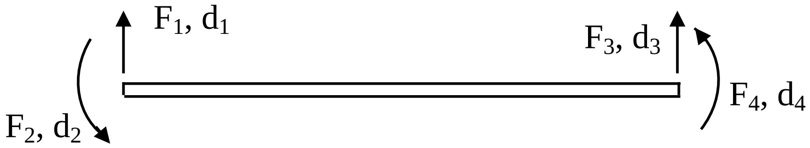


Figure 2.2: Coordinates of a Beam Element excluding axial forces

The mode shapes of the beam are generated from the general equation of the beam which is a cubic polynomial. The normalized mode shapes for a given beam element with four DOF numbered 1- 4 are

$$\phi_1(x) = 1 - 3\xi^2 + 2\xi^3 \tag{2.75a}$$

$$\phi_2(x) = l\xi - 2l\xi^2 + l\xi^3 \tag{2.75b}$$

$$\phi_3(x) = 3\xi^2 - 2\xi^3 \tag{2.75c}$$

$$\phi_4(x) = -l\xi^2 + l\xi^3 \tag{2.75d}$$

For the mode shapes the generalized masses m_{ij} (which form the elements of the inertia matrix) can be obtained from equation (2.74) while stiffness influence coefficients k_{ij} (which form the elements of the element stiffness matrix) can be obtained from

$$k_{ij} = \int_0^l EI\phi_i'' \phi_j'' dx_1 \tag{2.76}$$

By substituting equations (2.75a) – (2.75d) into equation (2.74) the mass matrix for a uniform element is obtained as

$$[m] = \frac{\mu l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{2.77}$$

This matrix is called consistent mass because it is based on the same functions used for the stiffness (Thomson and Dahleh 1998).

The accuracy of the finite element method depends on the assumed shape functions. The shape functions are picked from different base functions such as polynomials and trigonometric functions (Milsted and Hutchinson 1974, Christian 2009). They are carefully chosen to meet the following requirements amongst others (Hutton 2004, Rao 2005)

1) Vertex modes have unit magnitude at one vertex and zero at all others.
2) Edge modes have magnitude along one edge and zero at all edges and vertices.
3) Inter-element compatibility and degrees of continuity (C^0, C^1, …) e.t.c

The difference between models developed from different shape functions is their accuracy and rate of convergence (Faroughi and Ahmadian 2010). The accuracy of finite element method can be improved by increasing the number of elements while keeping the order of the elements fixed. This results in elements of smaller sizes (Zienkiwicz *et al* 2013). Another method is by increasing the order of the interpolation functions without any reduction in the size of the elements (Houmat 2009).

Another strategy for obtaining more accurate finite element models is by employing the inverse methods (Faroughi *et al* 2011). In the inverse method the criteria that must be satisfied by the element model is determined and used to form a parametric family of admissible mass and stiffness matrices. The elements are determined by minimizing the discretization error in the element formulation (Ahmadian *et al* 1998). This has the benefit of producing a better model for the element under consideration. NoTable contributors to this method include Kim (Kim 1993) and Fried I (Fried and Chavez 2004; Fried and Leong 2005).

2.7.4 **Rayleigh Ritz Method**

This is another method for obtaining approximate results in the dynamic analysis of structures. It also has the effect of reducing the number of degrees of freedom of a continuous system from infinity to a finite. In the application of Rayleigh-Ritz method the beam deflection is represented by a series of shape functions multiplied by constant coefficients. The chosen functions must satisfy the kinematic boundary conditions of the vibrating element. If the chosen equations are ψ_1, ψ_2,..,ψ_3 then the beams deflection curve, generalized masses and generalized stiffness coefficients are given as follows

$$u_2(x_2, t) = \sum_{j=1}^{n} \psi_j(x_1) q_j(t) \tag{2.78}$$

$$\bar{m}_{ij} = \int_0^l \mu \psi_i \psi_j \, dx_1 \tag{2.79}$$

$$\bar{k}_{ij} = \int_0^l EI \psi_i'' \psi_j'' \, dx_1 \tag{2.80}$$

After using equations (2.62) and (2.63) to generate the inertia matrix and stiffness matrix the analysis that follows is the same as in the Lagrange's method. The solution of the structures equations of motion is

$$\{q\} = [\gamma]\{A\cos(wt) + B\sin(wt)\} \tag{2.81}$$

and the final equation of vibration is written as

$$u_2(x_1, t) = \sum_{i=1}^{n} \phi_j(x_1)(A_i \cos \omega_i t + B_i \sin \omega_i t) \tag{2.82}$$

where $\phi_i(x_1) = \sum_{j=1}^{n} \psi_j(x_1)\gamma_{ij}$ (2.83)

Reducing the degrees of freedom of a continuous structure is like adding additional constraints on the structure hence the idealized structure is stiffer than the real structure. As a result the values of frequencies obtained by Rayleigh-Ritz method are large and amplitudes of vibration smaller than their exact values (Tauchert 1974). Rayleigh-Ritz method generally gives a good prediction of the values of the fundamental frequency and provides progressively poor results for the higher modes. To obtain better results there is need to choose good shape functions and increase the number of functions.

2.8 Dynamic Analysis of Building Frames

Most international codes allow the use of an equivalent static lateral force method for the practical design of real structure from the action of dynamic loads (Nassani, 2014). The estimation of the fundamental period of buildings is therefore important both for the design of new buildings and for the performance assessment of existing one (Clive *et al*, 2007; Chiauzzi *et al*, 2012). Empirical formula were proposed and adopted for calculating the fundamental periods of building frames such (ASCE 7-10)

$T = 0.0724h^{0.8}$ for steel moment resisting frames (2.84a)

$T = 0.0466h^{0.9}$ for concrete moment resisting frames (2.84b)

$T = 0.0731h^{0.75}$ for eccentrically braced steel frames (2.84c)

Where h is the total height of frame.

According to the Eurocode 8(2004)

$T = 0.075h^{0.75}$ for framed structures (2.84d)

In the static force procedure (UBC, 1997), the second method (known as method B) for calculating T is the Rayleigh equation

$$T = 2\pi\sqrt{\left(\sum_{i=1}^{n} \omega_i \delta_i^2\right) \div \left(g \sum_{i=1}^{n} f_i \delta_i\right)} \tag{2.85}$$

Where ω_i is the portion of the seismic dead load located at or assigned to level i, δ_i is the horizontal displacement at level i relative to the base due to applied force, g is the acceleration due to gravity, f_i is the lateral force at level i.

The Rayleigh equation even though is time consuming gives a better estimation of the fundamental frequency of frames. A hand calculated approach for the computation of frame deflections using a calculator has been suggested by Hsiao (2009). The method implements the demands of the Rayleigh equation but in a simplier manner thereby facilitating an easier determination of the fundamental frequencies/periods of low rise moment frames.

Several research has been carried out using numerical methods to determine the contribution of shear walls on the fundamental frequency (Crowley and Pinho, 2004, Masi and Vona, 2008; Cinitha *et al* 2012). For the case of reinforced concrete frame buildings the building height contributes more to the value of its period when compared to the other characteristics of the building like in-plan regularity, infills and shear wall distribution (Kose 2009).

2.9 Use of Lumped mass idealisation

All real structures have distributed masses and consequently have infinite degrees of freedom. Their analysis is tedious and may require a lot of mathematical manipulations. The dynamic analysis of structures is made simpler using a lumped mass idealization (Sule 2011). The results from such analysis are usually approximate (Gasic *et al* 2014). In the lumping of masses effort is made to preserve the total mass of the structure even though this alone cannot ensure the quality of the solution (Iyer 1993). The use of diagonal lumped mass matrices simplifies the program coding and results in significant reduction in computer memory and computational effort.

In finite element analysis, researchers have found out that the use of consistent mass matrix did not always lead to an improved accuracy that justifies the additional computational effort (Clough 1971, Washizu 1971). For this reason Key and Beisinger developed a method for deriving a diagonal mass matrix from the standard consistent mass matrix for elements with linear and cubic shape functions (Key and Beisinger 1971). The use of lumped mass matrix has been extended to the dynamic finite element analysis of plates and shells (Iyer *et al* 2003). However it is a general belief that consistent matrix lead to a more accurate solution (Nandi and Bosu 2004). The diagonal mass matrix is still employed because of its lesser computational effort.

Efforts have also been made at developing simplified models for predicting the natural frequencies of lumped mass structures. The concept of using shear wave in a solid prismatic bar was used by Sule (Sule 2011) in studying and obtaining approximate natural frequency of vibration of beams under lateral vibration. Osadebe (1999) also proposed a model for calculating the natural frequencies of some beams.

2.10 Summary of Literature Review

One of the products of a dynamic analysis is the natural frequency of a structure. The natural frequency of the structure is a property of the structure that depends exclusively on its mass distribution and stiffness. It can be obtained for continuous structures (structures with a continuous distribution of mass) using the Hamilton's principle. This principle is an energy theorem that is an extension of the well known principle of minimum potential energy for dynamic analysis. It can be used to obtain exact results in the dynamic analysis of structures. Unfortunately it is quite difficult to obtain the differential equation of motions for complex structures using this principle, hence approximate methods are sought after. ‘Lagrange equations’ was one of them. Using this method, the continuous mass of the structure are lumped together at selected nodes (junctions/coordinates) about which we are interested in. This has the advantage of reducing the degree of freedom of the continuous structure from an infinite value to a finite value and so eases the analysis. But because the lumping of the continuous mass at selected nodes of the structure has altered the mass distribution (inertia matrix) of the structure the results are approximate and depends largely on the pattern and extent of redistribution

Another approximate method is the Rayleigh-ritz method which is an extension of the Rayleigh method. In this method shape functions are used to depict the displacement configuration of the system under vibration. While a single function is used in the Rayleigh function, a series of shape functions multiplied by constant coefficients are used in the Rayleigh-ritz approach. The use of a series instead of a singular function improves the results of the analysis as the series will better represent the displacement of the structure more than a single function. The strain energy and kinetic energy of the vibrating system are formulated with the chosen shape function. If the shape function depicts accurately the displacement configuration of the system then the result of the analysis will be accurate else it becomes approximate. What the chosen shape functions do is to apply a particular mass distribution on the system. The applied mass distribution will only be exact i.e. depict the continuous and uniform mass distribution if the chosen shape function depicts accurately the displacement of the real structure. The finite element method just like the Rayleigh-ritz approach uses a shape function to depict the displacement configuration of the system and as such the mass distribution. The product of the chosen shape function is the consistency matrix which is an inertia matrix formulated from the chosen shape function. While in the Rayleigh-ritz method the accuracy of results can be improved upon by increasing the number of terms in the shape function, the finite element method can also achieve this by subdividing the element into smaller elements hence making the chosen shape function a better approximation of the displacement configuration and hence improving the result of the analysis.

Osadebe's model (Osadebe 1999) and that of Sule (Sule 2011) are empirical formulae for estimating the natural frequency of simple beams. The building codes are also replete with empirical equations for estimating the natural frequency of building frames.

We observe that the approximate methods such as Lagrange (by mass lumping), Rayleigh-ritz and the finite element method all modify the mass distribution of the structure. While in the use of the Lagrange's equations, simple redistribution on the basis of the centre of gravity can be done. The Rayleigh-ritz and finite element method carries out a more complex mass formulation resulting in the consistency matrix. The effort of these methods is to obtain a mass distribution (inertia matrix) representative of the real structure and so produce exact results. In the use of the Lagrange's equation the masses are lumped. Efforts have been made at finding better patterns of lumping that will minimize errors due to the lumping of continuous mass.

Using evenly spaced lumps and increasing the number of lumps are the results of such studies. We can see that all the approximate methods hover around the distribution of the mass of the structure; work has not been done on possible redistribution of the stiffness of the structure so as to counter the errors due to the poor or non-representative distribution of the mass. We therefore need to explore the possibility of modification in the structure's stiffness distribution.

Stiffness in its simplest form is the deformation per unit force and can be likened to the unit response of a system to external forces. For a simple bar, Hooke's law is written as

$$F = ke \tag{2.86}$$

Where F is the applied longitudinal force, e is the enlongation and k the elastic constant or stiffness. The stiffness is a characteristic of the bar as it expresses the relationship between force applied on the bar and the corresponding deformation that results. Because equation (2.66) does not take care of other variables that affect k such as cross-sectional area of the bar and bar length, Hooke's law can be rewritten as

$$\sigma = E\varepsilon \tag{2.87}$$

Where $\sigma = stress = {}^{F}/_{A}$, $\varepsilon = strain = {}^{e}/_{l}$ and E is the young's modulus. By comparing equations (3.86) and (3.87) we find out that

$$k = {}^{EA}/_{l} \tag{2.88}$$

The stiffness from equation (2.88) is a property of the bar and captures the material property E (young's modulus is a property of the material of the bar), cross-sectional area A and length l are geometrical properties of the bar.

In deriving equations (2.86) and (2.87) force is applied at specified points (nodes) and deflection is also measured at those points (nodes). Hence in calculating or writing the stiffness of a bar , it has to be written with respect to the defined nodes. For a structure that consists of an assemblage of elements such as frames, the structures cannot be one dimensional as in equation (2.86) but mult-dimensional because of the presence of more than one node (see section 2.6.2). However the multi-dimensional stiffness [k] provides the force per unit deformation of the assembled elements at the nodes considered in the writing of the stiffness matrix.

Consider a uniform bar made of a homogenous material and having four nodes (1 - 4).

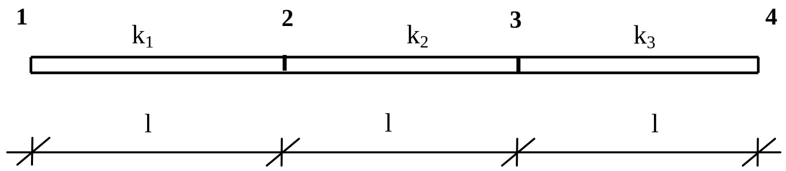


Figure 2.3: A uniform beam subdivided into three equal segments

The bar can be said to be divided into segments and if the length of each segment/element l is equal, each segment will have a stiffness

$$k = k_1 = k_2 = k_3 = \frac{EA}{l} \tag{2.89}$$

If the beam's stiffness is altered in such a way that

$$k_1 = \emptyset_1 k$$

$$k_2 = \emptyset_2 k \tag{2.90}$$

$$k_3 = \emptyset_3 k$$

And $\emptyset_1 \neq \emptyset_2 \neq \emptyset_3$ (2.91)

The response of the structure to external loads will no longer be the same as before and one can say that the stiffness distribution of the structure has been modified (or the stiffness of the structure has been redistributed). This is the intended meaning of modification in stiffness distribution in the context of this work.

Chapter 3

METHODOLOGY

The two essential components that determine the vibration of structural systems are:-

i) The structure's mass distribution

ii) The structure' s stiffness

These properties are captured in the structure's inertia matrix and stiffness matrix respectively. The prominent role these elements play can easily be appreciated by taking a look at the equations of motion of a vibrating system or the structural dynamics' eigenvalue problem.

In the use of Lagrange's method, a solid mass is treated as particles connected together by mass-less elements. When the continuous mass of an element are lumped at specified nodes, these lumped masses are used in formulating the diagonal mass matrix of the element. The corresponding stiffness matrix is formulated with respect to the coordinates of the lumped masses. Hence there is a relationship between the way lumped masses depicting a continuous system are positioned and the stiffness matrix of the system. The product of the mass matrix and the inverse of structure stiffness matrix (the flexibility matrix) give the dynamical matrix from which the natural frequencies and mode shapes are obtained (see equations (2.35), (2.39) – (2.40)). This can also be observed in the othogonality relations for discrete masses. By substituting equation (2.48) into equation (2.49) we obtain

$$[\phi]^T[k][\phi] = [\omega^2]\{[\phi]^T[m][\phi]\} \quad (3.1)$$

where $[\omega^2]$ is a diagonal matrix containing the square of the circular frequencies of vibration ω_1, ω_2, ω_3 etc. From equation (3.1) it would be seen that if $[\phi]$ and $[\omega^2]$

which are the two products of a dynamic analysis are kept constant, any change in the inertia matrix [m] must be accompanied by a corresponding change in the stiffness matrix [k] and vice versa.

In the analysis of continuous systems using the Hamilton's principle the way the structure's stiffness and mass distribution interplay in the determination of the system's circular frequencies and mode shapes is evident if equation (2.60b) is substituted into equation (2.60d).

$$EA \int_0^l \phi_j'^2 dx_1 = \omega_j^2 \mu \int_0^l \phi_j^2 dx_1 \tag{3.2}$$

If the mode shape ϕ_j and circular frequency ω_j are kept constant, then any variation in mass distribution μ will have a corresponding change in the element rigidity EA. Likewise by substituting equation (2.69b) into equation (2.69d)

$$EI \int_0^l \phi_j''^2 dx_1 = \omega_j^2 \mu \int_0^l \phi_j^2 dx_1 \tag{3.3}$$

If the mode shape ϕ_j and circular frequency ω_j are kept constant, then any variation in mass distribution μ will have a corresponding change in the element rigidity EI.

The use of either the finite element method or the Rayleigh-ritz approach also involve the evaluation of equations similar to equations (3.1) – (3.3); hence the results of a structural dynamic analysis of a structure rests squarely on how the mass of the structure is distributed and on the structure's stiffness matrix.

The important question is how is the mass of the structure distributed? They are continuous. If the continuous system is represented with a set of lumped masses connected by masses elements, do I hope to get an exact solution (exact value of circular frequency and shape function)? Presented in Appendix H is a rigid body dynamics for a lumped mass encaste beam under longitudinal and lateral vibration showing its departure from the correct value of natural frequency at different possible positions of the lumped mass(es).

From equations (3.1) – (3.3) a possible way of achieving this is by altering the structure's stiffness distribution. The lumped mass beam would be reduced to finite elements with masses lumped at its ends see the Figure 3.1

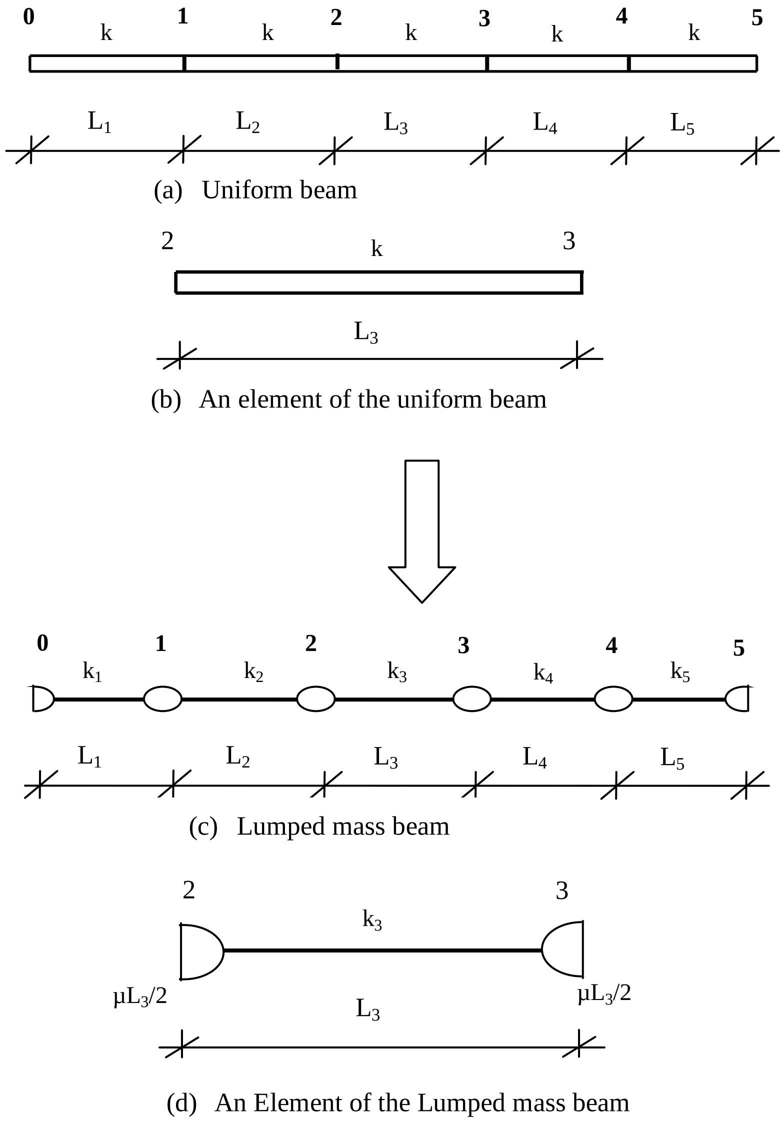


Figure 3.1: Lump mass idealizations of a uniform beam (Tauchert 1974)

(μ is the mass per unit length of the beams)

Beams of different end restraint will be analysed under free vibration to determine the distribution of the inertia forces causing vibration. This can be obtained by finding the second derivative of equation (2.59) with respect to time and multiplying it with an elementary mass μdx of the vibrating beam (μ is the mass per unit length of the beam). The stiffness of the beam will be taken to be constant (as expected) which implies that the stiffness at time t = 0 will be the same at any other time during vibration. Equilibrium equations of any element of the vibrating beam will be written and the deformation of the element at the ends at time t = 0 equated to the deformation of a corresponding lumped massed element with similar position on the vibrating element and the stiffness value of the lumped mass element calculated. The rotational inertia of the lumped mass may or may not be ignored or may be computed as a fraction of the total mass.

This can be achieved by comparing two equations. One is the force equilibrium equation written as

$$\{F\} + [S]\{D\} = 0 \quad (3.4)$$

(when no external force acts at the element's nodes)

$$\{F\} + [S]\{D\} = \{F^*\} \quad (3.5)$$

(when the external force vector {F*} acts at the element's nodes)

Where {F} is the vector of fixed end forces generated when nodal displacements are restrained. [S] is the element stiffness matrix and {D} a vector of nodal displacements (Okonkwo 2012).

The second is the equation of motion of a vibrating system written simply as (see section 2.7.1)

$$[m]\{\ddot{x}\} + [k]\{x\} = 0 \tag{3.6}$$

(when no external force acts at the element's nodes)

$$[m]\{\ddot{x}\} + [k]\{x\} = \{F\} \tag{3.7}$$

(when the external force vector {F} acts at the element's nodes)

Where [m] is the inertia matrix, [k] is the element stiffness matrix and {x} a vector of nodal displacements.

By comparing equations (3.4) and (3.5) with (3.6) and (3.7) we see a lot of similarities. Even though equations (3.4) and (3.5) have been largely applied in statics, it can also be applied in dynamics if the equations for the vector of fixed end moments/forces {F} can be formulated. The real structure (continuous system) will be analyzed using the Hamilton's principle and the equations for the fixed end forces {F} and nodal displacements {D} formulated for any arbitrary segment of a vibrating beam at time t = 0. This will be substituted into equation (3.5) to get the vector of nodal force {P} that is causing the vibration.

$$\{F\} + [S]\{D\} = \{P\} \tag{3.8}$$

[k] in equations (3.6) and (3.7) will be taken as the stiffness matrix of the lump-massed beam. If a vibrating element of the real beam (beam with continuous mass) and that of a corresponding element of a lump-massed beam are to be equivalent then their deformation must be equal and the force acting on their nodes {P} will also be equal. Therefore

$$\{D\} = \{x\} \tag{3.9}$$

$$[m]\{\ddot{x}\} + [k]\{x\} = \{P\} \tag{3.10}$$

Equation (3.10) can then be solved to obtain the value of [k] which will not be the same as [S]. Since [k] will be solved for different segments (elements) of the vibrating beam, it is not expected to give the same results but will vary depending on the position of the selected segment in the vibrating beam. This gives rise to the modification of the beam stiffness distribution. The lumping of masses at the left and right ends of a segment already has an established principle of lumping 50% of mass to the left and another 50% to the right (see Figure 3.1d).

3.1 Formulation of the equations for fixed end forces for segments of a bar under free longitudinal vibration

The partial differential equation governing the free longitudinal vibration of an elastic bar of uniform cross-section is (see section 2.7.2)

$$c^2 u'' = \ddot{u} \qquad (3.10)$$

where $c^2 = \frac{EA}{\mu}$ (3.11)

E is the young's modulus of the material of the bar and μ is the mass per unit length of the bar.

For a normal mode vibration where each particle in the bar vibrates harmonically with a circular frequency w

$$u(x,t) = \emptyset(x)\sin(\omega t + \delta) \qquad (3.12)$$

By differentiating equation (3.12) twice with respect to x we obtain

$$u''(x,t) = \emptyset''(x)\sin(\omega t + \delta) \qquad (3.13)$$

Equation (3.12) and (3.13) are substituted into equation (3.10) to obtain

$$\emptyset'' + \frac{\omega^2}{c^2}\emptyset = 0 \qquad (3.14)$$

The general solution of equation (3.14) is

$$\emptyset = C_1 \cos\frac{\omega x}{c} + C_2 \sin\frac{\omega x}{c} \tag{3.15}$$

3.1.1 For a Clamped (fixed-fixed) bar or pinned (pinned-pinned) bar

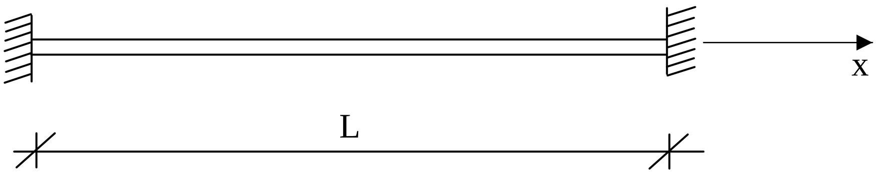


Figure 3.2: An illustration of a Clamped bar

At the boundaries (see Figure 3.2)

$$\emptyset(0) = \emptyset(L) = 0 \tag{3.16}$$

By substituting equation (3.15) into equation (3.16) we obtain

$$C_1 = 0 \tag{3.17}$$

$$C_1 \cos\frac{wL}{c} + C_2 \sin\frac{wL}{c} = 0 \tag{3.18}$$

For a non-trivial solution $C_2 \neq 0$ hence

$$\sin\frac{wL}{c} = 0 \tag{3.19}$$

From equation (3.19) it would be seen that the bar has infinite number of natural frequencies given by

$$w_j = \frac{j\pi c}{L} = j\pi\sqrt{\frac{EA}{\mu L^2}} \tag{3.20}$$

$j = 1,2,3, \dots, \infty$

By taking C_2 to be equal to unity

$$\emptyset_j = \sin\frac{j\pi x}{L} \tag{3.21}$$

Or $\emptyset_j = \sin\gamma_1 x$, where $\gamma_1 = \frac{j\pi}{L}$

$j = 1,2,3, \dots, \infty$

Substituting equation (3.20) and (3.21) into (3.12) and summing it for all normal modes

$$u(x,t) = \sum_{j=1}^{\infty} \emptyset_j \left(A_j \cos\omega_j t + B_j \sin\omega_j t\right) \tag{3.22}$$

Equation (3.22) is the equation of motion for the longitudinal vibration of a bar. To find the acceleration and hence the inertia forces due to vibration there is need to find the second derivative of equation (3.22) with respect to time.

$$\ddot{u}(x,t) = \sum_{j=1}^{\infty} -\omega_j^2 \emptyset_j \left(A_j \cos\omega_j t + B_j \sin\omega_j t\right) \tag{3.23}$$

Substituting equation (3.21) into (3.23) at time t = 0 will give

$$\ddot{u}(x,0) = \sum_{j=1}^{\infty} -\omega_j^2 A_j \sin\frac{j\pi x}{L}$$

$$= \sum_{j=1}^{\infty} -\omega_j^2 A_j \sin\gamma_1 x \tag{3.24}$$

where $\gamma_1 = \frac{j\pi}{L} = \frac{\omega_j}{c}$ (3.25)

By treating the longitudinally vibrating bar like a beam segment pinned at both ends, it is possible to obtain the fixed end forces (axial) forces of an arbitrary segment of the bar (see Figure 3.3).

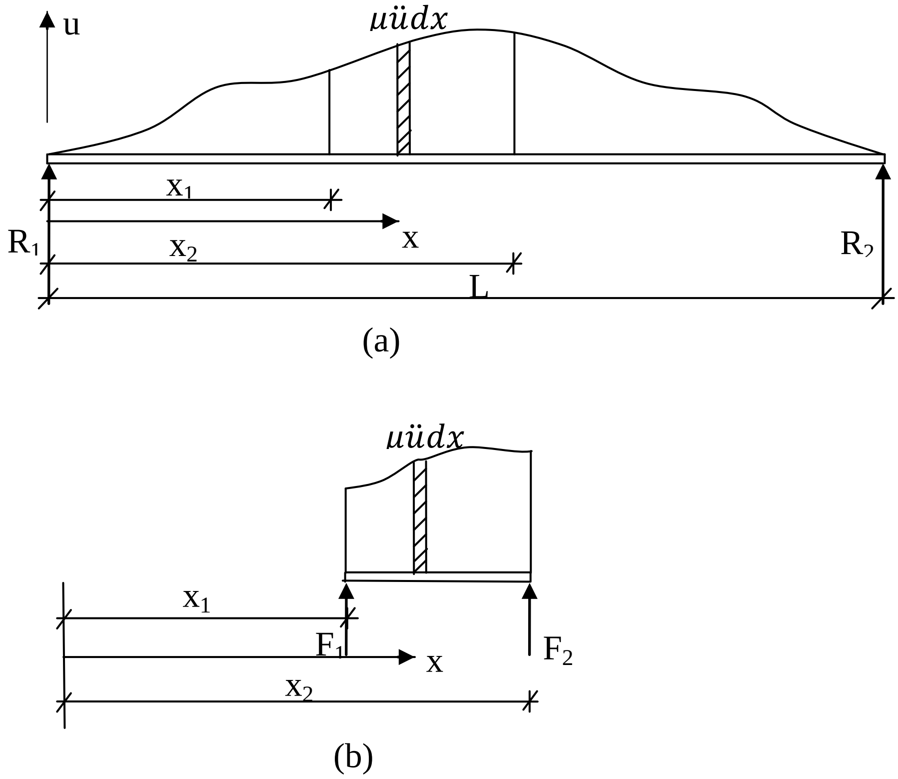


Figure 3.3

(a) A bar under longitudinal vibration due to inertial forces $\mu\ddot{u}$

(b) A segment of the bar under longitudinal vibration due to inertial forces $\mu\ddot{u}$

The forces at the ends of the isolated segment are F_1 and F_2.

Using the equations of external equilibrium

$\sum M_2 = 0; \quad F_1(x_2 - x_1) + \int_{x_1}^{x_2} \mu\ddot{u}(x_2 - x)dx = 0$

$$F_1 = \frac{-1}{x_2 - x_1}\int_{x_1}^{x_2} \mu\ddot{u}(x_2 - x)dx \qquad (3.26)$$

$\sum F_y = 0; \quad F_1 + F_2 + \int_{x_1}^{x_2} \mu\ddot{u}dx = 0$

$$F_2 = -\int_{x_1}^{x_2} \mu\ddot{u}dx - F_1 \qquad (3.27)$$

It is necessary to carry out the integration of the different component functions of equations (3.26) and (3.27) separately.

$$\int_{x_1}^{x_2} \mu\ddot{u}x\, dx = \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu \int_{x_1}^{x_2} \emptyset_j x\, dx$$

$$= \sum_{j=1}^{\infty} -\omega^2 A_j \mu \int_{x_1}^{x_2} x \sin\gamma_1 x \ dx$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\gamma_1^2} (-\gamma_1 x_2 \cos\gamma_1 x_2 + \sin\gamma_1 x_2 + \gamma_1 x_1 \cos\gamma_1 x_1 - \sin\gamma_1 x_1)$$

(3.28)

$$\int_{x_1}^{x_2} \mu \ddot{u} x_2 \, dx = \mu x_2 \int_{x_1}^{x_2} \ddot{u} \, dx$$

$$= \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu \, x_2 \int_{x_1}^{x_2} \sin\gamma_1 x \; dx$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu x_2}{\gamma_1} (\cos\gamma_1 x_2 + \cos\gamma_1 x_1) \tag{3.29}$$

By substituting equations (3.28) and (3.29) into equation (3.26) we obtain

$$F_1 = \frac{1}{(x_2 - x_1)} \sum_{j=1}^{\infty} \frac{\omega^2 A_j \mu}{\gamma_1^2} (\gamma_1 x_2 \cos\gamma x_1 - \gamma_1 x_1 \cos\gamma x_1 - \sin\gamma_1 x_2 + \sin\gamma_1 x_1) \tag{3.30}$$

From equation (3.29)

$$\int_{x_1}^{x_2} \mu \ddot{u} \, dx = \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu \int_{x_1}^{x_2} \sin\gamma_1 x \; dx$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\gamma_1} (-\cos\gamma_1 x_2 + \cos\gamma_1 x_1) \tag{3.31}$$

By substituting equation (3.31) and (3.30) into equation (3.27) we obtain the value of the second end force F_2

$$F_2 = \sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\gamma_1^2} \left[\frac{-\gamma_1 L \cos\gamma_1 x_2 + \gamma L \cos\gamma_1 x_1}{L} - \frac{\gamma_1 x_2 \cos\gamma_1 x_1 - \gamma_1 x_1 \cos\gamma_1 x_1 - \sin\gamma_1 x_2 + \sin\gamma_1 x_1}{x_2 - x_1} \right]$$

(3.32)

Having obtained the equations of the fixed end forces F_1 and F_2, there is need to express them in terms of EA rather than ω_j

From equation (3.25)

$$\gamma_1 L = j\pi \tag{3.33}$$

By substituting equation (3.11) into (3.25)

$$\gamma_1 = w_j \sqrt{\frac{\mu}{EA}}$$

$$\frac{\omega_j^2}{\gamma_1^2} = \frac{EA}{\mu} \tag{3.34}$$

The length of the bar L can be normalized to be equal to unity and the distances x_1 and x_2 expressed in dimensionless units.

Let

$$\xi_1 = \frac{x_1}{L} \tag{3.35}$$

$0 \leq x_1 < 1$

$$\xi_2 = \frac{x_2}{L} \tag{3.36}$$

$0 < x_2 \leq 1$

By substituting equations (3.33) – (3.36) into equations (3.30) and (3.32) we obtain

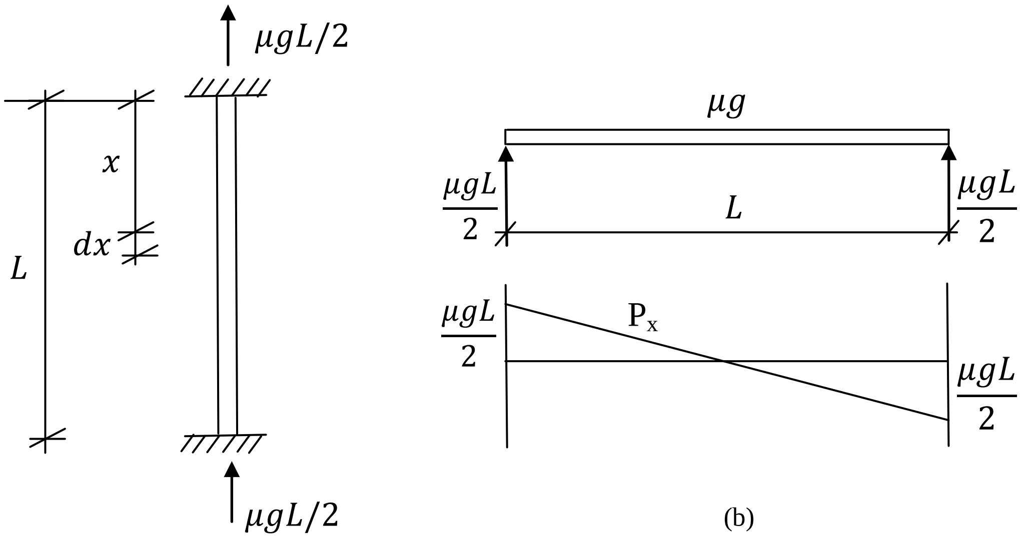
$$F_1 = \frac{EA}{L(\xi_2 - \xi_1)} \sum_{j=1}^{\infty} A_j \left[j\pi\xi_2 \cos j\pi\xi_1 - j\pi\xi_1 \cos j\pi\xi_1 - \sin j\pi\xi_2 + \sin j\pi\xi_1 \right] \tag{3.37}$$

and

$$F_2 = \frac{EA}{L} \sum_{j=1}^{\infty} A_j \left[-j\pi \cos j\pi\xi_2 + j\pi \cos j\pi\xi_1 - \frac{j\pi\,\xi_2 \cos j\pi\,\xi_1 - j\pi\,\xi_1 \cos j\pi\,\xi_1 - \sin j\pi\,\xi_2 + \sin j\pi\,\xi_1}{\xi_2 - \xi_1} \right] \tag{3.38}$$

Equations (3.37) and (3.38) are the equations of the fixed end forces on an arbitrary segment of a clamped bar under free vibration. In order to evaluate these equations, there is need to derive an expression for A_j.

Recall that the constant A_j depends on the initial conditions of the vibrating bar.



(a)

A vertical uniform bar under its self weight

(b)

The vertical bar represented as a horizontal bar with the self weight acting transversely to enable easy calculation of the axial force P at any point along its length (See proof in Appendix J)

Figure 3.4: A vertical uniform bar clamped at both ends and acted upon by its self weight

The axial force at any point x along the length of the bar is given by (see Figure 3.4)

$$P_x = \mu g \left(\frac{L}{2} - x\right) \tag{3.39}$$

μ is the mass per unit length of the bar and g is the acceleration due to gravity.

At any distance x an element of the bar is under an axial force P_x. This is illustrated with Figure 3.5

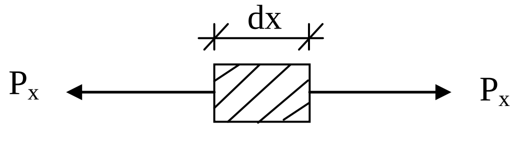


Figure 3.5: An infinitesimal element of the bar under axial stress

If the axial deformation on the infinitesimal element dx is du, then from Hooke's law

$$P_x = EA\frac{du}{dx} \tag{3.40}$$

By equating equation (3.39) to equation (3.40)

$$du = \frac{\mu g}{EA}\left(\frac{L}{2} - x\right)dx$$

$$u = \frac{\mu g}{EA}\int_0^x \left(\frac{L}{2} - x\right)dx$$

$$= \frac{\mu g}{2EA}(Lx - x^2) \tag{3.41}$$

Equation (3.41) is the equation of the deformation/longitudinal displacement of the bar under its weight. Let this displacement u of the bar at time t = 0 be equal to

$$u(x,0) = \frac{e}{L}(Lx - x^2) \tag{3.42}$$

Where e is a dimensionless constant equal to $\frac{\mu g L}{2EA}$

By substituting equation (3.21) and (3.42) into the equation for A_j given as equation (2.61a)

$$A_j = \frac{\mu}{M_j}\int_0^L \frac{e}{L}(Lx - x^2)\sin\gamma_1 x\,dx \tag{3.42}$$

Where $\gamma_1 = \frac{j\pi}{L} = \frac{\omega_j}{c}$ as stated earlier

$$A_j = \frac{\mu e}{M_j L}\int_0^L Lx\sin\gamma_1 x - x^2\sin\gamma_1 x\,dx$$

$$= \frac{\mu e}{M_j L}\left[\frac{L(-\gamma_1 L\cos\gamma_1 L + \sin\gamma_1 L)}{\gamma_1^2} - \frac{-\gamma_1^2 L^2\cos\gamma_1 L + 2\gamma_1 L\sin\gamma_1 L + 2\cos\gamma_1 L - 2}{\gamma_1^3}\right]$$

$$= \frac{\mu e L^2}{M_j}\left(\frac{2 - \gamma_1 L\sin\gamma_1 L - 2\cos\gamma_1 L}{\gamma_1^3 L^3}\right) \tag{3.43}$$

Equation (3.43) above is an expression for the constant A_j. As can be seen, it is a function of the generalized mass Mj.

The generalized mass can be expressed as (from equation (2.60b))

$$M_j = \mu \int_0^L \emptyset_j^2 \, dx \tag{3.44}$$

By substituting equation (3.21) into equation (3.44)

$$M_j = \mu \int_0^L \sin^2 \gamma_1 x \, dx$$

$$= \frac{\mu}{\gamma_1}\left(\frac{\gamma_1 L}{2} - \frac{\sin 2\gamma_1 L}{4}\right)$$

$$= \frac{\mu L}{2} \tag{3.45}$$

(since $\gamma_1 L = j\pi$)

By substituting equation (3.33) and (3.45) into (3.43) and simplifying

$$A_j = 2eL\left(\frac{2-2(-1)^j}{j^3\pi^3}\right) \tag{3.46}$$

$$j = 1,2,3, \dots, \infty$$

Equation (3.46) above is an expression for the constant A_j for a bar under an initial displacement caused by its self weight. Equation (3.46) can be substituted into the equation (3.37) and (3.38) to obtain the values of the fixed end forces F_1 and F_2.

3.1.2 **For a Clamped-free bar**

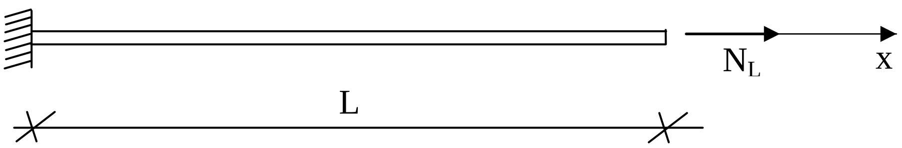


Figure 3.6: An illustration of a Clamped-free bar

At the boundaries (see equation 2.53 and Figure 3.6)

$$\emptyset(0) = 0 \tag{3.47}$$

$$N_L = [EAu']_{x=L}$$

$$= EA\emptyset'(L) \tag{3.48}$$

For a free vibration $N_L = 0$, hence

$$\emptyset'(L) = 0 \tag{3.49}$$

By substituting equation (3.15) into equation (3.47) and (3.49) we obtain

$$C_1 = 0 \tag{3.50}$$

$$\frac{\omega}{c} C_2 \cos\frac{\omega L}{c} = 0 \tag{3.51}$$

For a non-trivial solution $C_2 \neq 0$ hence

$$\cos\frac{\omega L}{c} = 0 \tag{3.52}$$

From equation (3.52) it would be seen that the bar has infinite number of natural frequencies given by

$$w_j = \frac{i\pi c}{2L} = \frac{i\pi}{2}\sqrt{\frac{EA}{\mu L^2}} \tag{3.53}$$

$i = 1,3,5,7,9, \dots, \infty \qquad j = 1,2,3,4,5, \dots, \infty$

By substituting equation (3.50) and (3.53) into (3.15) and taking C_2 to be equal to unity

$$\emptyset_j = \sin\frac{i\pi x}{2L} \tag{3.54}$$

Or $\emptyset_j = \sin\gamma_2 x$, where $\gamma_2 = \frac{i\pi}{2L}$

$i = 1,3,5,7,9, \dots, \infty$

Substituting equation (3.53) and (3.54) into (3.12) and summing it for all normal modes

$$u(x,t) = \sum_{j=1}^{\infty} \emptyset_j \left(A_j \cos\omega_j t + B_j \sin\omega_j t\right) \tag{3.55}$$

Equation (3.55) is the equation of motion for the longitudinal vibration of a clamp-free bar. The second derivative of equation (3.55) with respect to time

$$\ddot{u}(x,t) = \sum_{j=1}^{\infty} -\omega_j^2 \emptyset_j \left(A_j \cos\omega_j t + B_j \sin\omega_j t\right) \tag{3.56}$$

Substituting equation (3.54) into (3.56) at time t = 0 will give

$$\ddot{u}(x,t) = \sum_{j=1}^{\infty} -\omega_j^2 A_j \sin\frac{i\pi x}{2L}$$

$$= \sum_{j=1}^{\infty} -\omega_j^2 A_j \sin\gamma_2 x \tag{3.57}$$

where $\gamma_2 = \frac{i\pi}{2L} = \frac{\omega_j}{c}$ (3.58)

By treating the longitudinally vibrating bar like a beam supported at one end, it is possible to obtain the fixed end forces (axial) forces of an arbitrary segment of the bar (see Figure 3.7).

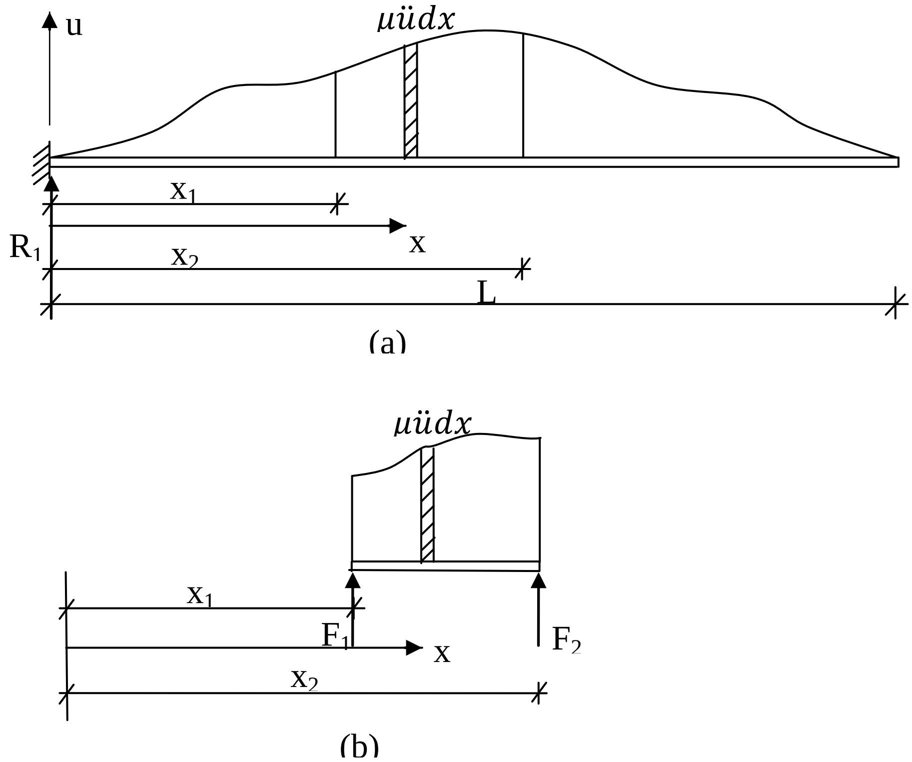


Figure 3.7

(a) A clamped-free bar under longitudinal vibration due to inertial forces $\mu\ddot{u}$

(b) A segment of the clamped-free bar under longitudinal vibration due to inertial forces $\mu\ddot{u}$

There are two possible cases for any arbitrary segment of the vibrating bar

a) **Case 1**

When $0 \leq x_1 < L$ and $0 < x_2 < L$

The forces at the ends of the isolated segment are F_1 and F_2 and the solution is largely the same as for the case of a fixed-fixed bar solved earlier.

Using the equations of external equilibrium

$$\sum M_2 = 0; \quad F_1(x_2 - x_1) + \int_{x_1}^{x_2} \mu\ddot{u}(x_2 - x)dx = 0$$

$$F_1 = \frac{-1}{x_2 - x_1}\int_{x_1}^{x_2} \mu\ddot{u}(x_2 - x)dx \tag{3.59}$$

$$\sum F_y = 0; \quad F_1 + F_2 + \int_{x_1}^{x_2} \mu\ddot{u}dx = 0$$

$$F_2 = -\int_{x_1}^{x_2} \mu\ddot{u}dx - F_1 \tag{3.60}$$

$$\int_{x_1}^{x_2} \mu\ddot{u}x\, dx = \sum_{j=1}^{\infty} -\omega^2 A_j \mu \int_{x_1}^{x_2} x \sin\gamma_2 x\ dx$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\gamma_2^2}(-\gamma_2 x_2 \cos\gamma_2 x_2 + \sin\gamma_2 x_2 + \gamma_2 x_1 \cos\gamma_2 x_1 - \sin\gamma_2 x_1) \tag{3.61}$$

$$\int_{x_1}^{x_2} \mu\ddot{u}x_2\, dx = \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu\, x_2 \int_{x_1}^{x_2} \sin\gamma_2 x\ dx$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu x_2}{\gamma_2}(\cos\gamma_2 x_2 + \cos\gamma_2 x_1) \tag{3.62}$$

By substituting equations (3.61) and (3.62) into equation (3.59) we obtain

$$F_1 = \frac{1}{(x_2 - x_1)}\sum_{j=1}^{\infty} \frac{\omega^2 A_j \mu}{\gamma_2^2}(\gamma_2 x_2 \cos\gamma_2 x_1 - \gamma_2 x_1 \cos\gamma_2 x_1 - \sin\gamma_2 x_2 + \sin\gamma_2 x_1) \tag{3.63}$$

$$\int_{x_1}^{x_2} \mu\ddot{u}\, dx = \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu \int_{x_1}^{x_2} \sin\gamma_2 x\ dx$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\gamma_2}(-\cos\gamma_2 x_2 + \cos\gamma_2 x_1) \tag{3.64}$$

By substituting equation (3.63) and (3.64) into equation (3.60) we obtain the value of the second end force F_2

$$F_2 = \sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\gamma_2^2} \left[\frac{-\gamma_2 L \cos \gamma_2 x_2 + \gamma_2 L \cos \gamma_2 x_1}{L} - \frac{\gamma_2 x_2 \cos \gamma_2 x_1 - \gamma_2 x_1 \cos \gamma_2 x_1 - \sin \gamma_2 x_2 + \sin \gamma_2 x_1}{x_2 - x_1}\right] \tag{3.65}$$

The fixed end forces F_1 and F_2, can be expressed in terms of the axial rigidity EA rather than the circular frequency w_j

From equation (3.58)

$$\gamma_2 L = \frac{i\pi}{2} \tag{3.66}$$

$i = 1,3,5,7,9, \dots, \infty$

By substituting equation (3.11) into (3.58)

$$\frac{\omega_j^2}{\gamma_2^2} = \frac{EA}{\mu} \tag{3.67}$$

There is also need to normalize our distances so that the length of the bar L becomes equal to unity and the distances x_1 and x_2 expressed in dimensionless units.

By substituting equations (3.35) – (3.36) and (3.66) – (3.67) into equations (3.63) and (3.65) we obtain

$$F_1 = \frac{EA}{L(\xi_2 - \xi_1)} \sum_{j=1}^{\infty} A_j \left[\frac{i\pi\xi_2}{2} \cos \frac{i\pi\xi_1}{2} - \frac{i\pi\xi_1}{2} \cos \frac{i\pi\xi_1}{2} - \sin \frac{i\pi\xi_2}{2} + \sin \frac{i\pi\xi_1}{2}\right] \tag{3.68}$$

$$F_2 = \frac{EA}{L} \sum_{j=1}^{\infty} A_j \left[-\frac{i\pi}{2} \cos \frac{i\pi\xi_2}{2} + \frac{i\pi}{2} \cos \frac{i\pi\xi_1}{2} - \frac{\frac{i\pi\xi_2}{2} \cos \frac{i\pi\xi_1}{2} - \frac{i\pi\xi_1}{2} \cos \frac{i\pi\xi_1}{2} - \sin \frac{i\pi\xi_2}{2} + \sin \frac{i\pi\xi_1}{2}}{\xi_2 - \xi_1}\right] \tag{3.69}$$

$j = 1,2,3,4,5, \dots, \infty$

$i = 1,3,5,7,9, \dots, \infty$

Equations (3.68) and (3.69) are the equations of the fixed end forces on an arbitrary segment of a clamped-free bar under free vibration.

b) **Case II**

When $0 \leq x_1 < L$ and $x_2 = L$

Since the far end of the bar is free, $F_2 = 0$

For vertical force equilibrium

$$F_1 + \int_{x_1}^{L} \mu \ddot{u} dx = 0 \tag{3.70}$$

$$F_1 = -\int_{x_1}^{L} \mu \ddot{u} dx$$

$$= -\sum_{j=1}^{\infty} -\omega_j^2 A_j \mu \int_{x1}^{L} \sin \gamma_2 x \, dx$$

$$= \sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\gamma_2} (-\cos \gamma_2 L + \cos \gamma_2 x) \tag{3.71}$$

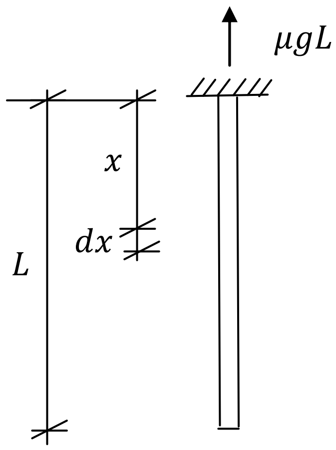
By substituting equations (3.35) – (3.36) and (3.66) – (3.67) into equations (3.71)

$$F_1 = \frac{EA}{2L} \sum_{j=1}^{\infty} A_j \, i\pi \left(-\cos \frac{i\pi}{2} + \cos \frac{i\pi \xi_1}{2}\right) \tag{3.72}$$

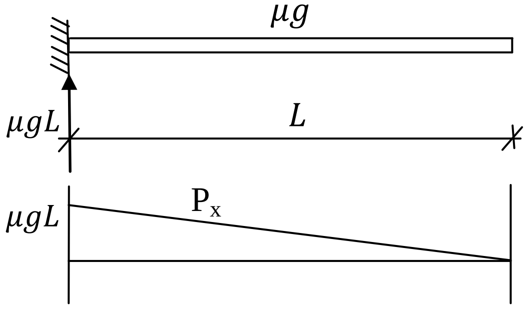
Where $j = 1,2,3,4,5, \dots, \infty$

$i = 1,3,5,7,9, \dots, \infty$

In order to evaluate the equations for F_1 and F_2, there is need to derive an expression for A_j for a clamped-free bar.



(a)
A vertical uniform bar
under its self weight



(b)
The vertical bar represented as a horizontal
bar with the self weight acting transversely
to enable easy calculation of the axial force P
at any point along its length

Figure 3.8: A vertical uniform clamped-free bar
acted upon by its self weight

The axial force at any point x along the length of the bar is given by (see Figure 3.8)

$$P_x = \mu g(L - x) \tag{3.73}$$

where μ is the mass per unit length of the bar and g is the acceleration due to gravity.

From Hooke's law the equilibrium equation for an infinitesimal element of the bar is

By equating equation (3.73) to equation (3.40)

$$du = \frac{\mu g}{EA}(L - x)dx$$

$$u = \frac{\mu g}{EA}\int_0^x (L - x)dx$$

$$= \frac{\mu g}{EA}\left(Lx - \frac{x^2}{2}\right) \tag{3.74}$$

Equation (3.74) is the equation of the deformation/longitudinal displacement of the clamped-free bar under its self weight. Let this displacement u of the bar at time t = 0 be equal to

$$u(x,0) = \frac{f}{L}\left(Lx - \frac{x^2}{2}\right) \tag{3.75}$$

Where f is a dimensionless constant equal to $\frac{\mu g L}{EA}$

By substituting equation (3.54) and (3.75) into the equation for A_j given as equation (2.61a)

$$A_j = \frac{\mu}{M_j}\int_0^L \frac{f}{L}\left(Lx - \frac{x^2}{2}\right)\sin\gamma_2 x\,dx \tag{3.76}$$

Where $\gamma_2 = \frac{j\pi}{2L} = \frac{\omega_j}{c}$ as stated earlier

$$A_j = \frac{\mu f}{M_j L}\int_0^L Lx\sin\gamma_2 x - \frac{x^2}{2}\sin\gamma_2 x\,dx$$

$$= \frac{\mu f}{M_j L}\left[\frac{L(-\gamma_2 L\cos\gamma_2 L + \sin\gamma_2 L)}{\gamma_2^2} - \frac{-\gamma_2^2 L^2\cos\gamma_2 L + 2\gamma_2 L\sin\gamma_2 L + 2\cos\gamma_2 L - 2}{2\gamma_2^3}\right]$$

$$= \frac{\mu f L^2}{M_j}\left(\frac{-\gamma_2^2 L^2\cos\gamma_2 L - 2\cos\gamma_2 L + 2}{2\gamma_2^3 L^3}\right) \tag{3.77}$$

Equation (3.43) above is an expression for the constant A_j for a clamped-free bar having an initial displacement due to its self weight.

To obtain the generalized mass M_j we substitute equation (3.54) into (3.44)

$$M_j = \mu\int_0^L \sin^2\gamma_2 x\,dx$$

$$= \frac{\mu}{\gamma_2}\left(\frac{\gamma_2 L}{2} - \frac{\sin 2\gamma_2 L}{4}\right)$$

$$= \frac{\mu L}{2} \tag{3.78}$$

(since $\gamma_2 L = j\pi/2$)

By substituting equation (3.66) and (3.78) into (3.77) and simplifying

$$A_j = 2fL\left(\frac{-\frac{i^2\pi^2}{4}\cos\frac{i\pi}{2} - 2\cos\frac{i\pi}{2} + 2}{\frac{2i^3\pi^3}{8}}\right) = \frac{16fL}{i^3\pi^3} \tag{3.79}$$

$j = 1,2,3,4,5, \ldots, \infty$

$i = 1,3,5,7,9, \ldots, \infty$

Equation (3.79) above is an expression for the constant A_j for a clamped-free bar under an initial displacement caused by its self weight.

Presented below is a summary of the end forces obtained for arbitrary segments of a bar under longitudinal vibration.

3.1.3 For a Free-free bar (a bar unrestrained at both ends)

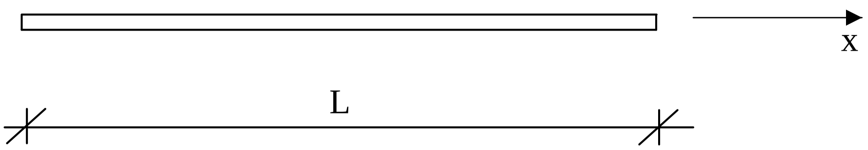


Figure 3.9: An illustration of a Free-free bar

At the boundaries (see Figure 3.2)

$$\emptyset'(0) = \emptyset'(L) = 0 \tag{3.80}$$

By substituting equation (3.15) into equation (3.80) we obtain

$$\frac{w}{c} C_2 = 0 \tag{3.81}$$

$$-\frac{w}{c}C_1\sin\frac{wL}{c}+\frac{w}{c}C_2\cos\frac{wL}{c}=0 \qquad (3.82)$$

Put in matrix format

$$\begin{bmatrix} 0 & \frac{w}{c} \\ -\frac{w}{c}\sin\frac{wL}{c} & \frac{w}{c}\cos\frac{wL}{c} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}=0 \qquad (3.83)$$

For a non-trivial solution the coefficients of the constants C_1 and C_2 must be zero hence

$$\frac{w^2}{c^2}\sin\frac{wL}{c}=0 \qquad (3.84)$$

From equation (3.84) it would be seen that the bar has infinite number of natural frequencies given by

$$w_j=\frac{j\pi c}{L}=j\pi\sqrt{\frac{EA}{\mu L^2}} \qquad (3.85)$$

$j=0,1,2,3,\ldots,\infty$

The calculated natural frequencies are the same with that of a fixed-fixed bar except that the lowest frequency for a free-free is zero (the condition for a rigid body motion).

By taking C_1 to be equal to unity C_2 can be calculated from equation (3.82) to be

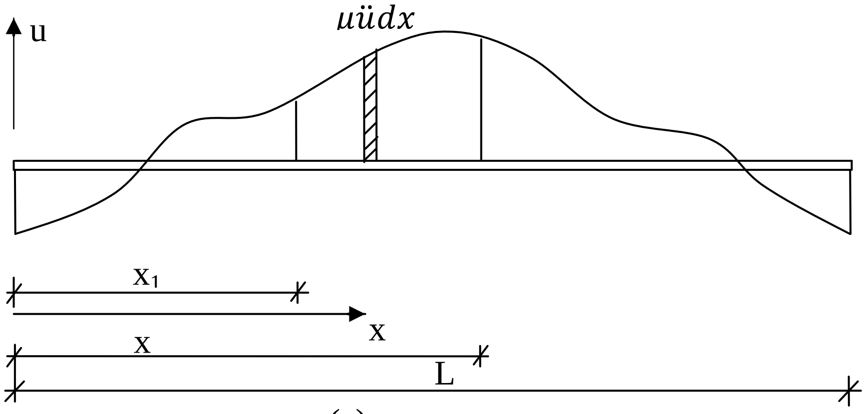
$$C_2=\tan\frac{wL}{c}=0 \qquad (3.86)$$

C_2 is equal to zero for all possible values of $\frac{wL}{c}$.

$$\emptyset_j=\cos\gamma_3 x,\ \text{where}\ \gamma_3=\frac{j\pi}{L} \qquad (3.86a)$$

$j=0,1,2,3,\ldots,\infty$

By treating the longitudinally vibrating bar like a horizontal beam segment free at both ends, it is possible to obtain the fixed end forces (axial) forces of an arbitrary segment of the free-free bar (see Figure 3.4).



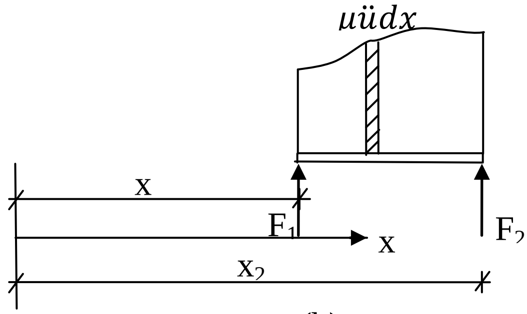


Figure 3.10

(a) A bar under longitudinal vibration due to inertial forces $\mu\ddot{u}$

(b) A segment of the bar under longitudinal vibration due to inertial forces $\mu\ddot{u}$

The forces at the ends of the isolated segment are F_1 and F_2.

Using the equations of external equilibrium

$\sum M_2 = 0; \quad F_1(x_2 - x_1) + \int_{x_1}^{x_2} \mu\ddot{u}(x_2 - x)dx = 0$

$$F_1 = \frac{-1}{x_2 - x_1}\int_{x_1}^{x_2} \mu\ddot{u}(x_2 - x)dx \tag{3.87}$$

$\sum F_y = 0; \quad F_1 + F_2 + \int_{x_1}^{x_2} \mu\ddot{u}dx = 0$

$$F_2 = -\int_{x_1}^{x_2} \mu\ddot{u}dx - F_1 \tag{3.88}$$

By carrying out the integration of the different component functions of equations (3.87) and (3.88) .

$\int_{x_1}^{x_2} \mu\ddot{u}x\, dx = \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu \int_{x_1}^{x_2} \emptyset_j x\, dx$

$$= \sum_{j=1}^{\infty} -\omega^2 A_j \mu \int_{x_1}^{x_2} x \cos \gamma_3 x \ dx$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\gamma_3^2} (-\gamma_3 x_2 \sin \gamma_3 x_2 + \cos \gamma_3 x_2 - \gamma_3 x_1 \sin \gamma_3 x_1 - \cos \gamma_3 x_1) \ . \qquad .$$

.(3.89)

$$\int_{x_1}^{x_2} \mu \ddot{u} x_2 \ dx = \mu x_2 \int_{x_1}^{x_2} \ddot{u} \ dx$$

$$= \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu \, x_2 \int_{x_1}^{x_2} \cos \gamma_3 x \ dx$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu x_2}{\gamma_3} (\sin \gamma_3 x_2 - \sin \gamma_3 x_1) \qquad (3.90)$$

By substituting equations (3.89) and (3.90) into equation (3.87) we obtain

$$F_1 = \frac{1}{(x_2 - x_1)} \sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\gamma_3^2} (-\gamma_3 x_2 \sin \gamma_3 x_1 + \gamma_3 x_1 \sin \gamma x_1 - \cos \gamma_3 x_2 + \cos \gamma_3 x_1)$$

(3.91)

$$\int_{x_1}^{x_2} \mu \ddot{u} \ dx = \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu \int_{x_1}^{x_2} \cos \gamma_1 x \ dx$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\gamma_3} (\sin \gamma_3 x_2 - \sin \gamma_3 x_1) \qquad (3.92)$$

By substituting equation (3.92) and (3.91) into equation (3.88) we obtain the value of the second end force F_2

$$F_2 = \sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\gamma_3^2} \left[\frac{\gamma_3 L \sin \gamma_3 x_2 - \gamma_3 L \sin \gamma_3 x_1}{L} - \frac{-\gamma_3 x_2 \sin \gamma_1 x_1 + \gamma_3 x_1 \sin \gamma_3 x_1 - \cos \gamma_3 x_2 + \cos \gamma_3 x_1}{x_2 - x_1} \right]$$

(3.93)

In other to express the fixed end forces F_1 and F_2 in terms of EA rather than w_j we substitute equations (3.33 – 3.36) into equations (3.92) and (3.93) to obtain

$$F_1 = \frac{EA}{L(\xi_2 - \xi_1)} \sum_{j=1}^{\infty} A_j \left[-j\pi\xi_2 \sin j\pi\xi_1 + j\pi\xi_1 \sin j\pi\xi_1 - \cos j\pi\xi_2 + \cos j\pi\xi_1 \right] \quad (3.94)$$

$$F_2 =$$

$$\frac{EA}{L} \sum_{j=1}^{\infty} A_j \left[j\pi \sin j\pi\xi_2 - j\pi \sin j\pi\xi_1 - \frac{-j\pi \, \xi_2 \sin j\pi \, \xi_1 + j\pi \, \xi_1 \sin j\pi \, \xi_1 - \cos j\pi \, \xi_2 + \cos j\pi \, \xi_1}{\xi_2 - \xi_1} \right]$$

(3.95)

Equations (3.94) and (3.95) are the equations of the fixed end forces on an arbitrary segment of a free-free bar under free vibration. In order to evaluate these equations, there is need to derive an expression for A_j.

Recall that the constant A_j depends on the initial conditions of the vibrating bar.

By treating the bar as a free bar under of the influence of gravitational force propped at the centre of gravity.

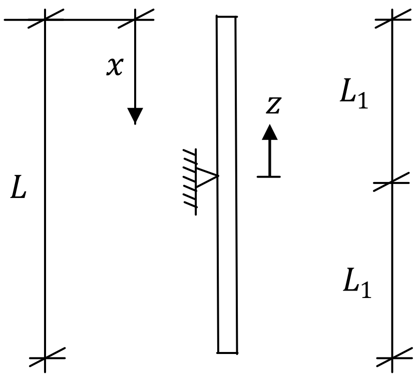


Figure 3.11: A vertical uniform bar under its self weight propped at the centre

From the equation for deflection of a uniform fixed-free bar under its self weight

$$u(z,0) = \frac{f}{L}\left(L_1 z - \frac{z^2}{2}\right) \tag{3.96}$$

where $f = \frac{\mu g L}{EA}$ (3.96a)

From Figure (3.5) we can infer that

$$z = L_1 - x \tag{3.97}$$

In order to express the displacement of the bar in terms of x rather than z we substitute equation (3.97) into (3.96) to obtain

$$u(x,0) = \frac{f}{2L_1}(L_1^2 - x^2) \tag{3.98}$$

By substituting equation (3.86a) and (3.98) into the equation for A_j given as equation (2.61a)

$$A_j = \frac{\mu}{M_j}\int_0^L \frac{f}{L_1}(L_1^2 - x^2)\cos\gamma_3 x\, dx \tag{3.99}$$

Where $\gamma_1 = \frac{j\pi}{L} = \frac{\omega_j}{c}$ as stated earlier

The integral has been multiplied by two to take care of the other half length L1. This will be correct for all symmetrical modes of vibration and is safe as we will later find out from that of fixed-fixed bar that A_j for all asymmetric modes of vibration is zero.

By evaluating the integral of equation (3.99) and simplifying

$$A_j = \frac{2\mu f L_1^2}{M_j}\left(\frac{-\cos\gamma_1 L_1}{\gamma_1^2 L_1^2} + \frac{\sin\gamma_1 L_1}{\gamma_1^3 L_1^3}\right) \tag{3.100}$$

But $L_1 = L/2$ (3.101)

Therefore

$$A_j = \frac{2\mu f L^2}{M_j}\left(\frac{-\cos\frac{\gamma_1 L}{2}}{\gamma_1^2 L^2} + \frac{2\sin\frac{\gamma_1 L}{2}}{\gamma_1^3 L^3}\right) \tag{3.102}$$

The generalized mass can be expressed as (from equation 2.60b)

By substituting equation (3.86a) into equation (3.44)

$$M_j = \mu\int_0^L \cos^2\gamma_1 x\, dx$$

$$= \frac{\mu L}{2} \tag{3.103}$$

(since $\gamma_1 L = j\pi$)

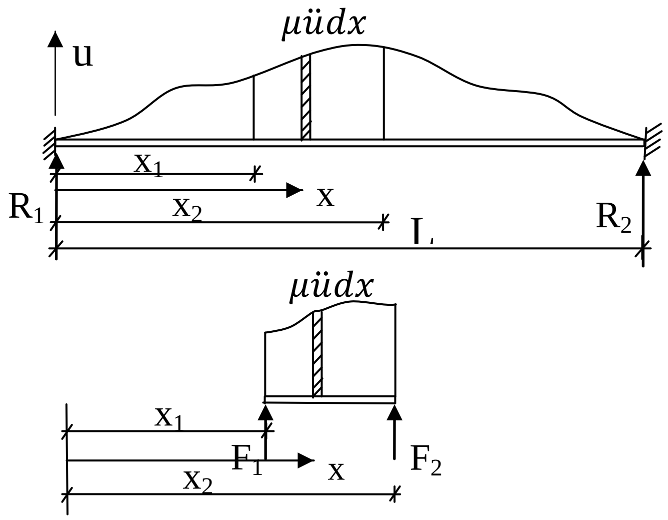
By substituting equation (3.33) and (3.103) into (3.102) and simplifying

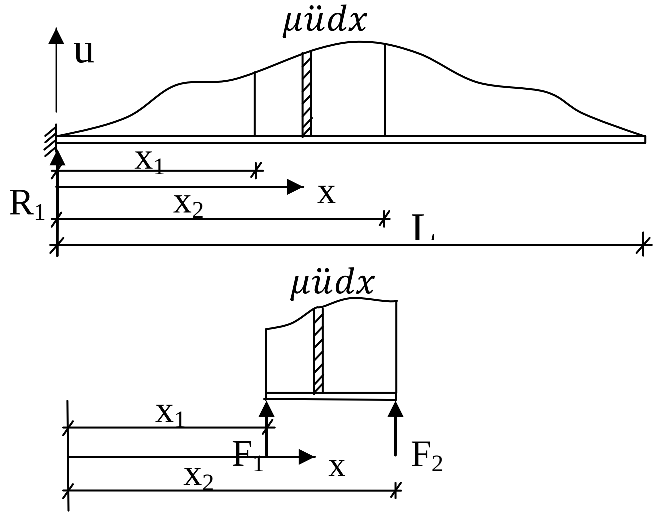
$$A_j = 4fL\left(\frac{-\cos\frac{j\pi}{2}}{j^2\pi^2} + \frac{2\sin\frac{j\pi}{2}}{j^3\pi^3}\right) \tag{3.104}$$

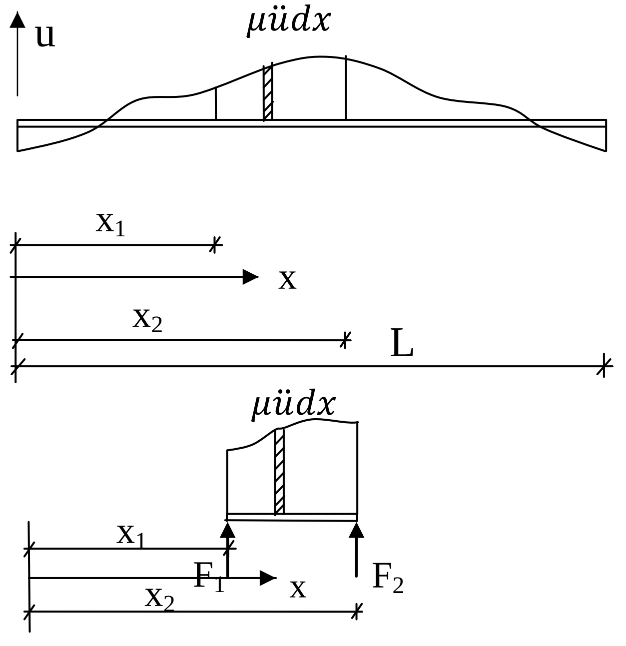
$j = 0,1,2,3, \ldots, \infty$

Equation (3.104) above is an expression for the constant A_j for a free-free bar under an initial displacement caused by its self weight. Equation (3.104) is substituted into the equation (3.94) and (3.95) to obtain the values of the fixed end forces F_1 and F_2.

Table 3.1: Summary of Fixed-end forces on a Segment of a Bar under Free Longitudinal vibration

S/N	Description	Remarks
1	Fixed-Fixed, Fixed-Pinned and Pinned-Pinned Bar  For $0 \leq \xi_1 < 1$ or $0 \leq x_1 < L$ $0 < \xi_2 \leq 1$ or $0 < x_2 \leq L$ $\xi_2 > \xi_1$	
	$F_1 = \frac{EA}{L(\xi_2-\xi_1)} \sum_{j=1}^{\infty} A_j \left[j\pi\xi_2 \cos j\pi\xi_1 - j\pi\xi_1 \cos j\pi\xi_1 - \sin j\pi\xi_2 + \sin j\pi\xi_1\right]$	See Equation (3.30)
	$F_2 = \frac{EA}{L} \sum_{j=1}^{\infty} A_j \left[-j\pi \cos j\pi\xi_2 + j\pi \cos j\pi\xi_1 - \frac{j\pi\xi_2 \cos j\pi\xi_1 - j\pi\xi_1 \cos j\pi\xi_1 - \sin j\pi\xi_2 + \sin j\pi\xi_1}{\xi_2-\xi_1}\right]$	See Equation (3.32)
	$A_j = 2eL\left(\frac{2-2(-1)^j}{j^3\pi^3}\right) \qquad j = 1,2,3,4,5,\ldots,\infty$	See Equation (3.46)

2	Fixed-Free and Pinned-Free Bar  	
	For **Case I**	
	$0 \le \xi_1 < 1$ or $0 \le x_1 < L$	
	$0 < \xi_2 \le 1$ or $0 < x_2 < L$	
	$\xi_2 > \xi_1$	
	$F_1 = \frac{EA}{L(\xi_2-\xi_1)} \sum_{j=1}^{\infty} A_j \left[\frac{i\pi\xi_2}{2}\cos\frac{i\pi\xi_1}{2} - \frac{i\pi\xi_1}{2}\cos\frac{i\pi\xi_1}{2} - \sin\frac{i\pi\xi_2}{2} + \sin\frac{i\pi\xi_1}{2}\right]$	See Equation (3.68)
	$F_2 = \frac{EA}{L}\sum_{j=1}^{\infty} A_j \left[-\frac{i\pi}{2}\cos\frac{i\pi\xi_2}{2} + \frac{i\pi}{2}\cos\frac{i\pi\xi_1}{2} - \frac{\frac{i\pi\xi_2}{2}\cos\frac{i\pi\xi_1}{2} - \frac{i\pi\xi_1}{2}\cos\frac{i\pi\xi_1}{2} - \sin\frac{i\pi\xi_2}{2} + \sin\frac{i\pi\xi_1}{2}}{\xi_2-\xi_1}\right]$	See Equation (3.69)
	For **Case II**	
	$0 \le \xi_1 < 1$ or $0 \le x_1 < L$	
	$\xi_2 = 1$ or $x_2 = L$	
	$F_1 = \frac{EA}{2L}\sum_{j=1}^{\infty} A_j\, i\pi \left(-\cos\frac{i\pi}{2} + \cos\frac{i\pi\xi_1}{2}\right)$	See Equation

	$F_2 = 0$ $A_j = \frac{16fL}{i^3\pi^3}$ $j = 1,2,3,4,5,\ldots,\infty$ $i = 1,3,5,7,9,\ldots,\infty$	(3.72) See Equation (3.79)
3	Free-Free Bar  $0 \leq \xi_1 \leq 1$ or $0 \leq x_1 \leq L$ $0 \leq \xi_2 \leq 1$ or $0 \leq x_2 \leq L$ $\xi_2 > \xi_1$ $F_1 = \frac{EA}{L(\xi_2-\xi_1)}\sum_{j=1}^{\infty} A_j\,[-j\pi\xi_2 \sin j\pi\xi_1 + j\pi\xi_1 \sin j\pi\xi_1 -$ $\cos j\pi\xi_2 + \cos j\pi\xi_1]$ $F_2 =$	See Equation (3.94) See Equation (3.95)

	$\frac{EA}{L}\sum_{j=1}^{\infty} A_j \Big[j\pi \sin j\pi\xi_2 - j\pi \sin j\pi\xi_1 - \frac{-j\pi\,\xi_2 \sin\, j\pi\,\xi_1 + j\pi\,\xi_1 \sin\, j\pi\,\xi_1 - \cos\, j\pi\,\xi_2 + \cos\, j\pi\,\xi_1}{\xi_2 - \xi_1}\Big]$ $A_j = 4fL\left(\frac{-\cos\frac{j\pi}{2}}{j^2\pi^2} + \frac{2\sin\frac{j\pi}{2}}{j^3\pi^3}\right)$ $j = 0,1,2,3,4,5,\ldots,\infty$	See Equation (3.104)
Note: F_1 and F_2 are axial forces in the bar, they were treated as support reactions to ease their calculation. $\xi_1 = \frac{x_1}{L}$ $\xi_2 = \frac{x_2}{L}$ e and f are dimensionless constants and can be taken as equal to unity		

3.2 Formulation of the equations for fixed end forces for segments of a beam under free lateral vibration

The partial differential equation governing the free lateral vibration of a beam is given by

$$EIu^{IV} + \mu\ddot{u} = 0 \tag{3.105}$$

By taking a trial solution of

$$u(x,t) = \emptyset(x)(A\cos\omega t + B\sin\omega t) \tag{3.106a}$$

$$u^{IV}(x,t) = \emptyset^{IV}(x)(A\cos\omega t + B\sin\omega t) \tag{3.106b}$$

$$\ddot{u}(x,t) = -\omega^2\emptyset(x)(A\cos\omega t + B\sin\omega t) \tag{3.106c}$$

Substitute equation (3.106b) and (3.106c) into equation (3.105)

$\emptyset^{IV} - \frac{\mu\omega^2}{EI}\emptyset = 0$ or

$$\emptyset^{IV} - \beta^4\emptyset = 0 \quad (3.107)$$

where $\beta^4 = \frac{\mu\omega^2}{EI}$ (3.108)

The solution of equation (3.107) is given as

$$\emptyset(x) = C_1 \cosh\beta x + C_2 \sinh\beta x + C_3 \cos\beta x + C_4 \sin\beta x \quad (3.109)$$

3.2.1 For a beam clamped at both ends (Fixed-fixed beam)

An illustration of this beam is shown in Figure 3.2. Since the displacement and rotation are zero at both ends

$\emptyset(0) = 0$

$\emptyset'(0) = 0$

$\emptyset(L) = 0$ (3.110)

$\emptyset'(L) = 0$

By substituting equations (3.110) into (3.109) we obtain

$$C_1 + C_3 = 0 \quad (3.110a)$$

$$C_2\beta + C_4\beta = 0 \quad (3.110b)$$

$$C_1 \cosh\beta L + C_2 \sinh\beta L + C_3 \cos\beta L + C_4 \sin\beta L = 0 \quad (3.110c)$$

$$C_1\beta \sinh \beta L + C_2\beta \cosh \beta L - C_3\beta \sin \beta L + C_4\beta \cos \beta L = 0 \tag{3.110d}$$

Putting equations (3.110a) to (3.110b) in matrix form

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \beta & 0 & \beta \\ \cosh \beta L & \sinh \beta L & \cos \beta L & \sin \beta L \\ \beta \sinh \beta L & \beta \cosh \beta L & -\beta \sin \beta L & \beta \cos \beta L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0 \tag{3.111}$$

For a non-trivial solution the determinant of the coefficients of the constants C_1, C_2, C_3 and C_4 must be zero .

$$\therefore \beta(\beta \cos^2 \beta L + \beta \sin^2 \beta L) + \beta(-\beta \sinh \beta L \cdot \sin \beta L - \beta \cosh \beta L \cdot \cos \beta L) -$$

$$\beta(\beta \cosh \beta L \cdot \cos \beta L - \beta \sinh \beta L \cdot \cos \beta L) + \beta(\beta \cosh^2 \beta L - \beta \sinh^2 \beta L) = 0$$

$$1 - \cos \beta L \cosh \beta L = 0 \tag{3.112}$$

The first seven roots of equation (3.90) were obtained numerically using the bisection method (Chapra and Canale, 2007) as

$$\beta_1 L = 4.7300408,\ \beta_2 L = 7.8532047, \beta_3 L = 10.99560784,\ \beta_4 L = 14.1371655,$$

$$\beta_5 L = 17.278759658, \beta_6 L = 20.42035224563,$$

$$\beta_7 L = 23.561944902041 \tag{3.113}$$

From equation (3.108)

$$\omega = \beta^2 L^2 \sqrt{\frac{EI}{\mu L^4}} \tag{3.114}$$

By substituting equation (3.113) into (3.114) we obtain the first natural frequencies of a fixed-fixed beam as

$$\omega_1 = 22.37328597\sqrt{\frac{EI}{\mu L^4}},\ \omega_2 = 61.67282406\sqrt{\frac{EI}{\mu L^4}},\ \omega_3 = 120.9033918\sqrt{\frac{EI}{\mu L^4}}$$

$$\omega_4 = 199.8594484\sqrt{\frac{EI}{\mu L^4}},\ \omega_5 = 298.555535\sqrt{\frac{EI}{\mu L^4}},\ \omega_6 = 416.9907856\sqrt{\frac{EI}{\mu L^4}}\ ,$$

$$\omega_7 = 555.1652475\sqrt{\frac{EI}{\mu L^4}}, \qquad (3.115)$$

By taking C_1 to be equal to one, the other constants C_2, C_3 and C_4 can be obtained from equation (3.111) as follows

$$C_3 = -1 \qquad (3.116a)$$

$$C_2 = \frac{\cos\beta L - \cosh\beta L}{\sinh\beta L - \sin\beta L} \qquad (3.116b)$$

$$C_3 = \frac{\cosh\beta L - \cos\beta L}{\sinh\beta L - \sin\beta L} \qquad (3.116c)$$

By substituting equation (3.116a) – (3.116c) into equation (3.109) we obtain the equation of the jth mode shape of vibration

$$\emptyset_j(x) = \cosh\beta_j x + \left(\frac{\cos\beta_j L - \cosh\beta_j L}{\sinh\beta_j L - \sin\beta_j L}\right)\sinh\beta_j x - \cos\beta_j x + \left(\frac{\cosh\beta_j L - \cos\beta_j L}{\sinh\beta_j L - \sin\beta_j L}\right)\sin\beta_j x$$

$$\emptyset_j(x) = \cosh\beta_j x + a_{2j}\sinh\beta_j x - \cos\beta_j x + a_{4j}\sin\beta_j x \qquad (3.117)$$

Where

$$a_{2j} = \frac{\cos\beta_j L - \cosh\beta_j L}{\sinh\beta_j L - \sin\beta_j L} \qquad (3.118a)$$

$$a_{4j} = \frac{\cosh\beta_j L - \cos\beta_j L}{\sinh\beta_j L - \sin\beta_j L} \qquad (3.118b)$$

Equation (3.113) is the equation of the mode of vibration of a fixed-fixed beam. The first mode of vibration (j = 1) can be obtained by substituting $\beta_j L = \beta_1 L = 4.7300408$ into equation (3.113). The second mode of vibration (j = 2) can be obtained by substituting $\beta_j L = \beta_2 L = 7.8532047$ into the equation. Likewise the mode shape for the jth mode can be obtained by substituting the value of $\beta_j L$ into the equation.

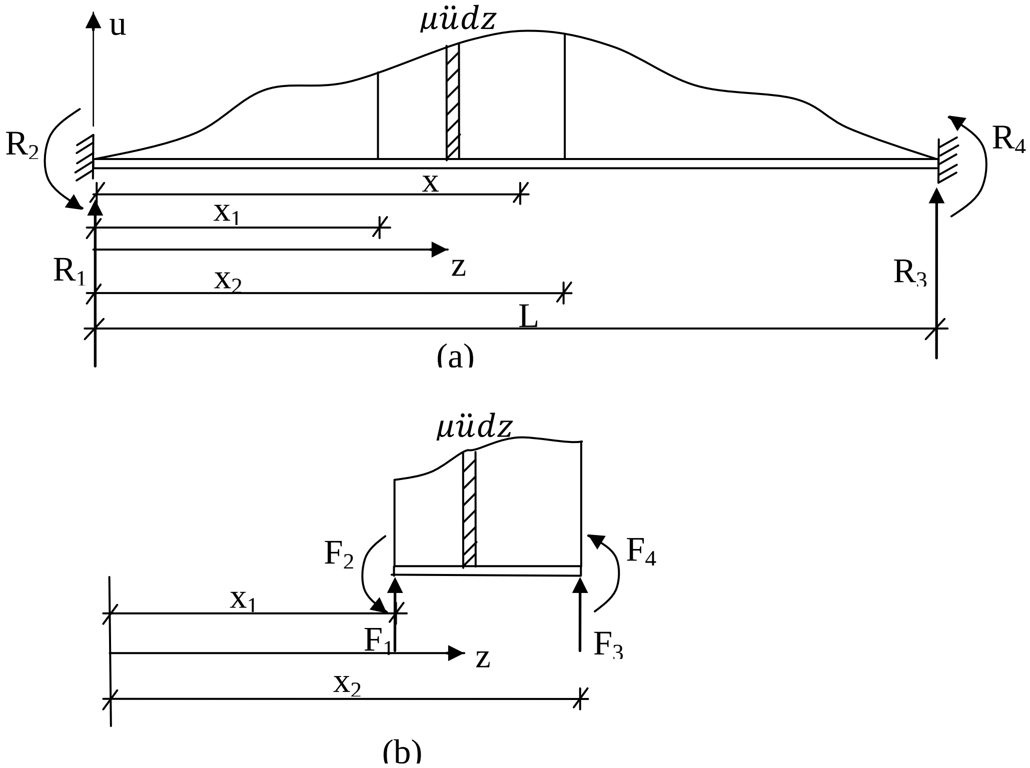


Figure 3.12

(a) A fixed-fixed beam under lateral vibration due to the inertial forces $\mu\ddot{u}$

(b) A segment of the beam under longitudinal vibration due to inertial forces $\mu\ddot{u}$

Figure 3.12 shows a fixed-fixed beam under inertia forces. A segment of the beam showed is being restrained by the fixed end forces $F_1 - F_4$. The reduced structure or basic system of the segment is shown in Figure 3.13.

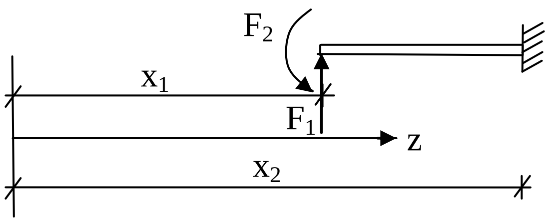


Figure 3.13: The reduced/basic structure of an arbitrary element of the vibrating beam

From equation (3.106c) and (3.117) the acceleration at any point in the vibrating beam is given by mode superposition as

$$\ddot{u} = \sum_{j=1}^{\infty} -\omega_j^2 \emptyset_j(x)\left(A_j \cos\omega_j t + B_j \sin\omega_j t\right) \tag{3.119}$$

The moment of an elementary force $\mu\ddot{u}dz$, at a distance z from the origin about an arbitrary point a distance x from the origin is given as

$$dM_x = \mu\, dz\, \ddot{u}(x - z) \tag{3.120}$$

$$M_x = \int_{x_1}^{x} \mu\ddot{u}(x - z)dz$$

$$= \mu x \int_{x_1}^{x} \ddot{u}\, dz - \mu \int_{x_1}^{x} \ddot{u}z\, dz \tag{3.121}$$

By integrating the component parts of equation (3.121) separately

$$\mu \int_{x_1}^{x} \ddot{u}z\, dz = \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu \int_{x_1}^{x} \emptyset_j z\, dz$$

$$= \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu \int_{x_1}^{x} \emptyset_j z\, dz$$

$$= \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu \int_{x_1}^{x} \left(z\cosh\beta_j z + a_{2j} z\sinh\beta_j z - z\cos\beta_j z + a_{4j} z\sin\beta_j z\right) dz \tag{3.122}$$

$$\int_{x_1}^{x} z\cosh\beta_j z\, dz = \frac{1}{\beta^2}\left(\beta_j x\sinh\beta_j x - \cosh\beta_j x - \beta_j x_1 \sinh\beta_j x_1 + \cosh\beta_j x_1\right) \tag{3.123}$$

$$\int_{x_1}^{x} z\sinh\beta_j z\, dz = \frac{1}{\beta^2}\left(\beta_j x\cosh\beta_j x - \sinh\beta_j x - \beta_j x_1 \cosh\beta_j x_1 + \sinh\beta_j x_1\right) \tag{3.124}$$

$$\int_{x_1}^{x} z\cos\beta_j z\, dz = \frac{1}{\beta^2}\left(\beta_j x\sin\beta_j x + \cos\beta_j x - \beta_j x_1 \sin\beta_j x_1 - \cos\beta_j x_1\right) \tag{3.125}$$

$$\int_{x_1}^{x} z\sin\beta_j z\, dz = \frac{1}{\beta^2}\left(-\beta_j x\cos\beta_j x + \sin\beta_j x + \beta_j x_1 \cos\beta_j x_1 - \sin\beta_j x_1\right) \tag{3.126}$$

By substituting equations (3.123) – (3.126) into equation (3.122) we obtain

$\mu \int_{x_1}^{x} \ddot{u} z\, dz = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^2} \left[\beta_j x\left(\sinh \beta_j x + a_{2j} \cosh \beta_j x - \sin \beta_j x - a_{4j} \cos \beta_j x\right) -\right.$

$\beta_j x_1\left(\sinh \beta_j x_1 + a_{2j} \cosh \beta_j x_1 - \sin \beta_j x_1 - a_{4j} \cos \beta_j x_1\right) - \left(\cosh \beta_j x +\right.$

$a_{2j} \sinh \beta_j x + \cos \beta_j x - a_{4j} \sin \beta_j x) + \left(\cosh \beta_j x_1 + a_{2j} \sinh \beta_j x_1 + \cos \beta_j x_1 -\right.$

$\left. a_{4j} \sin \beta_j x_1)\right]$ (3.127)

$\mu x \int_{x_1}^{x} \ddot{u}\, dz = \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu x \int_{x_1}^{x_2} \emptyset_j\, dz$

$$= \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu x \int_{x_1}^{x} \left(\cosh \beta_j z + a_{2j} \sinh \beta_j z - \cos \beta_j z + a_{4j} \sin \beta_j z\right) dz\ \ dz$$

(3.128)

We integrate the components of equation (3.128) separately

$\int_{x_1}^{x} \cosh \beta_j z\, dz = \frac{1}{\beta}\left(\sinh \beta_j x - \sinh \beta_j x_1\right)$ (3.129a)

$\int_{x_1}^{x} \sinh \beta_j z\, dz = \frac{1}{\beta}\left(\cosh \beta_j x - \cosh \beta_j x_1\right)$ (3.129b)

$\int_{x_1}^{x} \cos \beta_j z\, dz = \frac{1}{\beta}\left(\sin \beta_j x - \sin \beta_j x_1\right)$ (3.129c)

$\int_{x_1}^{x} \sin \beta_j z\, dz = \frac{1}{\beta}\left(- \cos \beta_j x + \cos \beta_j x_1\right)$ (3.129d)

By substituting equations (3.129a) – (3.129d) into equation (3.128) we obtain

$\mu x \int_{x_1}^{x} \ddot{u}\, dz = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu x}{\beta_j} \left(\sinh \beta_j x - \sinh \beta_j x_1 + a_{2j} \cosh \beta_j x - a_{2j} \cosh \beta_j x_1 -\right.$

$\left.\sin \beta_j x + \sin \beta_j x_1 - a_{4j} \cos \beta_j x + a_{4j} \cos \beta_j x_1\right)$ (3.130)

By substituting equations (3.128) and (3.130) into equations (3.121) we obtain

$M_x = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^2} \left[\beta_j x\left(- \sinh \beta_j x_1 - a_{2j} \cosh \beta_j x_1 + \sin \beta_j x_1 + a_{4j} \cos \beta_j x_1\right) +\right.$

$\beta_j x_1\left(\sinh \beta_j x_1 + a_{2j} \cosh \beta_j x_1 - \sin \beta_j x_1 - a_{4j} \cos \beta_j x_1\right) + \left(\cosh \beta_j x +\right.$

$a_{2j}\sinh\beta_j x + \cos\beta_j x - a_{4j}\sin\beta_j x) - \left(\cosh\beta_j x_1 + a_{2j}\sinh\beta_j x_1 + \cos\beta_j x_1 - a_{4j}\sin\beta_j x_1)\right]$ (3.131)

Equation (3.131) is the expression of the bending moment at a point x from the origin of a reduced segment of a fixed-fixed beam under free lateral vibration caused by inertia forces.

Using the principle of virtual work, there is need to obtain the equation of bending moments produced by unit values of the removed redundant forces.

For $F_1 = 1$ and $F_2 = 0$ (see Figure 3.13)

$$\bar{M}_1 = F_1(x - x_1)$$

$$= x - x_1 \quad (3.132)$$

$where \;\; x_1 \leq x \leq x_2$

Deformation at coordinate 1 (direction of force F_1) from the inertia forces is

$$\delta_{10} = \int_{x_1}^{x_2} \frac{\bar{M}_1 M_x}{EI} dx = \int_{x_1}^{x_2} \frac{M_x x - M_x x_1}{EI} dx \quad (3.133)$$

$$\int_{x_1}^{x_2} M_x x dx =$$

$$\sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^2}\Big[\left(-\sinh\beta_j x_1 - a_{2j}\cosh\beta_j x_1 + \sin\beta_j x_1 + a_{4j}\cos\beta_j x_1\right)\int_{x_1}^{x_2}\beta_j x^2 dx + \beta_j x_1\left(\sinh\beta_j x_1 + a_{2j}\cosh\beta_j x_1 - \sin\beta_j x_1 - a_{4j}\cos\beta_j x_1\right)\int_{x_1}^{x_2} x dx + \int_{x_1}^{x_2}\left(x\cosh\beta_j x + a_{2j} x\sinh\beta_j x + x\cos\beta_j x - a_{4j} x\sin\beta_j x\right)dx - \left(\cosh\beta_j x_1 + a_{2j}\sinh\beta_j x_1 + \cos\beta_j x_1 - a_{4j}\sin\beta_j x_1\right)\int_{x_1}^{x_2} x dx\Big] \quad (3.134)$$

By integrating the terms of equations of (3.134) separately

$$\int_{x_1}^{x_2} x\cosh\beta_j x\ \ dx = \frac{1}{\beta^2}\left(\beta_j x_2\sinh\beta_j x_2 - \cosh\beta_j x_2 - \beta_j x_1\sinh\beta_j x_1 + \cosh\beta_j x_1\right)$$

(3.135a)

$$\int_{x_1}^{x_2} x\sinh\beta_j x\ \ dx = \frac{1}{\beta^2}\left(\beta_j x_2\cosh\beta_j x_2 - \sinh\beta_j x_2 - \beta_j x_1\cosh\beta_j x_1 + \sinh\beta_j x_1\right)$$

(3.135b)

$$\int_{x_1}^{x_2} x\cos\beta_j x\ \ dx = \frac{1}{\beta^2}\left(\beta_j x_2\sin\beta_j x_2 + \cos\beta_j x_2 - \beta_j x_1\sin\beta_j x_1 - \cos\beta_j x_1\right)$$

(3.135c)

$$\int_{x_1}^{x_2} x\sin\beta_j x\ \ dx = \frac{1}{\beta^2}\left(-\beta_j x_2\cos\beta_j x_2 + \sin\beta_j x_2 + \beta_j x_1\cos\beta_j x_1 - \sin\beta_j x_1\right)$$

(3.135d)

By substituting equations (3.135a) – (3.135d) into equation (3.134) simplifying

$$\int_{x_1}^{x_2} M_x x dx = \sum_{j=1}^{\infty}\frac{-\omega_j^2 A_j\mu}{\beta_j^4}\Big[\frac{\beta_j^3(x_2^3-x_1^3)}{3}\big(-\sinh\beta_j x_1 - a_{2j}\cosh\beta_j x_1 + \sin\beta_j x_1 +$$

$$a_{4j}\cos\beta_j x_1\big) + \frac{\beta_j^3 x_1(x_2^2-x_1^2)}{2}\big(\sinh\beta_j x_1 + a_{2j}\cosh\beta_j x_1 - \sin\beta_j x_1 - a_{4j}\cos\beta_j x_1\big) +$$

$$\beta x_2\sinh\beta_j x_2 - \cosh\beta_j x_2 - \beta_j x_1\sinh\beta_j x_1 + \cosh\beta_j x_1 + \beta_j x_2\sin\beta_j x_2 +$$

$$\cos\beta_j x_2 - \beta x_1\sin\beta_j x_1 - \cos\beta_j x_1 + a_{2j}\big(\beta x_2\cosh\beta_j x_2 - \sinh\beta_j x_2 -$$

$$\beta_j x_1\cosh\beta_j x_1 + \sinh\beta_j x_1\big) - a_{4j}\big(-\beta x_2\cos\beta_j x_2 + \sin\beta_j x_2 + \beta_j x_1\cos\beta_j x_1 -$$

$$\sin\beta_j x_1\big) - \frac{\beta_j^2(x_2^2-x_1^2)}{2}\big(\cosh\beta_j x_1 + a_{2j}\sinh\beta_j x_1 + \cos\beta_j x_1 - a_{4j}\sin\beta_j x_1\big)\Big]$$

(3.136)

$$\int_{x_1}^{x_2} M_x x_1 dx = \sum_{j=1}^{\infty}\frac{-\omega_j^2 A_j\mu x_1}{\beta_j^2}\Big[\big(-\sinh\beta_j x_1 - a_{2j}\cosh\beta_j x_1 + \sin\beta_j x_1 +$$

$$a_{4j}\cos\beta_j x_1\big)\int_{x_1}^{x_2}\beta_j x dx + \beta_j x_1\big(\sinh\beta_j x_1 + a_{2j}\cosh\beta_j x_1 - \sin\beta_j x_1 -$$

$$a_{4j}\cos\beta_j x_1\big)\int_{x_1}^{x_2} dx + \int_{x_1}^{x_2}\big(\cosh\beta_j x + a_{2j}\sinh\beta_j x + \cos\beta_j x - a_{4j}\sin\beta_j x\big)dx -$$

$$\big(\cosh\beta_j x_1 + a_{2j}\sinh\beta_j x_1 + \cos\beta_j x_1 - a_{4j}\sin\beta_j x_1\big)\int_{x_1}^{x_2} dx\Big]$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu \beta x_1}{\beta_j^4} \Big[\frac{\beta_j^2 (x_2^2 - x_1^2)}{2} \big(-\sinh \beta_j x_1 - a_{2j} \cosh \beta_j x_1 + \sin \beta_j x_1 +$$

$$a_{4j} \cos \beta_j x_1 \big) + \beta_j^2 x_1 (x_2 - x_1) \big(\sinh \beta_j x_1 + a_{2j} \cosh \beta_j x_1 - \sin \beta_j x_1 -$$

$$a_{4j} \cos \beta_j x_1 \big) + \sinh \beta_j x_2 - \sinh \beta_j x_1 + \sin \beta_j x_2 - \sin \beta_j x_1 + a_{2j} \big(\cosh \beta_j x_2 -$$

$$\cosh \beta_j x_1 \big) - a_{4j} \big(-\cos \beta_j x_2 + \cos \beta_j x_1 \big) - \beta (x_2 - x_1) \big(\cosh \beta_j x_1 + a_{2j} \sinh \beta_j x_1 +$$

$$\cos \beta_j x_1 - a_{4j} \sin \beta_j x_1 \big) \Big] \qquad (3.137)$$

By substituting equations (3.136) and (3.137) into equation (3.133)

$$EI\delta_{10} = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^4} \Bigg[\beta_j^3 \left(\frac{(x_2^3 - x_1^3)}{3} - \frac{x_1 (x_2^2 - x_1^2)}{2} \right) \big(-\sinh \beta_j x_1 - a_{2j} \cosh \beta_j x_1 +$$

$$\sin \beta_j x_1 + a_{4j} \cos \beta_j x_1 \big) + \left(\frac{\beta_j^3 x_1 (x_2^2 - x_1^2)}{2} - \beta_j^3 x_1^2 (x_2 - x_1) \right) \big(\sinh \beta_j x_1 +$$

$$a_{2j} \cosh \beta_j x_1 - \sin \beta_j x_1 - a_{4j} \cos \beta_j x_1 \big) -$$

$$\left(\frac{\beta_j^2 (x_2^2 - x_1^2)}{2} - \beta_j^2 x_1 (x_2 - x_1) \right) \big(\cosh \beta_j x_1 + a_{2j} \sinh \beta_j x_1 + \cos \beta_j x_1 -$$

$$a_{4j} \sin \beta_j x_1 \big) + \beta x_2 \sinh \beta_j x_2 - \cosh \beta_j x_2 + \cosh \beta_j x_1 + \beta_j x_2 \sin \beta_j x_2 + \cos \beta_j x_2 -$$

$$\cos \beta_j x_1 - \beta_j x_1 \sinh \beta_j x_1 - \beta x_1 \sin \beta_j x_2 + a_{2j} \big(\beta x_2 \cosh \beta_j x_2 - \sinh \beta_j x_2 -$$

$$\beta_j x_1 \cosh \beta_j x_2 + \sinh \beta_j x_1 \big) - a_{4j} \big(-\beta x_2 \cos \beta_j x_2 + \sin \beta_j x_2 + \beta_j x_1 \cos \beta_j x_2 -$$

$$\sin \beta_j x_1 \big) \Bigg] \qquad (3.138)$$

Equation (3.138) above is the equation of deformation in coordinate 1 (direction of F_1) of a reduced segment of the vibrating fixed-fixed beam due to its inertia forces.

For $F_2 = 1$ and $F_1 = 0$ (see Figure 3.13)

$$\bar{M}_1 = -F_2$$

$$= -1 \qquad (3.139)$$

$where\ \ x_1 \leq x \leq x_2$

From the principle of virtual work deformation at coordinate 2 (direction of force F_2) from the inertia forces is

$$\delta_{20} = \int_{x_1}^{x_2} \frac{\bar{M}_2 M_x}{EI} dx = \int_{x_1}^{x_2} \frac{-M_x}{EI} dx \tag{3.140}$$

$$EI\delta_{20} = -\int_{x_1}^{x_2} M_x \, dx$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^2} \Big[\big(-\sinh\beta_j x_1 - a_{2j}\cosh\beta_j x_1 + \sin\beta_j x_1 + a_{4j}\cos\beta_j x_1\big)\int_{x_1}^{x_2} \beta_j x \, dx + \beta_j x_1\big(\sinh\beta_j x_1 + a_{2j}\cosh\beta_j x_1 - \sin\beta_j x_1 - a_{4j}\cos\beta_j x_1\big)\int_{x_1}^{x_2} dx + \int_{x_1}^{x_2}\big(\cosh\beta_j x + a_{2j}\sinh\beta_j x + \cos\beta_j x - a_{4j}\sin\beta_j x\big)\, dx - \big(\cosh\beta_j x_1 + a_{2j}\sinh\beta_j x_1 + \cos\beta_j x_1 - a_{4j}\sin\beta_j x_1\big)\int_{x_1}^{x_2} dx\Big]$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^3} \Big[\frac{\beta_j^2 (x_2^2 - x_1^2)}{2}\big(-\sinh\beta_j x_1 - a_{2j}\cosh\beta_j x_1 + \sin\beta_j x_1 + a_{4j}\cos\beta_j x_1\big) + \beta_j^2 x_1 (x_2 - x_1)\big(\sinh\beta_j x_1 + a_{2j}\cosh\beta_j x_1 - \sin\beta_j x_1 - a_{4j}\cos\beta_j x_1\big) + \sinh\beta_j x_2 - \sinh\beta_j x_1 + \sin\beta_j x_2 - \sin\beta_j x_1 + a_{2j}\big(\cosh\beta_j x_2 - \cosh\beta_j x_1\big) - a_{4j}\big(-\cos\beta_j x_2 + \cos\beta_j x_1\big) - \beta_j (x_2 - x_1)\big(\cosh\beta_j x + a_{2j}\sinh\beta_j x + \cos\beta_j x - a_{4j}\sin\beta_j x\big)\Big] \tag{3.141}$$

Equation (3.141) is the equation of deformation in coordinate 2 (direction of F_2) of a reduced segment of the vibrating fixed-fixed beam due to its inertia forces.

The compatibility equations of the segment (Figure 3.13) with respect to the coordinates 1 and 2 can be written as

$$\delta_{11}F_1 + \delta_{12}F_2 + \delta_{10} = 0$$

$$\delta_{21}F_1 + \delta_{22}F_2 + \delta_{20} = 0 \tag{3.142}$$

Or put in matrix form as

$$\begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \begin{bmatrix} \delta_{10} \\ \delta_{20} \end{bmatrix} = 0 \tag{3.142a}$$

Where the influence coefficient δ_{ij} is the deformation in coordinate i due to a unit load in coordinate j while δ_{i0} is the deformation in coordinate *i* due to inertia forces.

Using the principle of virtual work, these influence coefficients can be obtained as

$$\delta_{11} = \frac{l^3}{3EI} \ , \delta_{22} = \frac{l}{EI}, \delta_{12} = \delta_{21} = -\frac{l^2}{2EI} \tag{3.143}$$

By substituting equation (3.143) into (3.142a) and making F_1 and F_2 the subject of the formula

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \frac{l^3}{3EI} & -\frac{l^2}{2EI} \\ -\frac{l^2}{2EI} & \frac{l}{EI} \end{bmatrix}^{-1} \begin{bmatrix} -\delta_{10} \\ -\delta_{20} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12EI}{l^3} & \frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix} \begin{bmatrix} -\delta_{10} \\ -\delta_{20} \end{bmatrix} \tag{3.144}$$

But from Figure 3.13

$$l = x_2 - x_1 \tag{3.145}$$

By multiplying out the first row of equation (3.144) and substituting equation (3.145), the equation for F_1 is obtained as

$$F_1 = -\frac{12EI\delta_{10}}{l^3} - \frac{6EI\delta_{20}}{l^2} = -6\left(\frac{2EI\delta_{10}}{(x_2-x_1)^3} + \frac{EI\delta_{20}}{(x_2-x_1)^2}\right) \tag{3.146}$$

By substituting equation (3.138) and (3.141) into equation (3.146) and simplifying

$F_1 =$

$-6\sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^3}\Big[\beta_j^3\left(\frac{(x_2+x_1)(x_2^2-x_1^2)}{2}-\frac{2(x_2^3-x_1^3)}{3}\right)\left(-\sinh\beta_j x_1 - a_{2j}\cosh\beta_j x_1 +\right.$

$\sin\beta_j x_1 + a_{4j}\cos\beta_j x_1) + a_{2j}\left(\beta(x_1-x_2)\cosh\beta_j x_2 + 2\sinh\beta_j x_2 - \beta_j(x_2 -\right.$

$x_1)\cosh\beta_j x_1 - 2\sinh\beta_j x_1\big) - a_{4j}\left(-\beta(x_1-x_2)\cos\beta_j x_2 - 2\sin\beta_j x_2 +\right.$

$\beta_j(x_2-x_1)\cos\beta_j x_1 + 2\sin\beta_j x_1) + \beta(x_1-x_2)\sinh\beta_j x_2 -$

$\beta(x_2-x_1)\sinh\beta_j x_1 + \beta(x_1-x_2)\sin\beta_j x_2 - \beta(x_2-x_1)\sin\beta_j x_1 + 2\cosh\beta_j x_2 -$

$2\cosh\beta_j x_1 - 2\cos\beta_j x_2 + 2\cos\beta_j x_1\Big]$ (3.147)

$$F_1 = -6\sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^3} W_1 \tag{3.147a}$$

Where

$W_1 = \beta_j^3\left(\frac{(x_2+x_1)(x_2^2-x_1^2)}{2}-\frac{2(x_2^3-x_1^3)}{3}\right)\left(-\sinh\beta_j x_1 - a_{2j}\cosh\beta_j x_1 + \sin\beta_j x_1 +\right.$

$a_{4j}\cos\beta_j x_1) + a_{2j}\left(\beta(x_1-x_2)\cosh\beta_j x_2 + 2\sinh\beta_j x_2 - \beta_j(x_2-x_1)\cosh\beta_j x_1 -\right.$

$2\sinh\beta_j x_1) - a_{4j}\left(-\beta(x_1-x_2)\cos\beta_j x_2 - 2\sin\beta_j x_2 + \beta_j(x_2-x_1)\cos\beta_j x_1 +\right.$

$2\sin\beta_j x_1) + \beta(x_1-x_2)\sinh\beta_j x_2 - \beta(x_2-x_1)\sinh\beta_j x_1 + \beta(x_1-x_2)\sin\beta_j x_2 -$

$\beta(x_2-x_1)\sin\beta_j x_1 + 2\cosh\beta_j x_2 - 2\cosh\beta_j x_1 - 2\cos\beta_j x_2 + 2\cos\beta_j x_1$

(3.147b)

By multiplying out the second row of equation (3.144) and substituting equation (3.145), the equation for F_2 is obtained as

$$F_2 = -\frac{6EI\delta_{10}}{l^2} - \frac{4EI\delta_{20}}{l} = -2\left(\frac{3EI\delta_{10}}{(x_2-x_1)^2} + \frac{2EI\delta_{20}}{(x_2-x_1)}\right) \tag{3.148}$$

By substituting equation (3.138) and (3.141) into equation (3.148) and simplifying

$F_2 =$

$-2\sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^2}\Big[\beta_j^3\left(\frac{(2x_2+x_1)(x_2^2-x_1^2)}{2}-\frac{x_1(x_2-x_1)^2}{2}-(x_2^3-x_1^3)\right)\left(-\sinh\beta_j x_1 -\right.$

$a_{2j}\cosh\beta_j x_1+\sin\beta_j x_1+a_{4j}\cos\beta_j x_1\big)-$

$\left(\frac{\beta_j^2(x_2-x_1)^2}{2}\right)\left(\cosh\beta_j x_1+a_{2j}\sinh\beta_j x_1+\cos\beta_j x_1-a_{4j}\sin\beta_j x_1\right)+\beta_j(x_1-$

$x_2)\sinh\beta_j x_2-2\beta_j(x_2-x_2)\sinh\beta_j x_1+\beta_j(x_1-x_2)\sin\beta_j x_2-2\beta_j(x_2-$

$x_1)\sin\beta_j x_1-3\cos\beta_j x_2+3\cos\beta_j x_1+3\cosh\beta_j x_2-3\cosh\beta_j x_1+a_{2j}\big(\beta_j(x_1-$

$x_2)\cosh\beta_j x_2-2\beta_j(x_2-x_1)\cosh\beta_j x_1+3\sinh\beta_j x_2-3\sinh\beta_j x_1)-$

$a_{4j}\left(-\beta_j(x_1-x_2)\cos\beta_j x_2+2\beta(x_2-x_1)\cos\beta_j x_1-3\sin\beta_j x_2+3\sin\beta_j x_1\right)\Big]$

(3.149)

$$F_2 = -2\sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^2} W_2 \qquad (3.149a)$$

Where

$W_2 = \beta_j^3\left(\frac{(2x_2+x_1)(x_2^2-x_1^2)}{2}-\frac{x_1(x_2-x_1)^2}{2}-(x_2^3-x_1^3)\right)\left(-\sinh\beta_j x_1-a_{2j}\cosh\beta_j x_1+\right.$

$\sin\beta_j x_1+a_{4j}\cos\beta_j x_1\big)-\left(\frac{\beta_j^2(x_2-x_1)^2}{2}\right)\left(\cosh\beta_j x_1+a_{2j}\sinh\beta_j x_1+\cos\beta_j x_1-\right.$

$a_{4j}\sin\beta_j x_1\big)+\beta_j(x_1-x_2)\sinh\beta_j x_2-2\beta_j(x_2-x_2)\sinh\beta_j x_1+\beta_j(x_1-$

$x_2)\sin\beta_j x_2-2\beta_j(x_2-x_1)\sin\beta_j x_1-3\cos\beta_j x_2+3\cos\beta_j x_1+3\cosh\beta_j x_2-$

$3\cosh\beta_j x_1+a_{2j}\big(\beta_j(x_1-x_2)\cosh\beta_j x_2-2\beta_j(x_2-x_1)\cosh\beta_j x_1+3\sinh\beta_j x_2-$

$3\sinh\beta_j x_1)-a_{4j}\big(-\beta_j(x_1-x_2)\cos\beta_j x_2+2\beta_j(x_2-x_1)\cos\beta_j x_1-3\sin\beta_j x_2+$

$3\sin\beta_j x_1)$ (3.149b)

Equation (3.147a) and (3.149a) are the equations of the fixed end forces F_1 and F_2 on a segment of a fixed-fixed beam under vibration. There is need to determine the other end forces F_3 and F_4 needed to keep this segment in equilibrium (see Figure 3.12b).

For vertical equilibrium

$$F_1 + F_3 + \int_{x_1}^{x_2} \mu \ddot{u} dx = 0 \tag{3.150}$$

$$\therefore F_3 = -F_1 - \int_{x_1}^{x_2} \mu \ddot{u} dx \tag{3.151}$$

From equation (3.130)

$$\int_{x_1}^{x_2} \mu \ddot{u} dx = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j} \left(\sinh \beta_j x_2 - \sinh \beta_j x_1 + a_{2j} \cosh \beta_j x_2 - a_{2j} \cosh \beta_j x_1 - \sin \beta_j x_2 + \sin \beta_j x_1 - a_{4j} \cos \beta_j x_2 + a_{4j} \cos \beta_j x_1\right) \tag{3.152}$$

By substituting equations (3.152) and (3.147) into equation (3.151)

$$F_3 = \sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2 - x_1)^3} \Big[6W_1 + \beta_j^3 (x_2 - x_1)^3 \Big(\sinh \beta_j x_2 - \sinh \beta_j x_1 - \sin \beta_j x_2 + \sin \beta_j x_1 + a_{2j} \left(\cosh \beta_j x_2 - \cosh \beta_j x_1\right) - a_{4j} \left(\cos \beta_j x_2 - \cos \beta_j x_1\right)\Big)\Big] \tag{3.153}$$

When a system is in equilibrium, every part of it is in the same state, hence for the beam segment (see Figure 3.13b)

$$\sum M_{3,4} = 0; \quad F_1 (x_2 - x_1) - F_2 + \int_{x_1}^{x_2} \mu \ddot{u} dx \,(x_2 - x) - F_4 = 0$$

$$F_4 = F_1 (x_2 - x_1) - F_2 + \int_{x_1}^{x_2} \mu \ddot{u} dx \,(x_2 - x) \tag{3.154}$$

$$\int_{x_1}^{x_2} \mu \ddot{u} dx \,(x_2 - x) = \mu x_2 \int_{x_1}^{x_2} \ddot{u} dx - \mu \int_{x_1}^{x_2} \ddot{u} x \, dx \tag{3.155}$$

From equation (3.128)

$$\mu x_2 \int_{x_1}^{x_2} \ddot{u} dx = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu x_2}{\beta_j} \big(\sinh \beta_j x_2 - \sinh \beta_j x_1 + a_{2j} \cosh \beta_j x_2 - a_{2j} \cosh \beta_j x_1 - \sin \beta_j x_2 + \sin \beta_j x_1 - a_{4j} \cos \beta_j x_2 + a_{4j} \cos \beta_j x_1\big) \qquad (3.156)$$

From equation (3.127)

$$\mu \int_{x_1}^{x_2} \ddot{u} x \, dx = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^2} \big[\beta_j x_2 \big(\sinh \beta_j x_2 + a_{2j} \cosh \beta_j x_2 - \sin \beta_j x_2 - a_{4j} \cos \beta_j x_2\big) - \beta_j x_1 \big(\sinh \beta_j x_1 + a_{2j} \cosh \beta_j x_1 - \sin \beta_j x_1 - a_{4j} \cos \beta_j x_1\big) - \big(\cosh \beta_j x_2 + a_{2j} \sinh \beta_j x_2 + \cos \beta_j x_2 - a_{4j} \sin \beta_j x_2\big) + \big(\cosh \beta_j x_1 + a_{2j} \sinh \beta_j x_1 + \cos \beta_j x_1 - a_{4j} \sin \beta_j x_1\big)\big] \qquad (3.157)$$

By substituting equations (3.156) and (3.157) into equation (3.155) and simplifying

$$\int_{x_1}^{x_2} \mu \ddot{u} dx \, (x_2 - x) = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^2} \big[\beta_j x_2 \big(-\sinh \beta_j x_1 - a_{2j} \cosh \beta_j x_1 + \sin \beta_j x_1 + a_{4j} \cos \beta_j x_1\big) + \beta_j x_1 \big(\sinh \beta_j x_1 + a_{2j} \cosh \beta_j x_1 - \sin \beta_j x_1 - a_{4j} \cos \beta_j x_1\big) + \big(\cosh \beta_j x_2 + a_{2j} \sinh \beta_j x_2 + \cos \beta_j x_2 - a_{4j} \sin \beta_j x_2\big) - \big(\cosh \beta_j x_1 + a_{2j} \sinh \beta_j x_1 + \cos \beta_j x_1 - a_{4j} \sin \beta_j x_1\big)\big] \qquad (3.158)$$

By substituting equations (3.147), (3.149) and (3.158) into equation (3.154)

$$F_4 = \sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2 - x_1)^2} \big[-6W_1 + 2W_2 - \beta_j^3 (x_2 - x_1)^3 \big(-\sinh \beta_j x_1 - a_{2j} \cosh \beta_j x_1 + \sin \beta_j x_1 + a_{4j} \cos \beta_j x_1\big) + \big(\cosh \beta_j x_2 + a_{2j} \sinh \beta_j x_2 + \cos \beta_j x_2 - a_{4j} \sin \beta_j x_2\big) - \big(\cosh \beta_j x_1 + a_{2j} \sinh \beta_j x_1 + \cos \beta_j x_1 - a_{4j} \sin \beta_j x_1\big)\big] \qquad (3.159)$$

For us to be able to evaluate the equations for the fixed end forces $F_{1,}$ F_2, F_3 and F_4, there is need to derive an expression for A_j for a fixed-fixed beam.

a) **Derivation of the expression for A_j for a fixed-fixed beam**

Consider a uniform fixed-fixed beam under the action of its self weight.

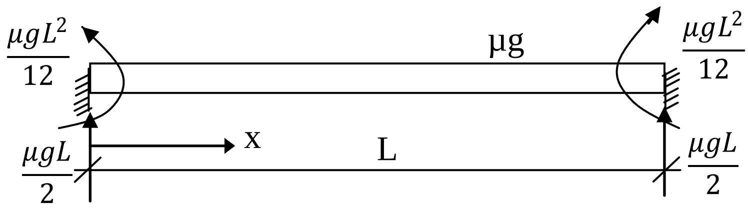


Figure 3.14: A Uniform fixed-fixed beam under the action of its self weight

The equation of the bending moment at any distance x from the left support is

$$M_x = \mu g\left(\frac{Lx}{2} - \frac{x^2}{2} - \frac{L^2}{12}\right) \qquad (3.160)$$

Where μ is the mass per unit length of the beam and g is the acceleration due to gravity.

From the equation of elastic curve (beam flexure equation)

$$\frac{M_x}{EI} = \frac{d^2y}{dx^2} \qquad (3.161)$$

By substituting equation (3.160) into (3.161) and solving for the deflection y

$$EIy' = \int_0^L M_x \, dx$$

$$= \int_0^L \mu g\left(\frac{Lx}{2} - \frac{x^2}{2} - \frac{L^2}{12}\right) dx$$

$$= \mu g\left(\frac{Lx^2}{4} - \frac{x^3}{6} - \frac{L^2 x}{12}\right) + c_1 \qquad (3.162)$$

$$EIy = \mu g\left(\frac{Lx^3}{12} - \frac{x^4}{24} - \frac{L^2 x^2}{24}\right) + c_1 x + c_2 \qquad (3.163)$$

Consider the boundary conditions

At $x = 0, \quad y = 0 \quad \therefore c_2 = 0$

At $x = 0, \quad y' = 0 \quad \therefore c_1 = 0$

Hence the equation for the static deformation or deflection of the uniform beam under its self weight is

$$EIy = \mu g \left(\frac{Lx^3}{12} - \frac{x^4}{24} - \frac{L^2x^2}{24}\right) \tag{3.164}$$

Let the initial deflection of the beam (at time t = 0) be

$$u(x,0) = \frac{\mu g}{EI}\left(\frac{Lx^3}{12} - \frac{x^4}{24} - \frac{L^2x^2}{24}\right)$$

$$= aL\left(\frac{2x^3}{L^3} - \frac{x^4}{L^4} - \frac{x^2}{L^2}\right) \tag{3.165}$$

Where a is dimensionless constant equal to $a = \frac{\mu g\, L^3}{24EI}$

From equation (2.61a)

$$A_j = \frac{\mu}{M_j}\int_0^L u(x,0)\emptyset_j \, dx$$

$$= \frac{\mu}{M_j}\int_0^L aL\left(\frac{2x^3}{L^3} - \frac{x^4}{L^4} - \frac{x^2}{L^2}\right)\left(\cosh\beta_j x + a_{2j}\sinh\beta_j x - \cos\beta_j x + a_{4j}\sin\beta_j x\right) dx$$

$$= \frac{a\mu L}{M_j}\int_0^L \left(\frac{2x^3}{L^3}\cosh\beta_j x + \frac{2x^3}{L^3}a_{2j}\sinh\beta_j x - \frac{2x^3}{L^3}\cos\beta_j x + \frac{2x^3}{L^3}a_{4j}\sin\beta_j x - \frac{x^4}{L^4}\cosh\beta_j x - \frac{x^4}{L^4}a_{2j}\sinh\beta_j x + \frac{x^4}{L^4}\cos\beta_j x - \frac{x^4}{L^4}a_{4j}\sin\beta_j x - \frac{x^2}{L^2}\cosh\beta_j x - \frac{x^2}{L^2}a_{2j}\sinh\beta_j x + \frac{x^2}{L^2}\cos\beta_j x - \frac{x^2}{L^2}a_{4j}\sin\beta_j x\right) \tag{3.166}$$

By integrating each term separately

$$\int_0^L x^3 \cosh\beta_j x\, dx = \left(\frac{\sinh\ \beta L}{\beta_j L} - \frac{3\cosh\ \beta L}{\beta_j^2 L^2} + \frac{6\sinh\ \beta L}{\beta_j^3 L^3} - \frac{6(\cosh\ \beta L - 1)}{\beta_j^4 L^4}\right) L^4 \qquad (3.167a)$$

$$\int_0^L x^3 \sinh\beta_j x\, dx = \left(\frac{\cosh\ \beta L}{\beta_j L} - \frac{3\sinh\ \beta L}{\beta_j^2 L^2} + \frac{6\cosh\ \beta L}{\beta_j^3 L^3} - \frac{6\sinh\ \beta L}{\beta_j^4 L^4}\right) L^4 \qquad (3.167b)$$

$$\int_0^L x^3 \cos\beta_j x\, dx = \left(\frac{\sin\ \beta L}{\beta_j L} + \frac{3\cos\ \beta L}{\beta_j^2 L^2} - \frac{6\sin\ \beta L}{\beta_j^3 L^3} - \frac{6(\cos\ \beta L - 1)}{\beta_j^4 L^4}\right) L^4 \qquad (3.167c)$$

$$\int_0^L x^3 \sin\beta_j x\, dx = \left(\frac{-\cos\ \beta L}{\beta_j L} + \frac{3\sin\ \beta L}{\beta_j^2 L^2} + \frac{6\cos\ \beta L}{\beta_j^3 L^3} - \frac{6\sin\ \beta L}{\beta_j^4 L^4}\right) L^4 \qquad (3.167d)$$

$$\int_0^L x^4 \cosh\beta_j x\, dx = \left(\frac{\sinh\ \beta L}{\beta_j L} - \frac{4\cosh\ \beta L}{\beta_j^2 L^2} + \frac{12\sinh\ \beta L}{\beta_j^3 L^3} - \frac{24\cosh\ \beta L}{\beta_j^4 L^4} + \frac{24\sinh\ \beta L}{\beta_j^5 L^5}\right) L^5$$

(3.167e)

$$\int_0^L x^4 \sinh\beta_j x\, dx = \left(\frac{\cosh\ \beta L}{\beta_j L} - \frac{4\sinh\ \beta L}{\beta_j^2 L^2} + \frac{12\cosh\ \beta L}{\beta_j^3 L^3} - \frac{24\sinh\ \beta L}{\beta_j^4 L^4} + \frac{24(\cosh\ \beta L - 1)}{\beta_j^5 L^5}\right) L^5$$

(3.167f)

$$\int_0^L x^4 \cos\beta_j x\, dx = \left(\frac{\sin\ \beta L}{\beta_j L} + \frac{4\cos\ \beta L}{\beta_j^2 L^2} - \frac{12\sin\ \beta L}{\beta_j^3 L^3} - \frac{24\cos\ \beta L}{\beta_j^4 L^4} + \frac{24\sin\ \beta L}{\beta_j^5 L^5}\right) L^5 \qquad (3.167g)$$

$$\int_0^L x^4 \sin\beta_j x\, dx = \left(\frac{-\cos\ \beta L}{\beta_j L} + \frac{4\sin\ \beta L}{\beta_j^2 L^2} + \frac{12\cos\ \beta L}{\beta_j^3 L^3} - \frac{24\sin\ \beta L}{\beta_j^4 L^4} - \frac{24(\cos\ \beta L - 1)}{\beta_j^5 L^5}\right) L^5$$

(3.159)

$$\int_0^L x^2 \cosh\beta_j x\, dx = \left(\frac{\sinh\ \beta L}{\beta_j L} - \frac{2\cosh\ \beta L}{\beta_j^2 L^2} + \frac{2\sinh\ \beta L}{\beta_j^3 L^3}\right) L^3 \qquad (3.167h)$$

$$\int_0^L x^2 \sinh\beta_j x\, dx = \left(\frac{\cosh\ \beta L}{\beta_j L} - \frac{2\sinh\ \beta L}{\beta_j^2 L^2} + \frac{2(\cosh\ \beta L - 1)}{\beta_j^3 L^3}\right) L^3 \qquad (3.167i)$$

$$\int_0^L x^2 \cos\beta_j x\, dx = \left(\frac{\sin\ \beta L}{\beta_j L} + \frac{2\cos\ \beta L}{\beta_j^2 L^2} - \frac{2\sin\ \beta L}{\beta_j^3 L^3}\right) L^3 \qquad (3.167j)$$

$$\int_0^L x^2 \sin\beta_j x\, dx = \left(\frac{-\cosh\ \beta L}{\beta_j L} + \frac{2\sin\ \beta L}{\beta_j^2 L^2} + \frac{2(\cos\ \beta L - 1)}{\beta_j^3 L^3}\right) L^3 \qquad (3.167k)$$

By substituting equations (3.167a – 3.167k) into equation (3.166) and simplifying

$$A_j = \frac{a\mu L^2}{M_j}\left[\frac{-2(\sinh\beta_j L - \sin\beta L)}{\beta_j^3 L^3} + \frac{12(\cosh\beta_j L - \cos\beta_j L)}{\beta_j^4 L^4} - \frac{24(\sinh\beta_j L - \sin\beta_j L)}{\beta_j^5 L^5} + \right.$$

$$\left. a_{2j}\left(\frac{-2(\cosh\beta_j L - \cos\beta_j L)}{\beta_j^3 L^3} + \frac{12(\sinh\beta_j L - \sin\beta_j L)}{\beta_j^4 L^4} - \frac{24(\cosh\beta_j L + \cos\beta_j L - 2)}{\beta_j^5 L^5}\right)\right] \quad (3.168)$$

Please note that $a_{2j} = -a_{4j}$

Equation (3.168) is the equation for the arbitrary constant A_j for a fixed-fixed beam.

In order to evaluate equation (3.168) there is need to derive an expression for the generalized mass Mj.

b) **Derivation of the expression for the generalized mass M_j for a fixed-fixed beam**

From equation (2.69b) the equation for the generalized mass is given as

$$M_j = \mu\int_0^L \emptyset_j^2\, dx \quad (3.169)$$

By substituting the general modal equation (3.117) into equation (3.169)

$$M_j = \mu\int_0^L\left(\cosh\beta_j x + a_{2j}\sinh\beta_j x - \cos\beta_j x + a_{4j}\sin\beta_j x\right)^2 dx$$

$$= \mu\int_0^L\left(\cosh^2\beta_j x + 2a_{2j}\sinh\beta_j x\cosh\beta_j x - 2\cosh\beta_j x\cos\beta_j x + \right.$$

$$2a_{4j}\sin\beta_j x\cosh\beta_j x + a_{2j}^2\sinh^2\beta_j x - 2a_{2j}\sinh\beta_j x\cos\beta_j x +$$

$$\left. 2a_{2j}a_{4j}\sin\beta_j x\sinh\beta_j x + \cos^2\beta_j x - 2a_{4j}\cos\beta_j x\sin\beta_j x + a_{4j}^2\sin^2\beta_j x\right)dx$$

(3.170)

In order to evaluate the integrals of the product of trigonometric and hyperbolic functions it is necessary to refer to the relationship between the two functions

$$\cosh i\theta = \cos\theta \quad (3.171a)$$

$$\sinh i\theta = i \sin\theta \tag{3.171b}$$

$$\sin i\theta = i \sinh\theta \tag{3.171c}$$

$$\cos i\theta = \cosh\theta \tag{3.171d}$$

(Stroud and Booth, 2007)

These together with trigonometric identities were used in evaluating the next set of integrals.

$$\int_0^L \cosh\beta_j x \cos\beta_j x \, dx = \frac{1}{2\beta_j}\left(\cosh\beta_j L \sin\beta_j L + \sinh\beta_j L \cos\beta_j L\right) \tag{3.172a}$$

$$\int_0^L \cosh\beta_j x \sin\beta_j x \, dx = \frac{1}{2\beta_j}\left(1 - \cos\beta_j L \cosh\beta_j L + \sin\beta_j L \sinh\beta_j L\right) \tag{3.172b}$$

$$\int_0^L \sinh\beta_j x \sin\beta_j x \, dx = \frac{1}{2\beta_j}\left(\cosh\beta_j L \sin\beta_j L - \sinh\beta_j L \cos\beta_j L\right) \tag{3.172c}$$

$$\int_0^L \sinh\beta_j x \cos\beta_j x \, dx = \frac{1}{2\beta_j}\left(\sinh\beta_j L \sin\beta_j L + \cosh\beta_j L \cos\beta_j L - 1\right) \tag{3.172d}$$

By substituting equations (3.172a – 3.172d) into equation (3.170) and evaluating the simple integrals we obtain

$$M_j =$$

$$\frac{\mu L}{2}\Bigg[2 + \frac{\sinh\ \beta L + \sin\ \beta L}{2\beta L} + a_{2j}^2\left(\frac{\sinh\ \beta L}{2\beta L} - 1\right) + a_{4j}^2\left(1 - \frac{\sin\ \beta L}{2\beta L}\right) -$$

$$2\left(\frac{\cosh\ \beta L\ \sin\ \beta L + \sinh\ \beta L\ \cos\ \beta L}{\beta L}\right) + 2a_{4j}\left(\frac{1 - \cos\ \beta L\ \cosh\ \beta L + \sin\ \beta L\ \sinh\ \beta L - \sin^2\beta L}{\beta L}\right) -$$

$$2a_{2j}\left(\frac{\sinh\ \beta L\ \sin\ \beta L + \cosh\ \beta L\ \cos\ \beta L - \sinh^2\beta L - 1}{\beta L}\right) + 2a_{2j}\, a_{4j}\left(\frac{\cosh\ \beta L\ \sin\ \beta L - \sinh\ \beta L\ \cos\ \beta L}{\beta L}\right)\Bigg]$$

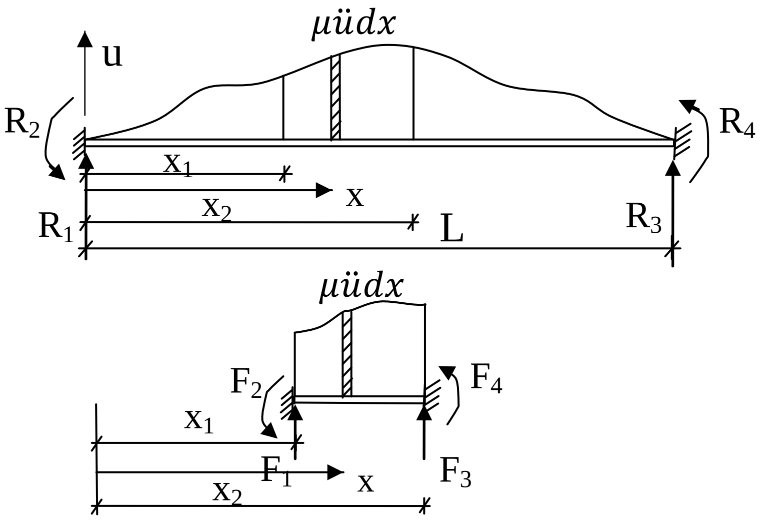
(3.173)

Equation (3.173) is the equation of the generalized mass of a fixed-fixed beam for the j[th] mode of vibration.

Having derived the equations of the fixed-end forces for a segment of a fixed-fixed beam under free vibration, a summary of the equations are presented in Table 3.2 below. In the Table the distances x_1 and x_2 were normalized using equations (3.35) and (3.36) and square of the j^{th} natural frequency ω_j^2 was eliminated from the equations by substituting equation (2.66a) which can be rewritten as

$$\frac{\omega_j^2}{\beta^4} = \frac{EI}{\mu} \tag{3.174}$$

Table 3.2: Summary of Fixed-end forces on a Segment of a Fixed-fixed Beam under Free Lateral vibration

S/N	Description	Remarks
1	Fixed-Fixed Beam  For $0 \leq \xi_1 < 1$ or $0 \leq x_1 < L$ $0 < \xi_2 \leq 1$ or $0 < x_2 \leq L$ $\xi_2 > \xi_1$ $\xi_1 = {}^{x_1}/_{L}$ $\xi_2 = {}^{x_2}/_{L}$ $F_1 = -6\sum_{j=1}^{\infty} \frac{EIA_j}{L^3(\xi_2-\xi_1)^3} W_1$ Where $W_1 = \beta_j^3 L^3 \left(\frac{(\xi_2+\xi_1)(\xi_2^2-\xi_1^2)}{2} - \frac{2(\xi_2^3-\xi_1^3)}{3}\right)\left(-\sinh\beta_j L\xi_1 -\right.$ $a_{2j}\cosh\beta_j L\xi_1 + \sin\beta_j L\xi_1 + a_{4j}\cos\beta_j L\xi_1) + a_{2j}\left(\beta L(\xi_1 -\right.$ $\xi_2)\cosh\beta_j L\xi_2 + 2\sinh\beta_j L\xi_2 - \beta_j L(\xi_2 - \xi_1)\cosh\beta_j L\xi_1 -$ $2\sinh\beta_j L\xi_1) - a_{4j}\left(-\beta_j L(\xi_1 - \xi_2)\cos\beta_j L\xi_2 - 2\sin\beta_j L\xi_2 +\right.$ $\beta_j L(\xi_2 - \xi_1)\cos\beta_j L\xi_1 + 2\sin\beta_j L\xi_1) + \beta_j L(\xi_1 -$	See Equation (3.147)

	$\xi_2) \sinh\beta_j L\xi_2 - \beta_j L(\xi_2 - \xi_1) \sinh\beta_j L\xi_1 + \beta_j L(\xi_1 -$ $\xi_2) \sin\beta_j L\xi_2 - \beta_j L(\xi_2 - \xi_1) \sin\beta_j L\xi_1 + 2\cosh\beta_j L\xi_2 -$ $2\cosh\beta_j L\xi_1 - 2\cos\beta_j L\xi_2 + 2\cos\beta_j L\xi_1$	
	$F_2 = -2\sum_{j=1}^{\infty} \frac{EIA_j}{L^2(\xi_2-\xi_1)^2} W_2$ where $W_2 =$ $\beta_j^3 L^3 \left(\frac{(2\xi_2+\xi_1)(\xi_2^2-\xi_1^2)}{2} - \frac{\xi_1(\xi_2-\xi_1)^2}{2} - (\xi_2^3 - \xi_1^3)\right)\left(-\sinh\beta_j L\xi_1 -\right.$ $a_{2j}\cosh\beta_j L\xi_1 + \sin\beta_j L\xi_1 + a_{4j}\cos\beta_j L\xi_1) -$ $\left(\frac{\beta_j^2 L^2(\xi_2-\xi_1)^2}{2}\right)\left(\cosh\beta_j L\xi_1 + a_{2j}\sinh\beta_j L\xi_1 + \cos\beta_j L\xi_1 -\right.$ $a_{4j}\sin\beta_j L\xi_1) + \beta_j L(\xi_1 - \xi_2) \sinh\beta_j L\xi_2 - 2\beta_j L(\xi_2 -$ $\xi_1) \sinh\beta_j L\xi_1 + \beta_j L(\xi_1 - \xi_2) \sin\beta_j L\xi_2 - 2\beta_j L(\xi_2 -$ $\xi_1) \sin\beta_j L\xi_1 - 3\cos\beta_j L\xi_2 + 3\cos\beta_j L\xi_1 + 3\cosh\beta_j L\xi_2 -$ $3\cosh\beta_j L\xi_1 + a_{2j}\left(\beta_j L(\xi_1 - \xi_2) \cosh\beta_j L\xi_2 - 2\beta_j L(\xi_2 -\right.$ $\xi_1) \cosh\beta_j L\xi_1 + 3\sinh\beta_j L\xi_2 - 3\sinh\beta_j L\xi_1) -$ $a_{4j}\left(-\beta_j L(\xi_1 - \xi_2) \cos\beta_j L\xi_2 + 2\beta_j L(\xi_2 - \xi_1) \cos\beta_j L\xi_1 -\right.$ $3\sin\beta_j L\xi_2 + 3\sin\beta_j L\xi_1)$	See Equation (3.149)
	$F_3 = \sum_{j=1}^{\infty} \frac{EIA_j}{L^3(\xi_2-\xi_1)^3}\left[6W_1 + \beta_j^3 L^3(\xi_2 - \xi_1)^3\left(\sinh\beta_j L\xi_2 -\right.\right.$ $\sinh\beta_j L\xi_1 - \sin\beta_j L\xi_2 + \sin\beta_j L\xi_1 + a_{2j}\left(\cosh\beta_j L\xi_2 -\right.$ $\left.\left.\cosh\beta_j L\xi_1) - a_{4j}\left(\cos\beta_j L\xi_2 - \cos\beta_j L\xi_1\right)\right)\right]$	See Equation (3.153)
	$F_4 =$ $\sum_{j=1}^{\infty} \frac{EIA_j}{L^2(\xi_2-\xi_1)^2}\left[-6W_1 + 2W_2 -\right.$	See Equation (3.159)

	$\beta_j^3 L^3 (\xi_2 - \xi_1)^3 \big(-\sinh \beta_j L\xi_1 - a_{2j} \cosh \beta_j L\xi_1 + \sin \beta_j L\xi_1 + a_{4j} \cos \beta_j L\xi_1\big) + \big(\cosh \beta_j L\xi_2 + a_{2j} \sinh \beta_j L\xi_2 + \cos \beta_j L\xi_2 - a_{4j} \sin \beta_j L\xi_2\big) - \big(\cosh \beta_j L\xi_1 + a_{2j} \sinh \beta_j L\xi_1 + \cos \beta_j L\xi_1 - a_{4j} \sin \beta_j L\xi_1\big)\Big]$	
	$A_j = \frac{a\mu L^2}{M_j}\Big[\frac{-2(\sinh \beta_j L - \sin \beta L)}{\beta_j^3 L^3} + \frac{12(\cosh \beta_j L - \cos \beta_j L)}{\beta_j^4 L^4} - \frac{24(\sinh \beta_j L - \sin \beta_j L)}{\beta_j^5 L^5} + a_{2j}\left(\frac{-2(\cosh \beta_j L - \cos \beta_j L)}{\beta_j^3 L^3} + \frac{12(\sinh \beta_j L - \sin \beta_j L)}{\beta_j^4 L^4} - \frac{24(\cosh \beta_j L + \cos \beta_j L - 2)}{\beta_j^5 L^5}\right)\Big]$	See Equation (3.168)
	$M_j = \frac{\mu L}{2}\Big[2 + \frac{\sinh \beta L + \sin \beta L}{2\beta L} + a_{2j}^2\left(\frac{\sinh \beta L}{2\beta L} - 1\right) + a_{4j}^2\left(1 - \frac{\sin \beta L}{2\beta L}\right) - 2\left(\frac{\cosh \beta L \sin \beta L + \sinh \beta L \cos \beta L}{\beta L}\right) + 2a_{4j}\left(\frac{1 - \cos \beta L \cosh \beta L + \sin \beta L \sinh \beta L - \sin^2 \beta L}{\beta L}\right) - 2a_{2j}\left(\frac{\sinh \beta L \sin \beta L + \cosh \beta L \cos \beta L - \sinh^2 \beta L - 1}{\beta L}\right) + 2a_{2j} a_{4j}\left(\frac{\cosh \beta L \sin \beta L - \sinh \beta L \cos \beta L}{\beta L}\right)\Big]$	See Equation (3.173)
	$a_{2j} = \frac{\cos \beta_j L - \cosh \beta_j L}{\sinh \beta_j L - \sin \beta_j L}$	See Equation (3.118a)
	$a_{4j} = \frac{\cosh \beta_j L - \cos \beta_j L}{\sinh \beta_j L - \sin \beta_j L}$	See Equation (3.118b)
	$\beta_1 L = 4.7300408 \quad \beta_2 L = 7.8532047\ , \beta_3 L = 10.99560784$ $\beta_4 L = 14.1371655\ ,\ \beta_5 L = 17.278759658$	See equation

	$\beta_6 L = 20.42035224563 \quad \beta_7 L = 23.561944902041$ etc.	(3.113)
Note: a is a dimensionless constant and can be taken as equal to unity		

3.2.2 For a beam clamped at one end and pinned at the other (Fixed-pinned beam)

If the beam is fixed at the near end (x = 0) and pinned at the far end (x = L), the boundary conditions are

$$\emptyset(0) = 0$$

$$\emptyset'(0) = 0$$

$$\emptyset(L) = 0 \tag{3.175}$$

$$\emptyset''(L) = 0$$

By substituting equations (3.175) into (3.109) we obtain

$$C_1 + C_3 = 0 \tag{3.175a}$$

$$C_2\beta + C_4\beta = 0 \tag{3.175b}$$

$$C_1 \cosh\beta L + C_2 \sinh\beta L + C_3 \cos\beta L + C_4 \sin\beta L = 0 \tag{3.175c}$$

$$C_1\beta^2 \cosh\beta L + C_2\beta^2 \sinh\beta L - C_3\beta^2 \cos\beta L - C_4\beta^2 \sin\beta L = 0 \tag{3.175d}$$

Putting equations (3.175a) to (3.175b) in matrix form

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \beta & 0 & \beta \\ \cosh \beta L & \sinh \beta L & \cos \beta L & \sin \beta L \\ \beta^2 \cosh \beta L & \beta^2 \sinh \beta L & -\beta^2 \cos \beta L & -\beta^2 \sin \beta L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0 \qquad (3.176)$$

For a non-trivial solution the determinant of the coefficients of the constants C_1, C_2, C_3 and C_4 must be zero .

$$\therefore \beta(-\beta^2 \sin \beta L \cdot \cos \beta L + \beta^2 \sin \beta L \cdot \cos \beta L) + \beta(-\beta^2 \sinh \beta L \cdot \cos \beta L + \beta^2 \sinh \beta L \cdot \cos \beta L) - \beta(-\beta^2 \sin \beta L \cdot \cosh \beta L - \beta^2 \sin \beta L \cdot \cosh \beta L) + \beta(\beta^2 \sinh \beta L \cdot \cosh \beta L - \beta^2 \sinh \beta L \cdot \cosh \beta L) = 0$$

$$\sinh \beta L \cos \beta L - \sin \beta L \cosh \beta L = 0 \qquad (3.177)$$

The first seven roots of equation (3.177) were obtained numerically using the bisection method as

$$\beta_1 L = 3.92660232, \beta_2 L = 7.06858275,\ \beta_3 L = 10.210176123,$$

$$\beta_4 L = 13.35176878,$$

$$\beta_5 L = 16.49336143135, \beta_6 L = 19.634954084937,$$

$$\beta_7 L = 22.77654673853 \qquad (3.178)$$

By substituting equations (3.178) into (3.114)

$$\omega_1 = 15.41820578\sqrt{\frac{EI}{\mu L^4}},\ \omega_2 = 49.96486209\sqrt{\frac{EI}{\mu L^4}},\ \omega_3 = 104.2476964\sqrt{\frac{EI}{\mu L^4}}$$

$$\omega_4 = 178.2697296\sqrt{\frac{EI}{\mu L^4}},\ \omega_5 = 272.0309713\sqrt{\frac{EI}{\mu L^4}},\ \omega_6 = 385.5314217\sqrt{\frac{EI}{\mu L^4}},$$

$$\omega_7 = 518.7710809\sqrt{\frac{EI}{\mu L^4}}, \qquad (3.179)$$

Equation (3.179) is the first seven natural frequencies of a fixed-pinned beam.

By taking C_1 to be equal to one, the other constants C_2, C_3 and C_4 can be obtained from equations (3.175) as follows

$$C_3 = -1 \tag{3.180a}$$

$$C_2 = \frac{\cos\ \beta L - \cosh\ \beta L}{\sinh\ \beta L - \sin\ \beta L} \tag{3.180b}$$

$$C_3 = \frac{\cosh\ \beta L - \cos\ \beta L}{\sinh\ \beta L - \sin\ \beta L} \tag{3.180c}$$

By substituting equation (3.180a – 3.180c) into equation (3.86) we obtain the equation of the jth mode shape of vibration

$$\emptyset_j(x) = \cosh \beta_j x + \left(\frac{\cos\ \beta_j L - \cosh\ \beta_j L}{\sinh\ \beta_j L - \sin\ \beta_j L}\right) \sinh \beta_j x - \cos \beta_j x + \left(\frac{\cosh\ \beta_j L - \cos\ \beta_j L}{\sinh\ \beta_j L - \sin\ \beta_j L}\right) \sin \beta_j x$$

$$\emptyset_j(x) = \cosh \beta_j x + b_{2j} \sinh \beta_j x - \cos \beta_j x + b_{4j} \sin \beta_j x \tag{3.181}$$

Where

$$b_{2j} = \frac{\cos\ \beta_j L - \cosh\ \beta_j L}{\sinh\ \beta_j L - \sin\ \beta_j L} \tag{3.182}$$

$$b_{4j} = \frac{\cosh\ \beta_j L - \cos\ \beta_j L}{\sinh\ \beta_j L - \sin\ \beta_j L} \tag{3.183}$$

Equation (3.181) is the equation of the jth mode of vibration of a fixed-pinned beam. The first mode of vibration (j = 1) can be obtained by substituting $\beta_j L = \beta_1 L = 3.92660232$ into equation (3.181). The second mode of vibration (j = 2) can be obtained by substituting $\beta_j L = \beta_2 L = 7.06858275$ into the equation. Likewise the mode shape for the jth mode can be obtained by substituting the value of $\beta_j L$ into the equation. Notice that a_{2j} and b_{2j} have the same equation, the only difference is that there values of B_jL are different. The same applies to a_{4j} and b_{4j}.

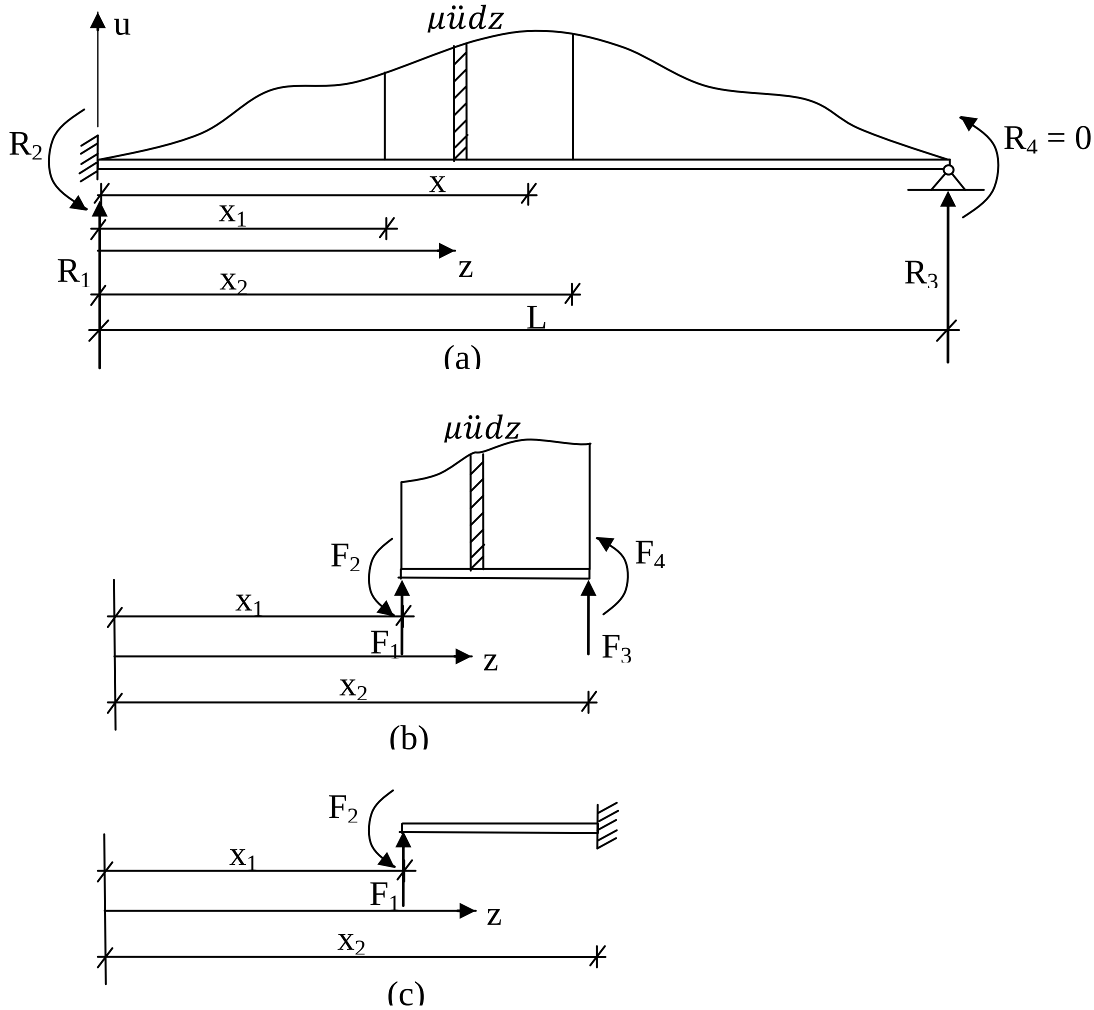


Figure 3.15

(a) A fixed-pinned beam under lateral vibration due to the inertial forces $\mu\ddot{u}$

(b) A segment of the beam under longitudinal vibration due to inertial forces $\mu\ddot{u}$

(c) The reduced/basic structure of an arbitrary element of the vibrating beam

Figure 3.15a shows a fixed-pinned beam under inertia forces. A segment of the beam showed is being restrained by the fixed end forces $F_1 – F_4$. The reduced structure of basic system is shown in Figure 3.15c.

The acceleration at any point in the vibrating beam as stated earlier (equation 3.119) is given by mode superposition as

$\ddot{u} = \sum_{j=1}^{\infty} -\omega_j^2 \emptyset_j(x)\left(A_j \cos\omega_j t + B_j \sin\omega_j t\right)$

For the segment of a fixed-pinned beam above (see figure 3.15b) there are two possible cases.

Case 1: $0 \leq x_1 < L;\;\; 0 < x_2 < L$

For this case the beam segment is largely the same as that of a fixed-fixed beam except that in the equation for mode shape Ø, a_{2j} and a_{4j} has been replaced by b_{2j} and b_{4j} and values of βjL defined earlier by equation (3.113) is now defined by equation (3.178). Compare equation (3.181) with (3.117).

Hence the equations for the fixed end forces (F_1 – F_4) for a segment of the fixed-fixed beam can be reproduced for this case of a fixed-pinned beam with a_{2j} and a_{4j} replaced by b_{2j} and b_{4j}.

From equation (3.147)

$$F_1 = -6\sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^3} W_1 \qquad (3.184a)$$

Where

$W_1 = \beta_j^3\left(\frac{(x_2+x_1)(x_2^2-x_1^2)}{2} - \frac{2(x_2^3-x_1^3)}{3}\right)\left(-\sinh\beta_j x_1 - b_{2j}\cosh\beta_j x_1 + \sin\beta_j x_1 + b_{4j}\cos\beta_j x_1\right) + b_{2j}\left(\beta(x_1-x_2)\cosh\beta_j x_2 + 2\sinh\beta_j x_2 - \beta_j(x_2-x_1)\cosh\beta_j x_1 - 2\sinh\beta_j x_1\right) - b_{4j}\left(-\beta(x_1-x_2)\cos\beta_j x_2 - 2\sin\beta_j x_2 + \beta_j(x_2-x_1)\cos\beta_j x_1 + 2\sin\beta_j x_1\right) + \beta(x_1-x_2)\sinh\beta_j x_2 - \beta(x_2-x_1)\sinh\beta_j x_1 + \beta(x_1-x_2)\sin\beta_j x_2 -$

$\beta(x_2 - x_1)\sin\beta_j x_1 + 2\cosh\beta_j x_2 - 2\cosh\beta_j x_1 - 2\cos\beta_j x_2 + 2\cos\beta_j x_1$

(3.184b)

From equation (3.149)

$$F_2 = -2\sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^2} W_2 \qquad (3.185a)$$

Where

$W_2 = \beta_j^3\left(\frac{(2x_2+x_1)(x_2^2-x_1^2)}{2} - \frac{x_1(x_2-x_1)^2}{2} - (x_2^3 - x_1^3)\right)\left(-\sinh\beta_j x_1 - b_{2j}\cosh\beta_j x_1 +\right.$

$\left.\sin\beta_j x_1 + b_{4j}\cos\beta_j x_1\right) - \left(\frac{\beta_j^2(x_2-x_1)^2}{2}\right)\left(\cosh\beta_j x_1 + b_{2j}\sinh\beta_j x_1 + \cos\beta_j x_1 -\right.$

$\left.b_{4j}\sin\beta_j x_1\right) + \beta_j(x_1 - x_2)\sinh\beta_j x_2 - 2\beta_j(x_2 - x_2)\sinh\beta_j x_1 + \beta_j(x_1 -$

$x_2)\sin\beta_j x_2 - 2\beta_j(x_2 - x_1)\sin\beta_j x_1 - 3\cos\beta_j x_2 + 3\cos\beta_j x_1 + 3\cosh\beta_j x_2 -$

$3\cosh\beta_j x_1 + b_{2j}\left(\beta_j(x_1 - x_2)\cosh\beta_j x_2 - 2\beta_j(x_2 - x_1)\cosh\beta_j x_1 + 3\sinh\beta_j x_2 -\right.$

$\left.3\sinh\beta_j x_1\right) - b_{4j}\left(-\beta_j(x_1 - x_2)\cos\beta_j x_2 + 2\beta_j(x_2 - x_1)\cos\beta_j x_1 - 3\sin\beta_j x_2 +\right.$

$\left.3\sin\beta_j x_1\right)$ (3.185b)

From equation (3.153)

$F_3 = \sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^3}\left[6W_1 + \beta_j^3(x_2 - x_1)^3\left(\sinh\beta_j x_2 - \sinh\beta_j x_1 - \sin\beta_j x_2 +\right.\right.$

$\left.\left.\sin\beta_j x_1 + b_{2j}\left(\cosh\beta_j x_2 - \cosh\beta_j x_1\right) - b_{4j}\left(\cos\beta_j x_2 - \cos\beta_j x_1\right)\right)\right]$ (3.186)

From equation (3.159)

$F_4 = \sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^2}\left[-6W_1 + 2W_2 - \beta_j^3(x_2 - x_1)^3\left(-\sinh\beta_j x_1 - b_{2j}\cosh\beta_j x_1 +\right.\right.$

$\left.\sin\beta_j x_1 + b_{4j}\cos\beta_j x_1\right) + \left(\cosh\beta_j x_2 + b_{2j}\sinh\beta_j x_2 + \cos\beta_j x_2 - b_{4j}\sin\beta_j x_2\right) -$

$\left.\left(\cosh\beta_j x_1 + b_{2j}\sinh\beta_j x_1 + \cos\beta_j x_1 - b_{4j}\sin\beta_j x_1\right)\right]$ (3.187)

Equations (3.184 – 3.187) are the equations of the fixed-end forces on a segment of a fixed-pinned beam under free lateral vibration.

Case II: $0 \leq x_1 < L$; $x_2 = L$

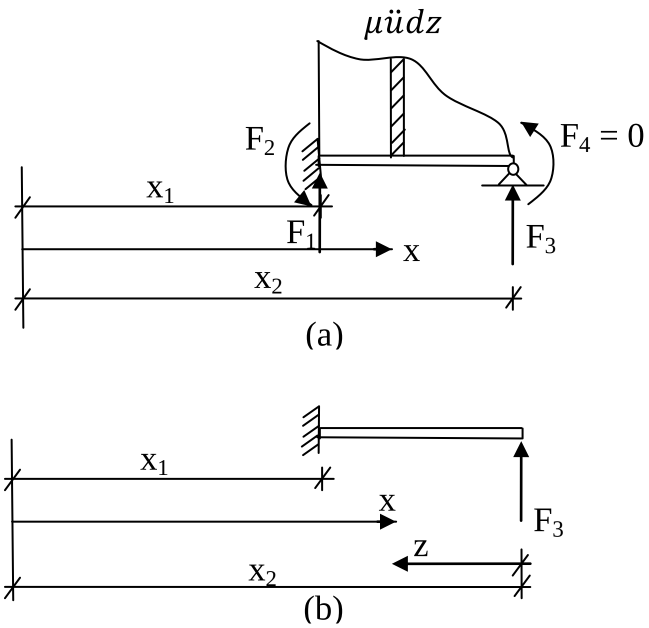


Figure 3.16

(a) A segment of the fixed-pinned beam with $x_2 = L$ under lateral vibration due to inertial forces $\mu\ddot{u}$

(b) The reduced/basic structure of an arbitrary element of the vibrating beam

If the reduced structure (Figure 3.16b) is mirrored laterally and its position measured from the right, it would look like the reduced structure of a segment of the fixed-fixed beam when $x_1 = 0$. Hence from equation (3.131) the equation of bending moment in z coordinates can be written as

$$M_z = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^2} \left[\beta_j z\left(-b_{2j} + b_{4j}\right) + \left(\cosh \beta_j z + b_{2j} \sinh \beta_j z + \cos \beta_j z - b_{4j} \sin \beta_j z\right) - 2\right] \quad (3.188)$$

From Figure 3.16b

$$z = L - x \tag{3.189}$$

By substituting equation (3.189) into (3.188) we can express the bending moment in terms of x.

$$M_x = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^2} \big[\beta_j (L-x)\left(-b_{2j} + b_{4j}\right) + \cosh\beta_j (L-x) + b_{2j} \sinh\beta_j (L-x) + \cos\beta_j (L-x) - b_{4j} \sin\beta_j (L-x) - 2\big] \tag{3.190}$$

By replacing the redundant force F_3 with a virtual load of unit value (see Figure 3.16b)

$$\bar{M}_3 = x_2 - x \tag{3.191}$$

From the principle of virtual work, the deformation at the direction (coordinate) of F_3 due to the inertia forces is

$$\delta_{30} = \int_{x_1}^{x_2} \frac{\bar{M}_3 M_x}{EI} dx = \int_{x_1}^{x_2} \frac{M_x x_2 - M_x x}{EI} dx \tag{3.192}$$

By integrating the components of equation (3.192) separately

$$\int_{x_1}^{x_2} M_x x \, dx = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^2} \Big[\beta_j\left(-b_{2j} + b_{4j}\right) \int_{x_1}^{x_2} (Lx - x^2)\, dx + \int_{x_1}^{x_2} \big(x \cosh\beta_j (L - x) + b_{2j} x \sinh\beta_j (L-x) + x \cos\beta_j (L-x) - b_{4j} x \sin\beta_j (L-x) - 2x\big)\, dx\Big]$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^4} \Big[\frac{\beta_j^3(-b_{2j} + b_{4j})}{6}\left(3L(x_2^2 - x_1^2) - 2(x_2^3 - x_1^3)\right) - \beta_j x_2 \sinh\beta_j (L - x_2) + \beta_j x_1 \sinh\beta_j (L-x_1) - \cosh\beta_j (L-x_2) + \cosh\beta_j (L-x_1) - \beta_j x_2 \sin\beta_j (L - x_2) + \beta_j x_1 \sin\beta_j (L-x_1) + \cos\beta_j (L-x_2) - \cos\beta_j (L-x_1) + b_{2j}\left(-\beta_j x_2 \cosh\beta_j (L-x_2) - \sinh\beta_j (L-x_2) + \beta_j x_1 \cosh\beta_j (L-x_1) + \right.$$

$\sinh\beta_j(L-x_1)) - b_{4j}\left(\beta_j x_2\cos\beta_j(L-x_2) + \sin\beta_j(L-x_2) - \beta_j x_1\cos\beta_j(L-x_1) - \sin\beta_j(L-x_1)\right) - \beta_j^2(x_2^2 - x_1^2)\Big]$ (3.193)

$$\int_{x_1}^{x_2} M_x x_2\,dx = \sum_{j=1}^{\infty}\frac{-\omega_j^2 A_j\mu x_2}{\beta_j^2}\left[\beta_j\left(b_{2j}+b_{4j}\right)\int_{x_1}^{x_2}(L-x)\,dx + \int_{x_1}^{x_2}\left(\cosh\beta_j(L-x) + b_{2j}\sinh\beta_j(L-x) + \cos\beta_j(L-x) - b_{4j}\sin\beta_j(L-x) - 2\right)dx\right]$$

$$= \sum_{j=1}^{\infty}\frac{-\omega_j^2 A_j\mu x_2}{\beta_j^4}\left[\frac{\beta_j^2(-b_{2j}+b_{4j})}{2}(2L(x_2-x_1) - x_2^2 + x_1^2) - \sinh\beta_j(L-x_2) + \sinh\beta_j(L-x_1) - \sin\beta_j(L-x_2) + \sin\beta_j(L-x_1) + 2\beta_j(x_2-x_1) - b_{2j}\left(\cosh\beta_j(L-x_2) - \cosh\beta_j(L-x_1)\right) + b_{4j}\left(-\cos\beta_j(L-x_2) + \cos\beta_j(L-x_1)\right)\right]$$

(3.194)

By substituting equations (3.193) and (3.194) into equation (3.192) we have

$$EI\delta_{30} = \sum_{j=1}^{\infty}\frac{-\omega_j^2 A_j\mu}{\beta_j^4}\left[\frac{\beta_j^3(-b_{2j}+b_{4j})(x_2-x_1)}{6}(3L(x_2-x_1) - x_2^2 - x_1x_2 + 2x_1^2) + \beta_j x_2\sinh\beta_j(L-x_1) + \beta_j x_2\sin\beta_j(L-x_1) - \beta_j x_1\sinh\beta(L-x_2) - \beta_j x_1\sin\beta_j(L-x_1) + \cosh\beta_j(L-x_2) - \cosh\beta_j(L-x_1) - \cos\beta_j(L-x_2) + \cos\beta_j(L-x_1) + b_{2j}\left(\beta_j x_2\cosh\beta_j(L-x_1) - \sinh\beta_j(L-x_2) - \beta_j x_1\cosh\beta_j(L-x_1) - \sinh\beta_j(L-x_1)\right) - b_{4j}\left(-\beta_j x_2\cos\beta(L-x_1) - \sin\beta_j(L-x_2) + \beta_j x_1\cos\beta_j(L-x_1) + \sin\beta_j(L-x_1)\right) + \beta_j^2(x_2^2 - x_1^2)\right]$$ (3.195)

By writing the beam's compatibility equations (see Figure 3.16b)

$$F_3\delta_{33} + \delta_{30} = 0$$

$$\therefore F_3 = \frac{-\delta_{30}}{\delta_{33}}$$ (3.196)

From the principle of virtual work, the influence coefficient δ_{33} (deformation in coordinate 3 due to a unit force at 3) can be determined as

$$\delta_{33} = \frac{l^3}{3EI} \qquad (3.197)$$

$$where\ l = x_2 - x_1 \qquad (3.198)$$

Substituting equation (3.195) and (3.197) into equation (3.196)

$$F_3 = \sum_{j=1}^{\infty} \frac{3\omega_j^2 A_j \mu}{\beta_j^4 l^3} \Big[\frac{\beta_j^3(-b_{2j}+b_{4j})(x_2-x_1)}{6} (3L(x_2 - x_1) - x_2^2 - x_1 x_2 + 2x_1^2) + \beta_j x_2 \sinh \beta_j (L - x_1) + \beta_j x_2 \sin \beta_j (L - x_1) - \beta_j x_1 \sinh \beta (L - x_2) - \beta_j x_1 \sin \beta_j (L - x_1) + \cosh \beta_j (L - x_2) - \cosh \beta_j (L - x_1) - \cos \beta_j (L - x_2) + \cos \beta_j (L - x_1) + b_{2j} \big(\beta_j x_2 \cosh \beta_j (L - x_1) - \sinh \beta_j (L - x_2) - \beta_j x_1 \cosh \beta_j (L - x_1) - \sinh \beta_j (L - x_1)\big) - b_{4j} \big(-\beta_j x_2 \cos \beta (L - x_1) - \sin \beta_j (L - x_2) + \beta_j x_1 \cos \beta_j (L - x_1) + \sin \beta_j (L - x_1)\big) + \beta_j^2 (x_2^2 - x_1^2) \Big] \qquad (3.198)$$

Equation (3.198) is an expression of the fixed end force F_3 on a segment of the fixed-pinned beam when x_2=L. Please note that this equation is simpler than it looks and this becomes obvious when x_2 = L is substituted into the equation.

$$F_3 = \sum_{j=1}^{\infty} \frac{3\omega_j^2 A_j \mu}{\beta_j^4 l^3} \Big[\frac{\beta_j^3(-b_{2j}+b_{4j})(L-x_1)}{6} (3L(L - x_1) - L^2 - x_1 L + 2x_1^2) + \beta_j L \sinh \beta_j (L - x_1) + \beta_j L \sin \beta_j (L - x_1) - \beta_j x_1 \sin \beta_j (L - x_1) - \cosh \beta_j (L - x_1) + \cos \beta_j (L - x_1) + b_{2j} \big(\beta_j L \cosh \beta_j (L - x_1) - \beta_j x_1 \cosh \beta_j (L - x_1) - \sinh \beta_j (L - x_1)\big) - b_{4j} \big(-\beta_j L \cos \beta (L - x_1) + \beta_j x_1 \cos \beta_j (L - x_1) + \sin \beta_j (L - x_1)\big) + \beta_j^2 (L^2 - x_1^2) \Big]$$

(3.198a)

Having obtained the equation for F_3, the equations of F_1 and F_2 can be obtained from the principles of static equilibrium from (see figure 3.16a)

$$F_2 = -M_{x=x1} - F_3(x_2 - x_1)$$

$$F_2 = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^4 l^2} \Big[\beta_j^3 l^2 (L - x_1)\big(-b_{2j} + b_{4j}\big) + \beta_j^2 l^2 \cosh \beta_j (L - x_1) +$$

$$b_{2j} \beta_j^2 l^2 \sinh \beta_j (L - x_1) + \beta_j^2 l^2 \cos \beta_j (L - x_1) - b_{4j} \beta_j^2 l^2 \sin \beta_j (L - x_1) - 2\beta_j^2 l^2 -$$

$$\frac{3(x_2-x_1)}{l} \Bigg(\frac{\beta_j^3(-a_{2j}+a_{4j})(L-x_1)}{6} (3L(L - x_1) - L^2 - x_1 L + 2x_1^2) + \beta_j L \sinh \beta_j (L - x_1) +$$

$$\beta_j L \sin \beta_j (L - x_1) - \beta_j x_1 \sin \beta_j (L - x_1) - \cosh \beta_j (L - x_1) + \cos \beta_j (L - x_1) +$$

$$b_{2j} \big(\beta_j L \cosh \beta_j (L - x_1) - \beta_j x_1 \cosh \beta_j (L - x_1) - \sinh \beta_j (L - x_1) \big) -$$

$$b_{4j} \big(-\beta_j L \cos \beta (L - x_1) + \beta_j x_1 \cos \beta_j (L - x_1) + \sin \beta_j (L - x_1) \big) + \beta_j^2 (L^2 - x_1^2) \Bigg) \Big] \tag{3.199}$$

For vertical equilibrium ($\sum F_y = 0$)

$$F_1 + F_3 + \int_{x_1}^{x_2} \mu \ddot{u} \, dx = 0$$

$$F_1 = -F_3 - \int_{x_1}^{x_2} \mu \ddot{u} \, dx \tag{3.200}$$

From equation (3.152)

$$\int_{x_1}^{x_2} \mu \ddot{u} dx =$$

$$\sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu \beta_j^3 l^3}{\beta_j^4 l^3} \big(\sinh \beta_j x_2 - \sinh \beta_j x_1 + b_{2j} \cosh \beta_j x_2 - b \cosh \beta_j x_1 - \sin \beta_j x_2 +$$

$$\sin \beta_j x_1 - b_{4j} \cos \beta_j x_2 + b_{4j} \cos \beta_j x_1) \tag{3.201}$$

By substituting equation (3.201) and (3.198a) into (3.200) we have

$$F_1 = \sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\beta_j^4 l^3} \Big[\beta_j^3 l^3 \big(\sinh\beta_j x_2 - \sinh\beta_j x_1 + b_{2j}\cosh\beta_j x_2 - b_{2j}\cosh\beta_j x_1 - \sin\beta_j x_2 + \sin\beta_j x_1 - b_{4j}\cos\beta_j x_2 + b_{4j}\cos\beta_j x_1\big) - 3\Big(\frac{\beta_j^3(-b_{2j}+b_{4j})(x_2-x_1)}{6}\big(3L(x_2 - x_1) - x_2^2 - x_1 x_2 + 2x_1^2\big) + \beta_j x_2 \sinh\beta_j(L-x_1) + \beta_j x_2 \sin\beta_j(L-x_1) - \beta_j x_1 \sinh\beta(L-x_2) - \beta_j x_1 \sin\beta_j(L-x_1) + \cosh\beta_j(L-x_2) - \cosh\beta_j(L-x_1) - \cos\beta_j(L-x_2) + \cos\beta_j(L-x_1) + b_{2j}\big(\beta_j x_2\cosh\beta_j(L-x_1) - \sinh\beta_j(L-x_2) - \beta_j x_1 \cosh\beta_j(L-x_1) - \sinh\beta_j(L-x_1)\big) - b_{4j}\big(-\beta_j x_2 \cos\beta(L-x_1) - \sin\beta_j(L-x_2) + \beta_j x_1 \cos\beta_j(L-x_1) + \sin\beta_j(L-x_1)\big) + \beta_j^2(x_2^2 - x_1^2)\Big)\Big] \quad (3.202)$$

In order to evaluate the equations for the fixed end forces F_1, F_2, F_3 and F_4, there is need to derive an expression for A_j for the fixed-pinned beam.

a) **Derivation of the expression for A_j for a fixed-pinned beam**

Consider a uniform fixed-pinned beam under the action of its self weight.

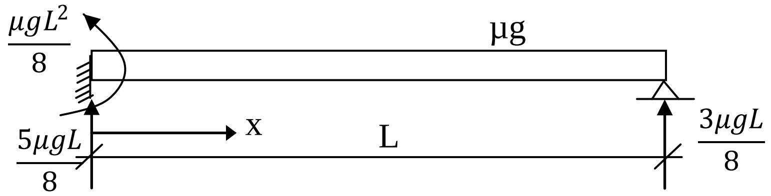


Figure 3.17: A Uniform fixed-pinned beam under the action of its self weight

The equation of the bending moment at any distance x from the left support is

$$M_x = \frac{\mu g}{2}\left(\frac{5Lx}{4} - x^2 - \frac{L^2}{4}\right) \quad (3.203)$$

Where μ is the mass per unit length of the beam and g is the acceleration due to gravity.

From the equation of elastic curve (beam flexure equation)

$$EIy' = \int M_x \, dx$$

$$= \int \frac{\mu g}{8}(5Lx - 4x^2 - L^2)\, dx$$

$$= \frac{\mu g}{8}\left(\frac{5Lx^2}{2} - \frac{4x^3}{3} - L^2 x\right) + c_1 \tag{3.204}$$

$$EIy = \frac{\mu g}{8}\left(\frac{5Lx^3}{6} - \frac{4x^4}{12} - \frac{L^2x^2}{2}\right) + c_1 x + c_2 \tag{3.205}$$

Consider the boundary conditions

At $x = 0$, $y = 0$ $\therefore c_2 = 0$

At $x = 0$, $y' = 0$ $\therefore c_1 = 0$

Hence the equation for the static deformation or deflection of the uniform beam under its self weight is

$$EIy = \frac{\mu g}{8}\left(\frac{5Lx^3}{6} - \frac{4x^4}{12} - \frac{L^2x^2}{2}\right) \tag{3.206}$$

Let the initial deflection of the beam (at time t = 0) be

$$u(x,0) = \frac{\mu g}{8EI}\left(\frac{5Lx^3}{6} - \frac{4x^4}{12} - \frac{L^2x^2}{2}\right)$$

$$= bL\left(\frac{5x^3}{L^3} - \frac{2x^4}{L^4} - \frac{3x^2}{L^2}\right) \tag{3.207}$$

Where b is a dimensionless constant equal to $\frac{\mu g L^3}{48EI}$

From equation (2.61a)

$A_j = \frac{\mu}{M_j} \int_0^L u(x,0) \emptyset_j \, dx$

$= \frac{\mu}{M_j} \int_0^L bL \left(\frac{5x^3}{L^3} - \frac{2x^4}{L^4} - \frac{3x^2}{L^2} \right) \left(\cosh \beta_j x + b_{2j} \sinh \beta_j x - \cos \beta_j x + \right.$

$\left. b_{4j} \sin \beta_j x \right) dx$

$= \frac{b\mu L}{M_j} \int_0^L \left(\frac{5x^3}{L^3} \cosh \beta_j x + \frac{5x^3}{L^3} b_{2j} \sinh \beta_j x - \frac{5x^3}{L^3} \cos \beta_j x + \frac{5x^3}{L^3} b_{4j} \sin \beta_j x - \right.$

$\frac{2x^4}{L^4} \cosh \beta_j x - \frac{2x^4}{L^4} b_{2j} \sinh \beta_j x + \frac{2x^4}{L^4} \cos \beta_j x - \frac{2x^4}{L^4} b_{4j} \sin \beta_j x - \frac{3x^2}{L^2} \cosh \beta_j x -$

$\left. \frac{3x^2}{L^2} b_{2j} \sinh \beta_j x + \frac{3x^2}{L^2} \cos \beta_j x - \frac{3x^2}{L^2} b_{4j} \sin \beta_j x \right)$ (3.208)

By taking advantage of equations (3.152) – (3.163) we can perform the integration of equation (3.208) to obtain

$A_j =$

$\frac{b\mu L^2}{M_j} \left[\frac{-(\cosh \beta_j L + \cos \beta L)}{\beta_j^2 L^2} + \frac{18(\cosh \beta_j L - \cos \beta_j L)}{\beta_j^4 L^4} - \frac{48(\sinh \beta_j L - \sin \beta_j L)}{\beta_j^5 L^5} + \right.$

$\left. b_{2j} \left(\frac{-2(\sinh \beta_j L + \sin \beta_j L)}{\beta_j^2 L^2} + \frac{18(\sinh \beta_j L - \sin \beta_j L)}{\beta_j^4 L^4} - \frac{48(\cosh \beta_j L + \cos \beta_j L - 2)}{\beta_j^5 L^5} \right) \right]$ (3.209)

Please note that $b_{2j} = -b_{4j}$

Equation (3.209) is the equation for the arbitrary constant A_j for a fixed-pinned beam.

In order to evaluate equation (3.209) there is need to derive an expression for the generalized mass Mj.

b) **Derivation of the expression for the generalized mass M_j for a fixed-pinned beam**

By substituting the general modal equation (3.181) into equation (3.165)

$$M_j = \mu \int_0^L \left(\cosh \beta_j x + b_{2j} \sinh \beta_j x - \cos \beta_j x + b_{4j} \sin \beta_j x\right)^2 dx$$

$$= \mu \int_0^L \left(\cosh^2 \beta_j x + 2b_{2j} \sinh \beta_j x \cosh \beta_j x - 2 \cosh \beta_j x \cos \beta_j x + \right.$$

$$2b_{4j} \sin \beta_j x \cosh \beta_j x + b_{2j}^2 \sinh^2 \beta_j x - 2b_{2j} \sinh \beta_j x \cos \beta_j x +$$

$$\left. 2b_{2j} b_{4j} \sin \beta_j x \sinh \beta_j x + \cos^2 \beta_j x - 2b_{4j} \cos \beta_j x \sin \beta_j x + b_{4j}^2 \sin^2 \beta_j x\right) dx \tag{3.210}$$

By substituting equations (3.168 – 3.171) into equation (3.210) and evaluating the simple integrals we obtain

$$M_j =$$

$$\frac{\mu L}{2}\left[2 + \frac{\sinh \beta L + \sin \beta L}{2\beta L} + b_{2j}^2 \left(\frac{\sinh \beta L}{2\beta L} - 1\right) + b_{4j}^2 \left(1 - \frac{\sin \beta L}{2\beta L}\right) - \right.$$

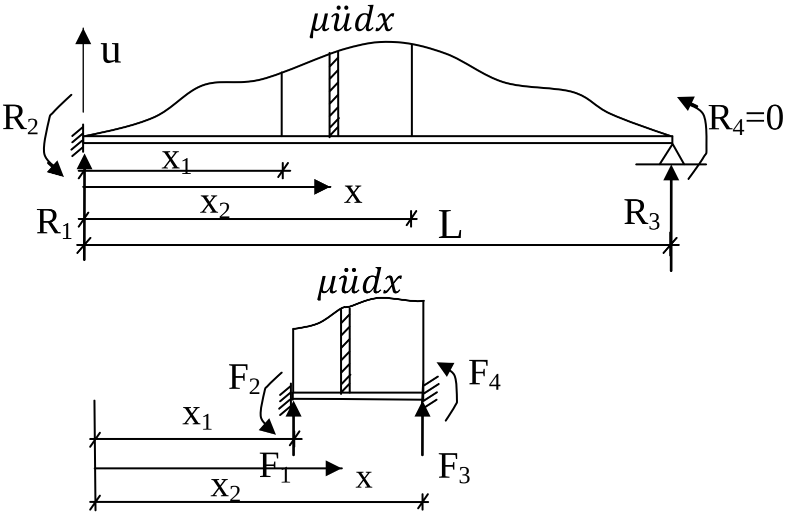
$$2\left(\frac{\cosh \beta L \sin \beta L + \sinh \beta L \cos \beta L}{\beta L}\right) + 2b_{4j} \left(\frac{1 - \cos \beta L \cosh \beta L + \sin \beta L \sinh \beta L - \sin^2 \beta L}{\beta L}\right) -$$

$$\left. 2b_{2j} \left(\frac{\sinh \beta L \sin \beta L + \cosh \beta L \cos \beta L - \sinh^2 \beta L - 1}{\beta L}\right) + 2b_{2j} b_{4j} \left(\frac{\cosh \beta L \sin \beta L - \sinh \beta L \cos \beta L}{\beta L}\right)\right] \tag{3.211}$$

Equation (3.211) is the equation of the generalized mass of a fixed-pinned beam for the jth mode of vibration.

Having derived the equations of the fixed-end forces for a segment of a fixed-pinned beam under free vibration, a summary of the derived equations are presented in Table 3.3 for a quick reference. In the Table the distances x_1 and x_2 were normalized using equations (3.35) and (3.36) and square of the jth natural frequency ω_j^2 was eliminated from the equations by substituting equation (3.173)

Table 3.3: Summary of Fixed-end forces on a Segment of a Fixed-pinned Beam under Free Lateral vibration

S/N	Description	Remarks
1	Fixed-pinned Beam  $\xi_1 = {}^{x_1}/_{L}$ $\xi_2 = {}^{x_2}/_{L}$ **CASE I** For $0 \leq \xi_1 < 1$ or $0 \leq x_1 < L$ $0 < \xi_2 \leq 1$ or $0 < x_2 < L$ $\xi_2 > \xi_1$ $F_1 = -6\sum_{j=1}^{\infty} \frac{EIA_j}{L^3(\xi_2-\xi_1)^3} W_1$ Where $W_1 = \beta_j^3 L^3 \left(\frac{(\xi_2+\xi_1)(\xi_2^2-\xi_1^2)}{2} - \frac{2(\xi_2^3-\xi_1^3)}{3}\right)\left(-\sinh\beta_j L\xi_1 - b_{2j}\cosh\beta_j L\xi_1 + \sin\beta_j L\xi_1 + b_{4j}\cos\beta_j L\xi_1\right) + b_{2j}\left(\beta L(\xi_1 - \xi_2)\cosh\beta_j L\xi_2 + 2\sinh\beta_j L\xi_2 - \beta_j L(\xi_2-\xi_1)\cosh\beta_j L\xi_1 -\right.$	See Equation (3.184)

	$2\sinh\beta_j L\xi_1) - b_{4j}\left(-\beta_j L(\xi_1-\xi_2)\cos\beta_j L\xi_2 - 2\sin\beta_j L\xi_2 + \beta_j L(\xi_2-\xi_1)\cos\beta_j L\xi_1 + 2\sin\beta_j L\xi_1\right) + \beta_j L(\xi_1-\xi_2)\sinh\beta_j L\xi_2 - \beta_j L(\xi_2-\xi_1)\sinh\beta_j L\xi_1 + \beta_j L(\xi_1-\xi_2)\sin\beta_j L\xi_2 - \beta_j L(\xi_2-\xi_1)\sin\beta_j L\xi_1 + 2\cosh\beta_j L\xi_2 - 2\cosh\beta_j L\xi_1 - 2\cos\beta_j L\xi_2 + 2\cos\beta_j L\xi_1$	
	$F_2 = -2\sum_{j=1}^{\infty}\frac{EIA_j}{L^2(\xi_2-\xi_1)^2}W_2$ Where $W_2 = \beta_j^3 L^3\left(\frac{(2\xi_2+\xi_1)(\xi_2^2-\xi_1^2)}{2} - \frac{\xi_1(\xi_2-\xi_1)^2}{2} - (\xi_2^3-\xi_1^3)\right)\left(-\sinh\beta_j L\xi_1 - b_{2j}\cosh\beta_j L\xi_1 + \sin\beta_j L\xi_1 + b_{4j}\cos\beta_j L\xi_1\right) - \left(\frac{\beta_j^2 L^2(\xi_2-\xi_1)^2}{2}\right)\left(\cosh\beta_j L\xi_1 + b_{2j}\sinh\beta_j L\xi_1 + \cos\beta_j L\xi_1 - b_{4j}\sin\beta_j L\xi_1\right) + \beta_j L(\xi_1-\xi_2)\sinh\beta_j L\xi_2 - 2\beta_j L(\xi_2-\xi_1)\sinh\beta_j L\xi_1 + \beta_j L(\xi_1-\xi_2)\sin\beta_j L\xi_2 - 2\beta_j L(\xi_2-\xi_1)\sin\beta_j L\xi_1 - 3\cos\beta_j L\xi_2 + 3\cos\beta_j L\xi_1 + 3\cosh\beta_j L\xi_2 - 3\cosh\beta_j L\xi_1 + b_{2j}\left(\beta_j L(\xi_1-\xi_2)\cosh\beta_j L\xi_2 - 2\beta_j L(\xi_2-\xi_1)\cosh\beta_j L\xi_1 + 3\sinh\beta_j L\xi_2 - 3\sinh\beta_j L\xi_1\right) - b_{4j}\left(-\beta_j L(\xi_1-\xi_2)\cos\beta_j L\xi_2 + 2\beta_j L(\xi_2-\xi_1)\cos\beta_j L\xi_1 - 3\sin\beta_j L\xi_2 + 3\sin\beta_j L\xi_1\right)$	See Equation (3.185)
	$F_3 = \sum_{j=1}^{\infty}\frac{EIA_j}{L^3(\xi_2-\xi_1)^3}\left[6W_1 + \beta_j^3 L^3(\xi_2-\xi_1)^3\left(\sinh\beta_j L\xi_2 - \sinh\beta_j L\xi_1 - \sin\beta_j L\xi_2 + \sin\beta_j L\xi_1 + b_{2j}\left(\cosh\beta_j L\xi_2 - \cosh\beta_j L\xi_1\right) - b_{4j}\left(\cos\beta_j L\xi_2 - \cos\beta_j L\xi_1\right)\right)\right]$	See Equation (3.186)

	$F_4 =$ $\sum_{j=1}^{\infty} \frac{EIA_j}{L^2(\xi_2-\xi_1)^2}\Big[-6W_1 + 2W_2 -$ $\beta_j^3 L^3(\xi_2-\xi_1)^3\big(-\sinh\beta_j L\xi_1 - b_{2j}\cosh\beta_j L\xi_1 + \sin\beta_j L\xi_1 +$ $b_{4j}\cos\beta_j L\xi_1\big) + \big(\cosh\beta_j L\xi_2 + b_{2j}\sinh\beta_j L\xi_2 + \cos\beta_j L\xi_2 -$ $b_{4j}\sin\beta_j L\xi_2\big) - \big(\cosh\beta_j L\xi_1 + b_{2j}\sinh\beta_j L\xi_1 + \cos\beta_j L\xi_1 -$ $b_{4j}\sin\beta_j L\xi_1\big)\Big]$ **CASE II** For $0 \le \xi_1 < 1$ or $0 \le x_1 < L$ $\xi_2 = 1$ or $x_2 = L$ $\xi_2 > \xi_1$ $F_1 = \sum_{j=1}^{\infty} \frac{EIA_j}{(x_2-x_1)^3}\Big[\beta_j^3 L^3(\xi_2-\xi_1)^3\big(\sinh\beta_j L\xi_2 - \sinh\beta_j L\xi_1 +$ $b_{2j}\cosh\beta_j L\xi_2 - b_{2j}\cosh\beta_j L\xi_1 - \sin\beta_j L\xi_2 + \sin\beta_j L\xi_1 -$ $b_{4j}\cos\beta_j L\xi_2 + b_{4j}\cos\beta_j L\xi_1\big) -$ $3\Big(\frac{\beta_j^3 L^3(-b_{2j}+b_{4j})(\xi_2-\xi_1)}{6}(3L(\xi_2-\xi_1) - \xi_2^2 - \xi_1\xi_2 + 2\xi_1^2) +$ $\beta_j L\xi_2\sinh\beta_j L(1-\xi_1) + \beta_j L\xi_2\sin\beta_j L(1-\xi_1) -$ $\beta_j L\xi_1\sinh\beta L(1-\xi_2) - \beta_j L\xi_1\sin\beta_j L(1-\xi_1) + \cosh\beta_j L(1-$ $\xi_2) - \cosh\beta_j L(1-\xi_1) - \cos\beta_j L(1-\xi_2) + \cos\beta_j L(1-\xi_1) +$ $b_{2j}\big(\beta_j L\xi_2\cosh\beta_j L(1-\xi_1) - \sinh\beta_j L(1-\xi_2) -$ $\beta_j L\xi_1\cosh\beta_j L(1-\xi_1) - \sinh\beta_j L(1-\xi_1)\big) -$ $b_{4j}\big(-\beta_j L\xi_2\cos\beta L(1-\xi_1) - \sin\beta_j L(1-\xi_2) +$	See Equation (3.187) See Equation (3.202)

	$\beta_j L\xi_1 \cos\beta_j L(1-\xi_1) + \sin\beta_j L(1-\xi_1)) + \beta_j^2(\xi_2^2 - \xi_1^2)\Big)\Big]$ $F_2 = \sum_{j=1}^{\infty} \frac{-EIA_j}{L^2(\xi_2-\xi_1)^2}\Big[\beta_j^3 L^3(\xi_2-\xi_1)^2(1-\xi_1)(-b_{2j}+b_{4j}) + \beta_j^2 L^2(\xi_2-\xi_1)^2 \cosh\beta_j L(1-\xi_1) + b_{2j}\beta_j^2 L^2(\xi_2-\xi_1)^2 \sinh\beta_j L(1-\xi_1) + \beta_j^2 L^2(\xi_2-\xi_1)^2 \cos\beta_j L(1-\xi_1) - b_{4j}\beta_j^2 L^2(\xi_2-\xi_1)^2 \sin\beta_j L(1-\xi_1) - 2\beta_j^2 L^2(\xi_2-\xi_1)^2 - 3\Big(\frac{\beta_j^3 L^3(-b_{2j}+b_{4j})(1-\xi_1)}{6}(3(1-\xi_1) - 1 - \xi_1 + 2\xi_1^2) + \beta_j L \sinh\beta_j L(1-\xi_1) + \beta_j L \sin\beta_j L(1-\xi_1) - \beta_j x_1 \sin\beta_j L(1-\xi_1) - \cosh\beta_j L(1-\xi_1) + \cos\beta_j L(1-\xi_1) + b_{2j}(\beta_j L \cosh\beta_j L(1-\xi_1) - \beta_j x_1 \cosh\beta_j L(1-\xi_1) - \sinh\beta_j L(1-\xi_1)) - b_{4j}(-\beta_j L \cos\beta L(1-\xi_1) + \beta_j x_1 \cos\beta_j L(1-\xi_1) + \sin\beta_j L(1-\xi_1)) + \beta_j^2 L^2(1-\xi_1^2)\Big)\Big]$	See Equation (3.199)
	$F_3 = \sum_{j=1}^{\infty} \frac{3EIA_j}{L^3(\xi_2-\xi_1)^3}\Big[\frac{\beta_j^3 L^3(-b_{2j}+b_{4j})(1-\xi_1)}{6}(3(1-\xi_1) - 1 - \xi_1 + 2\xi_1^2) + \beta_j L \sinh\beta_j L(1-\xi_1) + \beta_j L \sin\beta_j L(1-\xi_1) - \beta_j L\xi_1 \sin\beta_j L(1-\xi_1) - \cosh\beta_j L(1-\xi_1) + \cos\beta_j L(1-\xi_1) + b_{2j}(\beta_j L \cosh\beta_j L(1-\xi_1) - \beta_j L\xi_1 \cosh\beta_j L(1-\xi_1) - \sinh\beta_j L(1-\xi_1)) - b_{4j}(-\beta_j L \cos\beta L(1-\xi_1) + \beta_j L\xi_1 \cos\beta_j L(1-\xi_1) + \sin\beta_j L(1-\xi_1)) + \beta_j^2 L^2(1-\xi_1^2)\Big]$	See Equation (3.198a)

	$F_4 = 0$	
	$A_j = \frac{b\mu L^2}{M_j}\left[\frac{-(\cosh \beta_j L+\cos \beta L)}{\beta_j^2 L^2} + \frac{18(\cosh \beta_j L-\cos \beta_j L)}{\beta_j^4 L^4} - \frac{48(\sinh \beta_j L-\sin \beta_j L)}{\beta_j^5 L^5} + b_{2j}\left(\frac{-2(\sinh \beta_j L+\sin \beta_j L)}{\beta_j^2 L^2} + \frac{18(\sinh \beta_j L-\sin \beta_j L)}{\beta_j^4 L^4} - \frac{48(\cosh \beta_j L+\cos \beta_j L-2)}{\beta_j^5 L^5}\right)\right]$	See Equation (3.209)
	$M_j = \frac{\mu L}{2}\left[2 + \frac{\sinh \beta L+\sin \beta L}{2\beta L} + b_{2j}^2\left(\frac{\sinh \beta L}{2\beta L} - 1\right) + b_{4j}^2\left(1 - \frac{\sin \beta L}{2\beta L}\right) - 2\left(\frac{\cosh \beta L \sin \beta L+\sinh \beta L \cos \beta L}{\beta L}\right) + 2b_{4j}\left(\frac{1-\cos \beta L \cosh \beta L+\sin \beta L \sinh \beta L-\sin^2 \beta L}{\beta L}\right) - 2b_{2j}\left(\frac{\sinh \beta L \sin \beta L+\cosh \beta L \cos \beta L-\sinh^2 \beta L-1}{\beta L}\right) + 2b_{2j} b_{4j}\left(\frac{\cosh \beta L \sin \beta L-\sinh \beta L \cos \beta L}{\beta L}\right)\right]$	See Equation (3.211)
	$b_{2j} = \frac{\cos \beta_j L-\cosh \beta_j L}{\sinh \beta_j L-\sin \beta_j L}$ $b_{4j} = \frac{\cosh \beta_j L-\cos \beta_j L}{\sinh \beta_j L-\sin \beta_j L}$	See Equation (3.182)
	$\beta_1 L = 3.92660232$ $\beta_2 L = 7.06858275$, $\beta_3 L = 10.210176123$	See Equation (3.183)
	$\beta_4 L = 13.35176878$, $\beta_5 L = 16.49336143135$ $\beta_6 L = 19.634954084937$ $\beta_7 L = 22.77654673853$ etc.	See equation (3.178)

Note: b is a dimensionless constant and can be taken as equal to unity		

3.2.3 For a beam pinned at both ends (Pinned-pinned beam)

If the beam is pinned at the near end (x = 0) and also pinned at the far end (x = L), the displacements and moments at the ends are zero hence the boundary conditions are

$$\emptyset(0) = 0$$

$$\emptyset''(0) = 0$$

$$\emptyset(L) = 0 \qquad (3.212)$$

$$\emptyset''(L) = 0$$

By substituting equations (3.212) into (3.109) we obtain

$$C_1 + C_3 = 0 \qquad (3.213a)$$

$$C_2\beta^2 - C_3\beta^2 = 0 \qquad (3.213b)$$

$$C_1 \cosh\beta L + C_2 \sinh\beta L + C_3 \cos\beta L + C_4 \sin\beta L = 0 \qquad (3.213c)$$

$$C_1\beta^2 \cosh\beta L + C_2\beta^2 \sinh\beta L - C_3\beta^2 \cos\beta L - C_4\beta^2 \sin\beta L = 0 \qquad (3.213d)$$

Putting equations (3.213a) to (3.213b) in matrix form

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \beta & -\beta^2 & 0 \\ \cosh\beta L & \sinh\beta L & \cos\beta L & \sin\beta L \\ \beta^2\cosh\beta L & \beta^2\sinh\beta L & -\beta^2\cos\beta L & -\beta^2\sin\beta L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0 \qquad (3.214)$$

For a non-trivial solution the determinant of the coefficients of the constants C_1, C_2, C_3 and C_4 must be zero.

$$\therefore\ \beta^2(-\beta^2 \sinh\beta L \cdot \sin\beta L - \beta^2 \sinh\beta L \cdot \sin\beta L) + \beta^2(-\beta^2 \sinh\beta L \cdot \sin\beta L - \beta^2 \sinh\beta L \cdot \sin\beta L) = 0$$

$$\sinh\beta L \cdot \sin\beta L = 0 \qquad (3.215)$$

The first seven roots of equation (3.177) were obtained using the bisection method as

$$\beta_1 L = 3.1415927, \beta_2 L = 6.2831854,\ \beta_3 L = 9.424777961,$$

$$\beta_4 L = 12.5663706144,$$

$$\beta_5 L = 15.70796326795, \qquad \beta_6 L = 18.8495559215388,$$

$$\beta_7 L = 21.99114857513 \qquad (3.216)$$

By substituting equations (3.216) into (3.114) we obtain the first seven natural frequencies of a pinned-pinned beam as

$$\omega_1 = 9.869604693\sqrt{\frac{EI}{\mu L^4}},\ \omega_2 = 39.47841877\sqrt{\frac{EI}{\mu L^4}},\ \omega_3 = 88.82643961\sqrt{\frac{EI}{\mu L^4}}$$

$$\omega_4 = 157.9136703\sqrt{\frac{EI}{\mu L^4}},\ \omega_5 = 246.7401098\sqrt{\frac{EI}{\mu L^4}},\ \omega_6 = 355.3057584\sqrt{\frac{EI}{\mu L^4}}\ ,$$

$$\omega_7 = 483.6106154\sqrt{\frac{EI}{\mu L^4}}, \qquad (3.217)$$

By taking

$$C_1 = C_3 = 0 \qquad (3.218a)$$

$$C_2 = 1 \qquad (3.218b)$$

C_4 can be obtained from equations (3.214) to be

$$C_4 = \frac{-\sinh\,\beta L}{\sin\,\beta L} \qquad (3.218c)$$

Equations (3.218a – 3.218c) are substituted into equation (3.109) to obtain the equation of the j^th^ mode shape of vibration for a pinned-pinned beam under lateral vibration.

$$\emptyset_j\,(x) = \sinh\beta_j\,x - \left(\frac{\sinh\,\beta_j\,L}{\sin\,\beta_j\,L}\right)\sin\beta_j\,x$$

$$\emptyset_j\,(x) = \sinh\beta_j\,x + c_{4j}\,\sin\beta_j\,x \qquad (3.219)$$

Where

$$c_{4j} = -\frac{\sinh\,\beta_j\,L}{\sin\,\beta_j\,L} \qquad (3.220)$$

Equation (3.219) is the equation of the j^{th} mode of vibration of a pinned-pinned beam. The first mode of vibration (j = 1) can be obtained by substituting $\beta_j\,L = \beta_1 L = 3.1415927$ into equation (3.219). The second mode of vibration (j = 2) can be obtained by substituting $\beta_j\,L = \beta_2 L = 6.2831854$ into the equation. Likewise the mode shape for the j^{th} mode can be obtained by substituting the value of $\beta_j\,L$ into the equation.

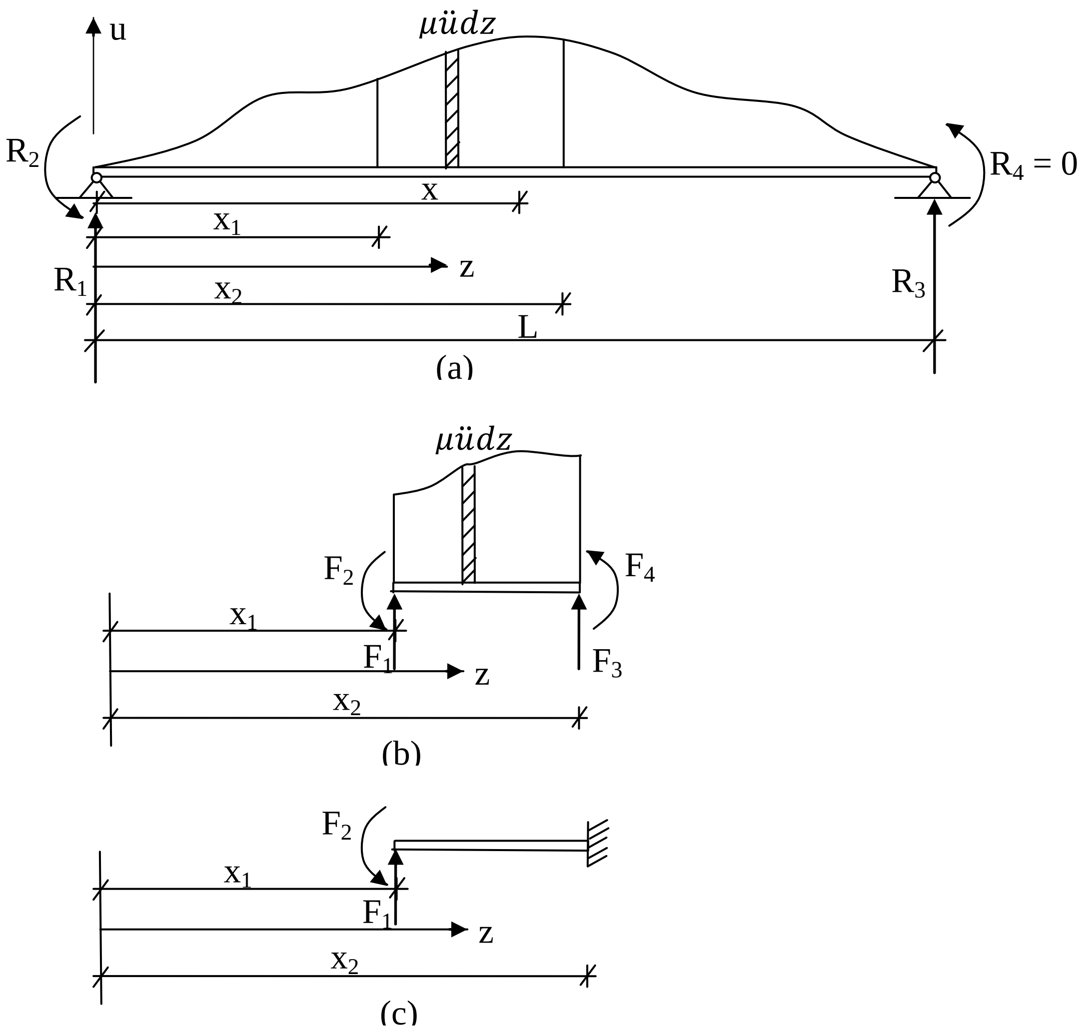


Figure 3.18

(a) A pinned-pinned beam under lateral vibration due to the inertial forces $\mu\ddot{u}$

(b) A segment of the beam under longitudinal vibration due to inertial forces $\mu\ddot{u}$

(c) The reduced/basic structure of an arbitrary element of the vibrating beam

Figure 3.18a shows a pinned-pinned beam under inertia forces. A segment of the beam showed is being restrained by the fixed end forces $F_1 - F_4$. The reduced structure or basic system is shown in Figure 3.18c.

The acceleration at any point in the vibrating beam as stated earlier is given by mode superposition as

$$\ddot{u} = \sum_{j=1}^{\infty} -\omega_j^2 \emptyset_j(x)\left(A_j \cos \omega_j t + B_j \sin \omega_j t\right)$$

The moment of an elementary force $\mu\ddot{u}dz$, at a distance z from the origin about an arbitrary point a distance x from the origin is given as (see Figure 3.18)

$$dM_x = \mu\, dz\, \ddot{u}(x - z)$$

$$M_x = \int_{x_1}^{x} \mu\ddot{u}(x - z)dz$$

$$= \mu x \int_{x_1}^{x} \ddot{u}\, dz + \mu \int_{x_1}^{x} \ddot{u}z\, dz \qquad (3.221)$$

By integrating the component parts of equation (3.221) separately

$$\mu \int_{x_1}^{x} \ddot{u}z\, dz = \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu \int_{x_1}^{x} \emptyset_j z\, dz$$

$$= \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu \int_{x_1}^{x} \left(z \sinh \beta_j z + c_{4j} z \sin \beta_j z\right) dz$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^2} \left[\beta_j x\left(\cosh \beta_j x - c_{4j} \cos \beta_j x\right) - \beta_j x_1\left(\cosh \beta_j x_1 - c_{4j} \cos \beta_j x_1\right) - \left(\sinh \beta_j x - c_{4j} \sin \beta_j x\right) + \left(\sinh \beta_j x_1 - c_{4j} \sin \beta_j x_1\right)\right] \qquad (3.222)$$

$$\mu x \int_{x_1}^{x} \ddot{u}\, dz = \sum_{j=1}^{\infty} -\omega_j^2 A_j \mu x \int_{x_1}^{x} \emptyset_j\, dz$$

$$= -\mu x \sum_{j=1}^{\infty} \omega_j^2 A_j \int_{x_1}^{x} \left(\sinh \beta_j z + c_{4j} \sin \beta_j z\right) dz$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu x}{\beta_j} \left(\cosh \beta_j x - \cosh \beta_j x_1 - c_{4j} \cos \beta_j x + c_{4j} \cos \beta_j x_1\right) \qquad (3.223)$$

By substituting equations (3.222) and (3.223) into equations (3.221) we obtain

$$M_x = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^2} \left[\beta_j x\left(-\cosh \beta_j x_1 + c_{4j} \cos \beta_j x_1\right) + \beta_j x_1\left(\cosh \beta_j x_1 - c_{4j} \cos \beta_j x_1\right) + \left(\sinh \beta_j x - c_{4j} \sin \beta_j x\right) - \left(\sinh \beta_j x_1 - c_{4j} \sin \beta_j x_1\right)\right] \qquad (3.224)$$

Equation (3.224) is the expression of the bending moment at a point x from the origin of a reduced segment of a pinned-pinned beam under free lateral vibration caused by its inertia forces.

Using the principle of virtual work, there is need to obtain the equation of bending moments produced by unit values of the removed redundant forces.

For $F_1 = 1$ and $F_2 = 0$ (see Figure 3.18c)

$$\bar{M}_1 = F_1(x - x_1)$$

$$= x - x_1 \qquad (3.225)$$

$$where \;\; x_1 \le x \le x_2$$

Deformation at coordinate 1 (direction of force F_1) from the inertia forces is

$$\delta_{10} = \int_{x_1}^{x_2} \frac{\bar{M}_1 M_x}{EI} dx = \int_{x_1}^{x_2} \frac{M_x x - M_x x_1}{EI} dx \qquad (3.226)$$

$$\int_{x_1}^{x_2} M_x x \, dx = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^2} \Big[\big(-\cosh\beta_j x_1 + c_{4j} \cos\beta_j x_1\big) \int_{x_1}^{x_2} \beta_j x^2 dx +$$

$$\beta_j x_1\big(\cosh\beta_j x_1 - c_{4j} \cos\beta_j x_1\big) \int_{x_1}^{x_2} x dx + \int_{x_1}^{x_2}\big(x \sinh\beta_j x - c_{4j} x \sin\beta_j x\big) dx -$$

$$\big(\sinh\beta_j x_1 - c_{4j} \sin\beta_j x_1\big) \int_{x_1}^{x_2} x dx\Big]$$

$$=$$

$$\sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^4} \Big[\frac{\beta_j^3 (x_2^3 - x_1^3)}{3}\big(-\cosh\beta_j x_1 + c_{4j} \cos\beta_j x_1\big) + \frac{\beta_j^3 x_1 (x_2^2 - x_1^2)}{2}\big(\cosh\beta_j x_1 -$$

$$c_{4j} \cos\beta_j x_1) + \big(\beta x_2 \cosh\beta_j x_2 - \sinh\beta_j x_2 - \beta_j x_1 \cosh\beta_j x_1 + \sinh\beta_j x_1\big) -$$

$$c_{4j}\big(-\beta x_2 \cos\beta_j x_2 + \sin\beta_j x_2 + \beta_j x_1 \cos\beta_j x_1 - \sin\beta_j x_1\big) - \frac{\beta_j^2 (x_2^2 - x_1^2)}{2}\big(\sinh\beta_j x_1 -$$

$$c_{4j} \sin\beta_j x_1)\Big] \qquad (3.227)$$

$$\int_{x_1}^{x_2} M_x x_1 dx = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu x_1}{\beta_j^2} \Big[\big(-\cosh \beta_j x_1 + c_{4j} \cos \beta_j x_1\big) \int_{x_1}^{x_2} \beta_j x dx + \beta_j x_1 \big(\cosh \beta_j x_1 - c_{4j} \cos \beta_j x_1\big) \int_{x_1}^{x_2} dx + \int_{x_1}^{x_2} \big(\sinh \beta_j x - c_{4j} \sin \beta_j x\big) dx - \big(\sinh \beta_j x_1 - c_{4j} \sin \beta_j x_1\big) \int_{x_1}^{x_2} dx \Big]$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu \beta x_1}{\beta_j^4} \Big[\frac{\beta_j^2 (x_2^2 - x_1^2)}{2} \big(-\cosh \beta_j x_1 + c_{4j} \cos \beta_j x_1\big) + \beta_j^2 x_1 (x_2 - x_1) \big(\cosh \beta_j x_1 - c_{4j} \cos \beta_j x_1\big) + \big(\cosh \beta_j x_2 - \cosh \beta_j x_1\big) - c_{4j} \big(-\cos \beta_j x_2 + \cos \beta_j x_1\big) - \beta (x_2 - x_1) \big(\sinh \beta_j x_1 - c_{4j} \sin \beta_j x_1\big) \Big] \tag{3.228}$$

By substituting equations (3.227) and (3.228) into equation (3.226)

$$EI\delta_{10} = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^4} \Bigg[\beta_j^3 \left(\frac{(x_2^3 - x_1^3)}{3} - \frac{x_1 (x_2^2 - x_1^2)}{2} \right) \big(-\cosh \beta_j x_1 + c_{4j} \cos \beta_j x_1\big) + \left(\frac{\beta_j^3 x_1 (x_2^2 - x_1^2)}{2} - \beta_j^3 x_1^2 (x_2 - x_1) \right) \big(\cosh \beta_j x_1 - c_{4j} \cos \beta_j x_1\big) - \left(\frac{\beta_j^2 (x_2^2 - x_1^2)}{2} - \beta_j^2 x_1 (x_2 - x_1) \right) \big(\sinh \beta_j x_1 - c_{4j} \sin \beta_j x_1\big) + \big(\beta x_2 \cosh \beta_j x_2 - \sinh \beta_j x_2 - \beta_j x_1 \cosh \beta_j x_2 + \sinh \beta_j x_1\big) - c_{4j} \big(-\beta x_2 \cos \beta_j x_2 + \sin \beta_j x_2 + \beta_j x_1 \cos \beta_j x_2 - \sin \beta_j x_1\big) \Bigg] \tag{3.229}$$

Equation (3.229) above is the equation of deformation in coordinate 1 (direction of F_1) of a reduced segment of the vibrating pinned-pinned beam due to its inertia forces.

For $F_2 = 1$ and $F_1 = 0$ (see Figure 3.14c)

$$\bar{M}_1 = -F_2$$

$$= -1 \tag{3.230}$$

$where\ \ x_1 \leq x \leq x_2$

From the principle of virtual work deformation at coordinate 2 (direction of force F_2) from the inertia forces is

$$\delta_{20} = \int_{x_1}^{x_2} \frac{\bar{M}_2 M_x}{EI} dx = \int_{x_1}^{x_2} \frac{-M_x}{EI} dx \tag{3.231}$$

$$EI\delta_{20} = -\int_{x_1}^{x_2} M_x \, dx$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^2} \Big[\big(-\cosh\beta_j x_1 + c_{4j} \cos\beta_j x_1\big) \int_{x_1}^{x_2} \beta_j x \, dx + \beta_j x_1\big(\cosh\beta_j x_1 - c_{4j} \cos\beta_j x_1\big) \int_{x_1}^{x_2} dx + \int_{x_1}^{x_2}\big(\sinh\beta_j x - c_{4j} \sin\beta_j x\big) \, dx - \big(\sinh\beta_j x_1 - c_{4j} \sin\beta_j x_1\big) \int_{x_1}^{x_2} dx\Big]$$

$$= \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^3} \Big[\frac{\beta_j^2 (x_2^2 - x_1^2)}{2}\big(-\cosh\beta_j x_1 + c_{4j} \cos\beta_j x_1\big) + \beta_j^2 x_1 (x_2 - x_1)\big(\cosh\beta_j x_1 - c_{4j} \cos\beta_j x_1\big) + \big(\cosh\beta_j x_2 - \cosh\beta_j x_1\big) - c_{4j}\big(-\cos\beta_j x_2 + \cos\beta_j x_1\big) - \beta_j (x_2 - x_1)\big(\sinh\beta_j x - c_{4j} \sin\beta_j x\big)\Big] \tag{3.232}$$

Equation (3.126) is the equation of deformation in coordinate 2 (direction of F_2) of a reduced segment of the vibrating pinned-pinned beam due to its inertia forces.

From the compatibility equations of the reduced segment, the equation of the redundant force F_1 can be written as (see equations 3.127) – (3.131)

$$F_1 = -\frac{12EI\delta_{10}}{l^3} - \frac{6EI\delta_{20}}{l^2} = -6\left(\frac{2EI\delta_{10}}{(x_2 - x_1)^3} + \frac{EI\delta_{20}}{(x_2 - x_1)^2}\right) \tag{3.233}$$

By substituting equations (3.229) and (3.232) into equation (3.233) and simplifying

$F_1 = -6\sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4(x_2-x_1)^3}\Big[\beta_j^3\left(\frac{(x_2+x_1)(x_2^2-x_1^2)}{2}-\frac{2(x_2^3-x_1^3)}{3}\right)(-\cosh\beta_j x_1 +$

$c_{4j}\cos\beta_j x_1) + \big(\beta(x_1-x_2)\cosh\beta_j x_2 + 2\sinh\beta_j x_2 - \beta_j(x_2-x_1)\cosh\beta_j x_1 -$

$2\sinh\beta_j x_1) - c_{4j}\big(-\beta(x_1-x_2)\cos\beta_j x_2 - 2\sin\beta_j x_2 + \beta_j(x_2-x_1)\cos\beta_j x_1 +$

$2\sin\beta_j x_1)\Big]$ (3.234)

$$F_1 = -6\sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4(x_2-x_1)^3}W_1 \qquad (3.234a)$$

Where

$W_1 = \beta_j^3\left(\frac{(x_2+x_1)(x_2^2-x_1^2)}{2}-\frac{2(x_2^3-x_1^3)}{3}\right)(-\cosh\beta_j x_1 + c_{4j}\cos\beta_j x_1) + \big(\beta(x_1 -$

$x_2)\cosh\beta_j x_2 + 2\sinh\beta_j x_2 - \beta_j(x_2-x_1)\cosh\beta_j x_1 - 2\sinh\beta_j x_1) - c_{4j}\big(-\beta(x_1 -$

$x_2)\cos\beta_j x_2 - 2\sin\beta_j x_2 + \beta_j(x_2-x_1)\cos\beta_j x_1 + 2\sin\beta_j x_1)$ (3.234b)

Likewise F_2 is obtained as

$$F_2 = -\frac{6EI\delta_{10}}{l^2} - \frac{4EI\delta_{20}}{l} = -2\left(\frac{3EI\delta_{10}}{(x_2-x_1)^2} + \frac{2EI\delta_{20}}{(x_2-x_1)}\right) \qquad (3.235)$$

By substituting equations (3.229) and (3.232) into equation (3.235) and simplifying

$F_2 =$

$-2\sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4(x_2-x_1)^2}\Big[\beta_j^3\left(\frac{(2x_2+x_1)(x_2^2-x_1^2)}{2}-\frac{x_1(x_2-x_1)^2}{2}-(x_2^3-x_1^3)\right)(-\cosh\beta_j x_1 +$

$c_{4j}\cos\beta_j x_1) - \left(\frac{\beta_j^2(x_2-x_1)^2}{2}\right)(\sinh\beta_j x_1 - c_{4j}\sin\beta_j x_1) + \big(\beta_j(x_1-x_2)\cosh\beta_j x_2 -$

$2\beta_j(x_2-x_1)\cosh\beta_j x_1 + 3\sinh\beta_j x_2 - 3\sinh\beta_j x_1) - c_{4j}\big(-\beta_j(x_1-x_2)\cos\beta_j x_2 +$

$2\beta(x_2-x_1)\cos\beta_j x_1 - 3\sin\beta_j x_2 + 3\sin\beta_j x_1)\Big]$ (3.236)

$$F_2 = -2\sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^2} W_2 \tag{3.236a}$$

Where

$$W_2 = \beta_j^3\left(\frac{(2x_2+x_1)(x_2^2-x_1^2)}{2} - \frac{x_1(x_2-x_1)^2}{2} - (x_2^3 - x_1^3)\right)\left(-\cosh\beta_j x_1 + c_{4j}\cos\beta_j x_1\right) - \left(\frac{\beta_j^2 (x_2-x_1)^2}{2}\right)\left(\sinh\beta_j x_1 - c_{4j}\sin\beta_j x_1\right) + \left(\beta_j (x_1 - x_2)\cosh\beta_j x_2 - 2\beta_j (x_2 - x_1)\cosh\beta_j x_1 + 3\sinh\beta_j x_2 - 3\sinh\beta_j x_1\right) - c_{4j}\left(-\beta_j (x_1 - x_2)\cos\beta_j x_2 + 2\beta(x_2 - x_1)\cos\beta_j x_1 - 3\sin\beta_j x_2 + 3\sin\beta_j x_1\right) \tag{3.236b}$$

Equation (3.234) and (3.236) are the equations of the fixed end forces F_1 and F_2 on a segment of a pinned-pinned beam under vibration. There is need to determine the other forces F_3 and F_4 needed to keep this segment in equilibrium (see Figure 3.18b).

For vertical equilibrium (as applied in fixed-fixed beam)

$$\therefore F_3 = -F_1 - \int_{x_1}^{x_2}\mu\ddot{u}dx \tag{3.237}$$

From equation (3.223)

$$\int_{x_1}^{x_2}\mu\ddot{u}dx = \sum_{j=1}^{\infty}\frac{-\omega_j^2 A_j \mu}{\beta_j}\left(\cosh\beta_j x_2 - \cosh\beta_j x_1 - c_{4j}\cos\beta_j x_2 + c_{4j}\cos\beta_j x_1\right) \quad . \tag{3.238}$$

By substituting equations (3.223) and (3.234) into equation (3.238)

$$F_3 = \sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^3}\left[6W_1 + \beta_j^3 (x_2 - x_1)^3\left(\cosh\beta_j x_2 - \cosh\beta_j x_1 - c_{4j}\left(\cos\beta_j x_2 - \cos\beta_j x_1\right)\right)\right] \tag{3.239}$$

When a system is in equilibrium, every part of it is in the same state, hence for the beam segment (see Figure 3.18b), the sum of the moment of forces about any point on the segment must be zero

$$\sum M_{3,4} = 0; \quad F_1(x_2 - x_1) - F_2 + \int_{x_1}^{x_2} \mu \ddot{u} dx \, (x_2 - x) - F_4 = 0$$

$$F_4 = F_1(x_2 - x_1) - F_2 + \mu x_2 \int_{x_1}^{x_2} \ddot{u} dx - \mu \int_{x_1}^{x_2} \ddot{u} x \, dx \qquad (3.240)$$

From equation (3.223)

$$\mu x_2 \int_{x_1}^{x_2} \ddot{u} dx = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu x_2}{\beta_j} \left(\cosh \beta_j x_2 - \cosh \beta_j x_1 - c_{4j} \cos \beta_j x_2 + c_{4j} \cos \beta_j x_1\right)$$

(3.241)

From equation (3.222)

$$\mu \int_{x_1}^{x_2} \ddot{u} x \, dx = \sum_{j=1}^{\infty} \frac{-\omega_j^2 A_j \mu}{\beta_j^2} \left[\beta_j x_2 \left(\cosh \beta_j x_2 - c_{4j} \cos \beta_j x_2\right) - \beta_j x_1 \left(\cosh \beta_j x_1 - c_{4j} \cos \beta_j x_1\right) - \left(\sinh \beta_j x_2 - c_{4j} \sin \beta_j x_2\right) + \left(\sinh \beta_j x_1 - c_{4j} \sin \beta_j x_1\right)\right] \qquad (3.242)$$

By substituting equations (3.234), (2.236), (3.241) and (3.242) into equation (3.240) and simplifying

$$F_4 = \sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2 - x_1)^2} \left[-6W_1 + 2W_2 - \beta_j^3 (x_2 - x_1)^3 \left(-\cosh \beta_j x_1 + c_{4j} \cos \beta_j x_1\right) + \left(\sinh \beta_j x_2 - c_{4j} \sin \beta_j x_2\right) - \left(\sinh \beta_j x_1 - c_{4j} \sin \beta_j x_1\right)\right] \qquad (3.243)$$

In order to evaluate the equations for the fixed end forces F_1, F_2, F_3 and F_4, there is need to derive an expression for A_j for a pinned-pinned beam.

a) **Derivation of the expression for A_j for a pinned-pinned beam**

Consider a uniform pinned-pinned beam under the action of its self weight.

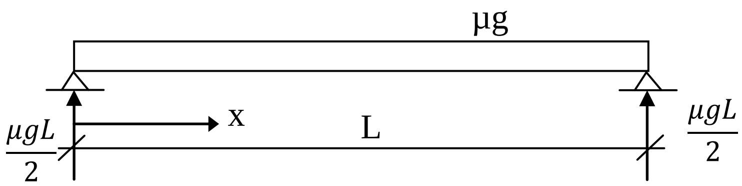


Figure 3.19: A Uniform pinned-pinned beam under the action of its self weight

From Figure 3.19 the equation of the bending moment at any distance x from the left support is

$$M_x = \frac{\mu g}{2}(Lx - x^2) \tag{3.244}$$

where µ is the mass per unit length of the beam and g is the acceleration due to gravity.

From the equation of elastic curve (beam flexure equation)

$$EIy' = \int M_x \, dx$$

$$= \int \frac{\mu g}{2}(Lx - x^2)\, dx$$

$$= \frac{\mu g}{2}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + c_1 \tag{3.245}$$

$$EIy = \frac{\mu g}{2}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + c_1 x + c_2 \tag{3.246}$$

Consider the boundary conditions

At $x = 0,\quad y = 0 \quad \therefore c_2 = 0$

At $x = 0,\quad y' = 0 \quad \therefore c_1 = -\frac{\mu g L^3}{24}$

Hence the equation for the static deformation or deflection of the uniform beam under its self weight is

$$EIy = \frac{\mu g}{2}\left(\frac{Lx^3}{6} - \frac{x^4}{12} - \frac{L^3 x}{12}\right) \tag{3.247}$$

Let the initial deflection of the beam (at time t = 0) be

$$u(x,0) = \frac{\mu g}{24EI}\left(\frac{Lx^3}{6} - \frac{x^4}{12} - \frac{L^3 x}{12}\right)$$

$$= cL\left(\frac{2x^3}{L^3} - \frac{x^4}{L^4} - \frac{x}{L}\right) \tag{3.248}$$

Where c is a dimensionless constant equal to $\frac{\mu g L^3}{24EI}$

From equation (2.61a)

$$A_j = \frac{\mu}{M_j}\int_0^L u(x,0)\emptyset_j \, dx$$

$$= \frac{\mu}{M_j}\int_0^L cL\left(\frac{2x^3}{L^3} - \frac{x^4}{L^4} - \frac{x}{L}\right)\left(\sinh\beta_j x + c_{4j}\sin\beta_j x\right) dx$$

$$= \frac{c\mu L}{M_j}\int_0^L \Big(\frac{2x^3}{L^3}\sinh\beta_j x + \frac{2x^3}{L^3} c_{4j}\sin\beta_j x - \frac{x^4}{L^4}\sinh\beta_j x - \frac{x^4}{L^4} c_{4j}\sin\beta_j x -$$

$$- \frac{x}{L}\sinh\beta_j x - \frac{x}{L} c_{4j}\sin\beta_j x\Big) \tag{3.249}$$

By taking advantage of equations (3.152) – (3.163) we can perform the integration of equation (3.249) to obtain

$$A_j = \frac{c\mu L^2}{M_j}\Big[\frac{-\sinh\beta_j L}{\beta_j^2 L^2} + \frac{12\sinh\beta_j L}{\beta_j^4 L^4} - \frac{24(\cosh\beta_j L - 1)}{\beta_j^5 L^5} + c_{2j}\Big(\frac{-\sin\beta_j L}{\beta_j^2 L^2} - \frac{12\sin\beta_j L}{\beta_j^4 L^4} -$$

$$\frac{24(\cos\beta_j L - 1)}{\beta_j^5 L^5}\Big)\Big] \tag{3.250}$$

Equation (3.250) is the equation for the arbitrary constant A_j for a pinned-pinned beam.

In order to evaluate equation (3.250) there is need to derive an expression for the generalized mass M_j.

b) **Derivation of the expression for the generalized mass M_j for a pinned-pinned beam**

By substituting the general modal equation (3.219) into equation (3.165)

$$M_j = \mu \int_0^L \left(\sinh \beta_j x + c_{4j} \sin \beta_j x\right)^2 dx$$

$$= \mu \int_0^L \left(\sinh^2 \beta_j x + 2c_{4j} \sin \beta_j x \sinh \beta_j x + c_{4j}^2 \sin^2 \beta_j x\right) dx \qquad (3.251)$$

By substituting equations (3.168 – 3.171) into equation (3.251) we obtain

$$M_j = \frac{\mu L}{2}\left[\left(\frac{\sinh \beta L}{2\beta L} - 1\right) + c_{4j}^2 \left(1 - \frac{\sin \beta L}{2\beta L}\right) + 2c_{4j}\left(\frac{\cosh \beta L \sin \beta L - \sinh \beta L \cos \beta L}{\beta L}\right)\right] \qquad (3.252)$$

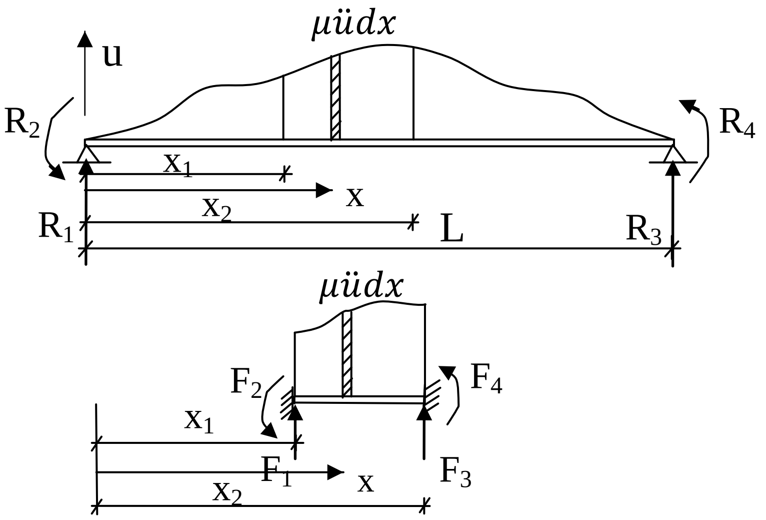
Equation (3.252) is the equation of the generalized mass of a pinned-pinned beam for the j^{th} mode of vibration.

Having derived the equations of the fixed-end forces for a segment of a pinned-pinned beam under free vibration, a summary of the equations are presented in Table 3.4. In the Table the distances x_1 and x_2 were normalized using equations (3.35) and (3.36) and square of the j^{th} natural frequency ω_j^2 was eliminated from the equations by substituting equation (3.173)

Table 3.4: Summary of Fixed-end forces on a Segment of a Pinned-pinned Beam under Free Lateral vibration

<table>
<tr><th>S/N</th><th>Description</th><th>Remarks</th></tr>
<tr><td>1</td><td>

Pinned-pinned Beam



For $0 \le \xi_1 < 1$ or $0 \le x_1 < L$

$0 < \xi_2 \le 1$ or $0 < x_2 \le L$

$\xi_2 > \xi_1$

$\xi_1 = {}^{x_1}/_{L}$

$\xi_2 = {}^{x_2}/_{L}$

$F_1 = -6\sum_{j=1}^{\infty} \frac{EIA_j}{L^3(\xi_2-\xi_1)^3} W_1$

Where

$W_1 = \beta_j^3 L^3 \left(\frac{(\xi_2+\xi_1)(\xi_2^2-\xi_1^2)}{2} - \frac{2(\xi_2^3-\xi_1^3)}{3}\right)\left(-\cosh\beta_j L\xi_1 + c_{4j}\cos\beta_j L\xi_1\right) + \left(\beta L(\xi_1-\xi_2)\cosh\beta_j L\xi_2 + 2\sinh\beta_j L\xi_2 - \beta_j L(\xi_2-\xi_1)\cosh\beta_j L\xi_1 - 2\sinh\beta_j L\xi_1\right) - c_{4j}\left(-\beta_j L(\xi_1 - \xi_2)\cos\beta_j L\xi_2 - 2\sin\beta_j L\xi_2 + \beta_j L(\xi_2-\xi_1)\cos\beta_j L\xi_1 + 2\sin\beta_j L\xi_1\right)$

</td><td>See Equation (3.234)</td></tr>
</table>

	$F_2 = -2\sum_{j=1}^{\infty} \frac{EIA_j}{L^2(\xi_2-\xi_1)^2} W_2$ Where $W_2 =$ $\beta_j^3 L^3 \left(\frac{(2\xi_2+\xi_1)(\xi_2^2-\xi_1^2)}{2} - \frac{\xi_1(\xi_2-\xi_1)^2}{2} -\right.$ $\left.(\xi_2^3-\xi_1^3)\right)\left(-\cosh\beta_j L\xi_1 + c_{4j}\cos\beta_j L\xi_1\right) -$ $\left(\frac{\beta_j^2 L^2(\xi_2-\xi_1)^2}{2}\right)\left(\sinh\beta_j L\xi_1 - c_{4j}\sin\beta_j L\xi_1\right) + \left(\beta_j L(\xi_1 -\right.$ $\xi_2)\cosh\beta_j L\xi_2 - 2\beta_j L(\xi_2-\xi_1)\cosh\beta_j L\xi_1 + 3\sinh\beta_j L\xi_2 -$ $3\sinh\beta_j L\xi_1) - c_{4j}\left(-\beta_j L(\xi_1-\xi_2)\cos\beta_j L\xi_2 + 2\beta_j L(\xi_2 -\right.$ $\xi_1)\cos\beta_j L\xi_1 - 3\sin\beta_j L\xi_2 + 3\sin\beta_j L\xi_1)$	See Equation (3.236)
	$F_3 = \sum_{j=1}^{\infty} \frac{EIA_j}{L^3(\xi_2-\xi_1)^3}\left[6W_1 + \beta_j^3 L^3(\xi_2-\xi_1)^3\left(\cosh\beta_j L\xi_2 -\right.\right.$ $\left.\left.\cosh\beta_j L\xi_1 - c_{4j}\left(\cos\beta_j L\xi_2 - \cos\beta_j L\xi_1\right)\right)\right]$	See Equation (3.239)
	$F_4 =$ $\sum_{j=1}^{\infty} \frac{EIA_j}{L^2(\xi_2-\xi_1)^2}\left[-6W_1 + 2W_2 -\right.$ $\beta_j^3 L^3(\xi_2-\xi_1)^3\left(-\cosh\beta_j L\xi_1 + c_{4j}\cos\beta_j L\xi_1\right) +$ $\left.\left(\sinh\beta_j L\xi_2 - c_{4j}\sin\beta_j L\xi_2\right) - \left(\sinh\beta_j L\xi_1 - c_{4j}\sin\beta_j L\xi_1\right)\right]$	See Equation (3.243)
	$A_j =$ $\frac{c\mu L^2}{M_j}\left[\frac{-\sinh\beta_j L}{\beta_j^2 L^2} + \frac{12\sinh\beta_j L}{\beta_j^4 L^4} - \frac{24(\cosh\beta_j L-1)}{\beta_j^5 L^5} + c_{4j}\left(\frac{-\sin\beta_j L}{\beta_j^2 L^2} -\right.\right.$ $\left.\left.\frac{12\sin\beta_j L}{\beta_j^4 L^4} - \frac{24(\cos\beta_j L-1)}{\beta_j^5 L^5}\right)\right]$	See Equation (3.250)

	$M_j =$ $\frac{\mu L}{2}\left[\left(\frac{\sinh\ \beta L}{2\beta L} - 1\right) + c_{4j}^2\left(1 - \frac{\sin\ \beta L}{2\beta L}\right) + 2c_{4j}\left(\frac{\cosh\ \beta L\ \sin\ \beta L - \sinh\ \beta L\ \cos\ \beta L}{\beta L}\right)\right]$	See Equation (3.252)
	$c_{4j} = \frac{-\sinh\ \beta_j L}{\sin\ \beta_j L}$	See Equation (3.218c)
	$\beta_1 L = 3.1415927\quad \beta_2 L = 6.2831854$, $\beta_3 L = 9.424777961$ $\beta_4 L = 12.5663706144$, $\beta_5 L = 15.70796326795$ $\beta_6 L = 18.8495559215388\quad \beta_7 L = 21.99114857513$ etc.	See Equation (3.216)
Note: c is a dimensionless constant and can be taken as equal to unity		

3.2.4 For a beam clamped at one end and free at the other (Fixed-free beam)

If the beam is fixed at the near end (x = 0) and pinned at the far end (x = L), the boundary conditions are

$\emptyset(0) = 0$

$\emptyset^{'}(0) = 0$

$\emptyset''(L) = 0$ (3.253)

$\emptyset^{'''}(L) = 0$

By substituting equations (3.174) into (3.109) we obtain

$$C_1 + C_3 = 0 \tag{3.254a}$$

$$C_2\beta + C_4\beta = 0 \tag{3.254b}$$

$$C_1\beta^2 \cosh\beta L + C_2\beta^2 \sinh\beta L - C_3\beta^2 \cos\beta L - C_4\beta^2 \sin\beta L = 0 \tag{3.254c}$$

$$C_1\beta^3 \sinh\beta L + C_2\beta^3 \cosh\beta L + C_3\beta^3 \sin\beta L - C_4\beta^3 \cos\beta L = 0 \tag{3.254d}$$

Putting equations (3.254a) to (3.254b) in matrix form

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \beta & 0 & \beta \\ \beta^2 \cosh\beta L & \beta^2 \sinh\beta L & -\beta^2 \cos\beta L & -\beta^2 \sin\beta L \\ \beta^3 \sinh\beta L & \beta^3 \cosh\beta L & \beta^3 \sin\beta L & -\beta^3 \cos\beta L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0 \tag{3.255}$$

For a non-trivial solution the determinant of the coefficients of the constants C_1, C_2, C_3 and C_4 must be zero .

$$\therefore \beta(\beta^5 \cos^2\beta L + \beta^5 \sin^2\beta L) + \beta(\beta^5 \sinh\beta L \cdot \sin\beta L + \beta^5 \cosh\beta L \cdot \cos\beta L) -$$
$$\beta(-\beta^5 \cosh\beta L \cdot \cos\beta L + \beta^5 \sinh\beta L \cdot \sin\beta L) + \beta(\beta^5 \cosh\beta L \cdot \cosh\beta L -$$
$$\beta^5 \sinh\beta L \cdot \sinh\beta L) = 0$$

$$1 + \cosh\beta L \cos\beta L = 0 \tag{3.256}$$

The first seven roots of equation (3.256) were obtained using the bisection method as

$$\beta_1 L = 1.87510407, \beta_2 L = 4.6940912,\ \beta_3 L = 7.854757439,$$

$$\beta_4 L = 10.9955407349,$$

$$\beta_5 L = 14.137168392, \qquad \beta_6 L = 17.27875953209,$$

$$\beta_7 L = 20.420352251042 \tag{3.257}$$

By substituting equations (3.257) into (3.114)

$$\omega_1 = 3.516015273\sqrt{\frac{EI}{\mu L^4}}\,,\ \omega_2 = 22.03449219\sqrt{\frac{EI}{\mu L^4}},\ \omega_3 = 61.69721443\sqrt{\frac{EI}{\mu L^4}}$$

$$\omega_4 = 120.9019159\sqrt{\frac{EI}{\mu L^4}},\ \omega_5 = 199.8595301\sqrt{\frac{EI}{\mu L^4}}\,,\ \omega_6 = 298.5555309\sqrt{\frac{EI}{\mu L^4}}\ ,$$

$$\omega_7 = 416.990786\sqrt{\frac{EI}{\mu L^4}}, \tag{3.258}$$

By taking C_1 to be equal to one, the other constants C_2, C_3 and C_4 can be obtained from equations (3.255) as follows

$$C_3 = -1 \tag{3.259a}$$

$$C_2 = \frac{-(\cosh\ \beta L + \cos\ \beta L)}{\sinh\ \beta L + \sin\ \beta L} \tag{3.259b}$$

$$C_3 = \frac{\cosh\ \beta L + \cos\ \beta L}{\sinh\ \beta L + \sin\ \beta L} \tag{3.259c}$$

By substituting equation (3.259a) – (3.259c) into equation (3.86) we obtain the equation of the j^{th} mode shape of vibration

$$\emptyset_j(x) = \cosh\beta_j x - \left(\frac{\cosh\ \beta_j L + \cos\ \beta_j L}{\sinh\ \beta_j L + \sin\ \beta_j L}\right)\sinh\beta_j x - \cos\beta_j x + \left(\frac{\cosh\ \beta_j L + \cos\ \beta_j L}{\sinh\ \beta_j L + \sin\ \beta_j L}\right)\sin\beta_j x$$

$$\emptyset_j(x) = \cosh\beta_j x + d_{2j}\ \sinh\beta_j x - \cos\beta_j x + d_{4j}\ \sin\beta_j x \tag{3.260}$$

Where

$$d_{2j} = \frac{-(\cosh\ \beta L + \cos\ \beta L)}{\sinh\ \beta L + \sin\ \beta L} \tag{3.261}$$

$$d_{4j} = \frac{\cosh\ \beta L + \cos\ \beta L}{\sinh\ \beta L + \sin\ \beta L} \tag{3.262}$$

Equation (3.260) is the equation of the jth mode of vibration of a fixed-free beam. The first mode of vibration (j = 1) can be obtained by substituting $\beta_j L = \beta_1 L = 1.87510407$ into equation (3.260). The second mode of vibration (j = 2) can be obtained by substituting $\beta_j L = \beta_2 L = 4.6940912$ into the equation. Likewise the mode shape for the jth mode can be obtained by substituting the value of $\beta_j L$ into the equation.

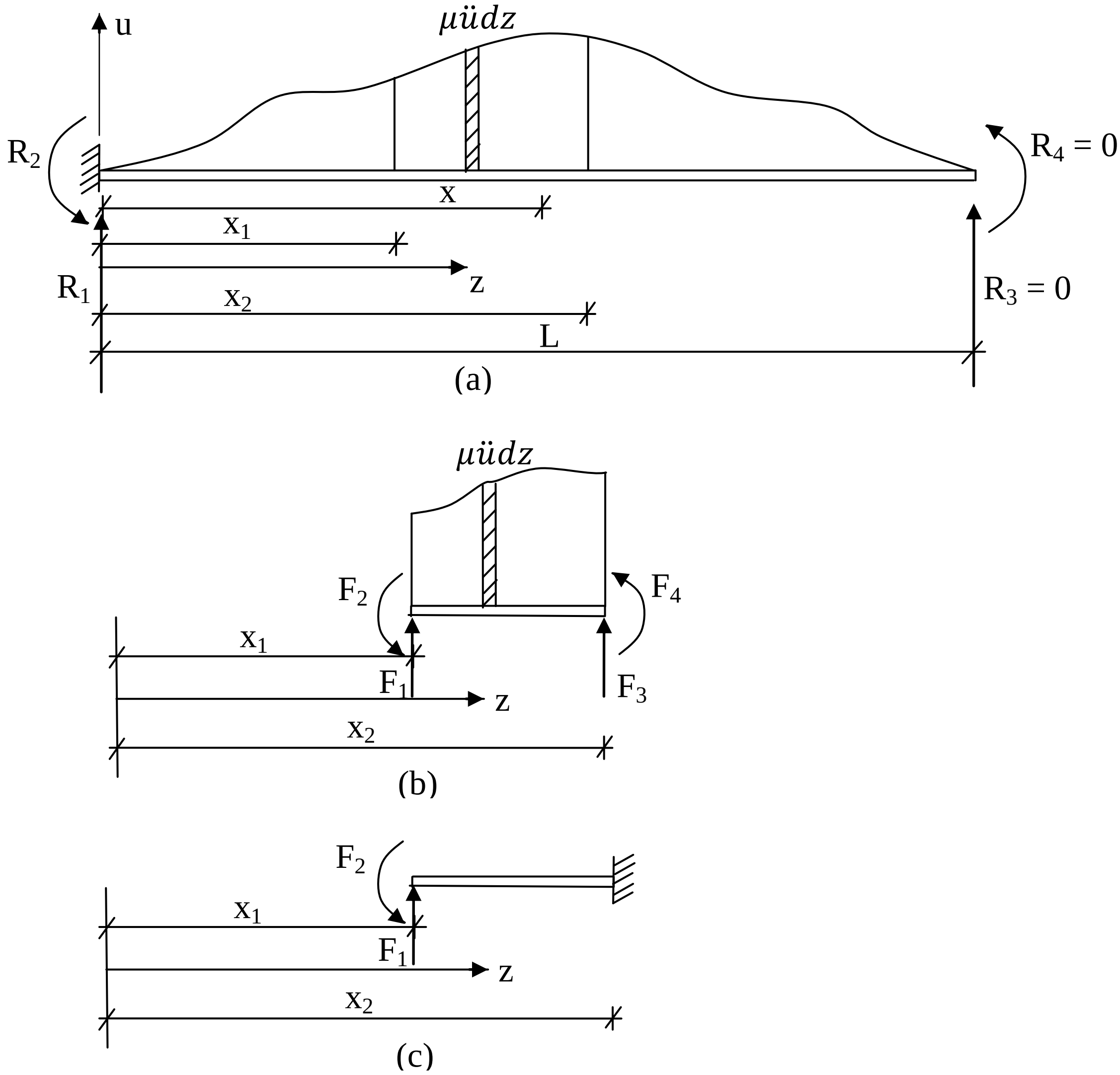


Figure 3.20

(a) A fixed-free beam under lateral vibration due to the inertial forces $\mu\ddot{u}$

(b) A segment of the beam under longitudinal vibration due to inertial forces $\mu\ddot{u}$

(c) The reduced/basic structure of an arbitrary element of the vibrating beam

Figure 3.20a shows a fixed-free beam under inertia forces. A segment of the beam showed is being restrained by the fixed end forces $F_1 - F_4$. The reduced structure of basic system is shown in Figure 3.20c.

The acceleration at any point in the vibrating beam as stated earlier (equation 3.119) is given by mode superposition as

$$\ddot{u} = \sum_{j=1}^{\infty} -\omega_j^2 \emptyset_j(x)\left(A_j \cos\omega_j t + B_j \sin\omega_j t\right)$$

For the segment of a fixed-free beam above (see figure 3.20b) the beam segment is largely the same as that of a fixed-fixed beam except that in the equation for mode shape Ø, a_{2j} and a_{4j} has been replaced it d_{2j} and d_{4j} and values of $\beta_j L$ defined earlier by equation (3.113) is now defined by equation (3.257). Compare equation (3.260) with (3.117).

Hence the equations for the fixed end forces ($F_1 - F_4$) for a segment of the fixed-fixed beam can be reproduced for this case of a fixed-free beam with a_{2j} and a_{4j} replaced by d_{2j} and d_{4j}.

From equation (3.132a)

$$F_1 = -6\sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2 - x_1)^3} W_1 \qquad (3.263a)$$

Where

$$W_1 = \beta_j^3\left(\frac{(x_2+x_1)(x_2^2-x_1^2)}{2} - \frac{2(x_2^3-x_1^3)}{3}\right)\left(-\sinh\beta_j x_1 - d_{2j}\cosh\beta_j x_1 + \sin\beta_j x_1 + d_{4j}\cos\beta_j x_1\right) + d_{2j}\left(\beta(x_1-x_2)\cosh\beta_j x_2 + 2\sinh\beta_j x_2 - \beta_j(x_2-x_1)\cosh\beta_j x_1 - 2\sinh\beta_j x_1\right) - d_{4j}\left(-\beta(x_1-x_2)\cos\beta_j x_2 - 2\sin\beta_j x_2 + \beta_j(x_2-x_1)\cos\beta_j x_1 + 2\sin\beta_j x_1\right) + \beta(x_1-x_2)\sinh\beta_j x_2 - \beta(x_2-x_1)\sinh\beta_j x_1 + \beta(x_1-x_2)\sin\beta_j x_2 - \beta(x_2-x_1)\sin\beta_j x_1 + 2\cosh\beta_j x_2 - 2\cosh\beta_j x_1 - 2\cos\beta_j x_2 + 2\cos\beta_j x_1$$

(3.263b)

From equation (3.134a)

$$F_2 = -2\sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^2} W_2 \tag{3.264a}$$

Where

$$W_2 = \beta_j^3\left(\frac{(2x_2+x_1)(x_2^2-x_1^2)}{2} - \frac{x_1(x_2-x_1)^2}{2} - (x_2^3 - x_1^3)\right)\left(-\sinh\beta_j x_1 - d_{2j}\cosh\beta_j x_1 + \sin\beta_j x_1 + d_{4j}\cos\beta_j x_1\right) - \left(\frac{\beta_j^2 (x_2-x_1)^2}{2}\right)\left(\cosh\beta_j x_1 + d_{2j}\sinh\beta_j x_1 + \cos\beta_j x_1 - d_{4j}\sin\beta_j x_1\right) + \beta_j (x_1 - x_2)\sinh\beta_j x_2 - 2\beta_j (x_2 - x_2)\sinh\beta_j x_1 + \beta_j (x_1 - x_2)\sin\beta_j x_2 - 2\beta_j (x_2 - x_1)\sin\beta_j x_1 - 3\cos\beta_j x_2 + 3\cos\beta_j x_1 + 3\cosh\beta_j x_2 - 3\cosh\beta_j x_1 + d_{2j}\left(\beta_j (x_1 - x_2)\cosh\beta_j x_2 - 2\beta_j (x_2 - x_1)\cosh\beta_j x_1 + 3\sinh\beta_j x_2 - 3\sinh\beta_j x_1\right) - d_{4j}\left(-\beta_j (x_1 - x_2)\cos\beta_j x_2 + 2\beta_j (x_2 - x_1)\cos\beta_j x_1 - 3\sin\beta_j x_2 + 3\sin\beta_j x_1\right) \tag{3.264b}$$

From equation (3.138)

$$F_3 = \sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^3}\left[6W_1 + \beta_j^3 (x_2 - x_1)^3\left(\sinh\beta_j x_2 - \sinh\beta_j x_1 - \sin\beta_j x_2 + \sin\beta_j x_1 + d_{2j}\left(\cosh\beta_j x_2 - \cosh\beta_j x_1\right) - d_{4j}\left(\cos\beta_j x_2 - \cos\beta_j x_1\right)\right)\right] \tag{3.265}$$

From equation (3.144)

$$F_4 = \sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^2}\left[-6W_1 + 2W_2 - \beta_j^3 (x_2 - x_1)^3\left(-\sinh\beta_j x_1 - d_{2j}\cosh\beta_j x_1 + \sin\beta_j x_1 + d_{4j}\cos\beta_j x_1\right) + \left(\cosh\beta_j x_2 + d_{2j}\sinh\beta_j x_2 + \cos\beta_j x_2 - d_{4j}\sin\beta_j x_2\right) - \left(\cosh\beta_j x_1 + d_{2j}\sinh\beta_j x_1 + \cos\beta_j x_1 - d_{4j}\sin\beta_j x_1\right)\right] \tag{3.266}$$

Equations (3.263) – (3.266) are the equations of the fixed-end forces on a segment of a fixed-free beam under free lateral vibration.

In order to evaluate the equations for the fixed end forces F_1, F_2, F_3 and F_4, there is need to derive an expression for A_j for a fixed-free beam.

a) **Derivation of the expression for A_j for a fixed-free beam**

Consider a uniform fixed-free beam under the action of its self weight.

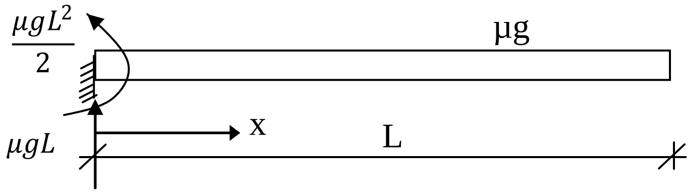


Figure 3.21: A Uniform fixed-free beam under the action of its self weight

The equation of the bending moment at any distance x from the left support is

$$M_x = \frac{\mu g}{2}(2Lx - x^2 - L^2) \tag{3.267}$$

Where μ is the mass per unit length of the beam and g is the acceleration due to gravity.

By substituting equation (3.267) into the equation of elastic curve (equation 3.146) and solving for the deflection y

$$EIy' = \int_0^L M_x \, dx$$

$$= \int_0^L \frac{\mu g}{2}(2Lx - x^2 - L^2)\, dx$$

$$= \frac{\mu g}{2}\left(Lx^2 - \frac{x^3}{3} - L^2 x\right) + c_1 \tag{3.268}$$

$$EIy = \frac{\mu g}{2}\left(\frac{Lx^3}{3} - \frac{x^4}{12} - \frac{L^2x^2}{2}\right) + c_1x + c_2 \tag{3.269}$$

Consider the boundary conditions

At $x = 0, \quad y = 0 \quad \therefore c_2 = 0$

At $x = 0, \quad y' = 0 \quad \therefore c_1 = 0$

Hence the equation for the static deformation or deflection of the uniform beam under its self weight is

$$EIy = \frac{\mu g}{2}\left(\frac{Lx^3}{3} - \frac{x^4}{12} - \frac{L^2x^2}{2}\right) \tag{3.270}$$

Let the initial deflection of the beam (at time t = 0) be

$$u(x,0) = \frac{\mu g}{2}\left(\frac{Lx^3}{3} - \frac{x^4}{12} - \frac{L^2x^2}{2}\right)$$

$$= dL\left(\frac{4x^3}{L^3} - \frac{x^4}{L^4} - \frac{6x^2}{L^2}\right) \tag{3.271}$$

Where *d* is dimensionless constant equal to $\frac{\mu g\, L^3}{24EI}$

From equation (2.61a)

$$A_j = \frac{\mu}{M_j}\int_0^L u(x,0)\emptyset_j \; dx$$

$$= \frac{\mu}{M_j}\int_0^L dL\left(\frac{4x^3}{L^3} - \frac{x^4}{L^4} - \frac{6x^2}{L^2}\right)\left(\cosh\beta_j x + d_{2j}\sinh\beta_j x - \cos\beta_j x +\right.$$

$$\left. d_{4j}\sin\beta_j x\right) dx$$

$$= \frac{d\mu L}{M_j} \int_0^L \Big(\frac{4x^3}{L^3} \cosh\beta_j x + \frac{4x^3}{L^3} d_{2j} \sinh\beta_j x - \frac{4x^3}{L^3} \cos\beta_j x + \frac{4x^3}{L^3} d_{4j} \sin\beta_j x -$$

$$\frac{x^4}{L^4} \cosh\beta_j x - \frac{x^4}{L^4} d_{2j} \sinh\beta_j x + \frac{x^4}{L^4} \cos\beta_j x - \frac{x^4}{L^4} d_{4j} \sin\beta_j x - \frac{6x^2}{L^2} \cosh\beta_j x -$$

$$\frac{6x^2}{L^2} d_{2j} \sinh\beta_j x + \frac{6x^2}{L^2} \cos\beta_j x - \frac{6x^2}{L^2} d_{4j} \sin\beta_j x \Big) \qquad (3.272)$$

By evaluating the integrals and simplifying

$$A_j =$$

$$\frac{d\mu L^2}{M_j} \Bigg[\frac{-3(\sinh\beta_j L - \sin\beta L)}{\beta_j L} + \frac{4(\cosh\beta_j L + \cos\beta_j L)}{\beta_j^2 L^2} - \frac{24(\sinh\beta_j L - \sin\beta_j L)}{\beta_j^5 L^5} +$$

$$d_{2j} \left(\frac{-3(\cosh\beta_j L + \cos\beta_j L)}{\beta_j L} + \frac{4(\sinh\beta_j L + \sin\beta_j L)}{\beta_j^2 L^2} - \frac{24(\cosh\beta_j L + \cos\beta_j L - 2)}{\beta_j^5 L^5} \right) \Bigg] \qquad (3.273)$$

Please note that $d_{2j} = -d_{4j}$

Equation (3.273) is the equation for the arbitrary constant A_j for a fixed-free beam.

In order to evaluate equation (3.273) there is need to derive an expression for the generalized mass Mj.

b) **Derivation of the expression for the generalized mass M_j for a fixed-free beam**

By substituting the general modal equation (3.260) into equation (3.165)

$$M_j = \mu \int_0^L \left(\cosh\beta_j x + d_{2j} \sinh\beta_j x - \cos\beta_j x + d_{4j} \sin\beta_j x \right)^2 dx$$

$$= \mu \int_0^L \big(\cosh^2\beta_j x + 2d_{2j} \sinh\beta_j x \cosh\beta_j x - 2\cosh\beta_j x \cos\beta_j x +$$

$$2d_{4j} \sin\beta_j x \cosh\beta_j x + d_{2j}^2 \sinh^2\beta_j x - 2d_{2j} \sinh\beta_j x \cos\beta_j x +$$

$$2d_{2j} d_{4j} \sin\beta_j x \sinh\beta_j x + \cos^2\beta_j x - 2d_{4j} \cos\beta_j x \sin\beta_j x + d_{4j}^2 \sin^2\beta_j x \big) dx$$

. . (3.274)

By substituting equations (3.168 – 3.171) into equation (3.274) and evaluating the simple integrals we obtain

$$M_j =$$

$$\frac{\mu L}{2}\Big[2 + \frac{\sinh\beta L + \sin\beta L}{2\beta L} + d_{2j}^2\left(\frac{\sinh\beta L}{2\beta L} - 1\right) + d_{4j}^2\left(1 - \frac{\sin\beta L}{2\beta L}\right) -$$

$$2\left(\frac{\cosh\beta L\sin\beta L + \sinh\beta L\cos\beta L}{\beta L}\right) + 2d_{4j}\left(\frac{1 - \cos\beta L\cosh\beta L + \sin\beta L\sinh\beta L - \sin^2\beta L}{\beta L}\right) -$$

$$2d_{2j}\left(\frac{\sinh\beta L\sin\beta L + \cosh\beta L\cos\beta L - \sinh^2\beta L - 1}{\beta L}\right) + 2d_{2j}d_{4j}\left(\frac{\cosh\beta L\sin\beta L - \sinh\beta L\cos\beta L}{\beta L}\right)\Big]$$

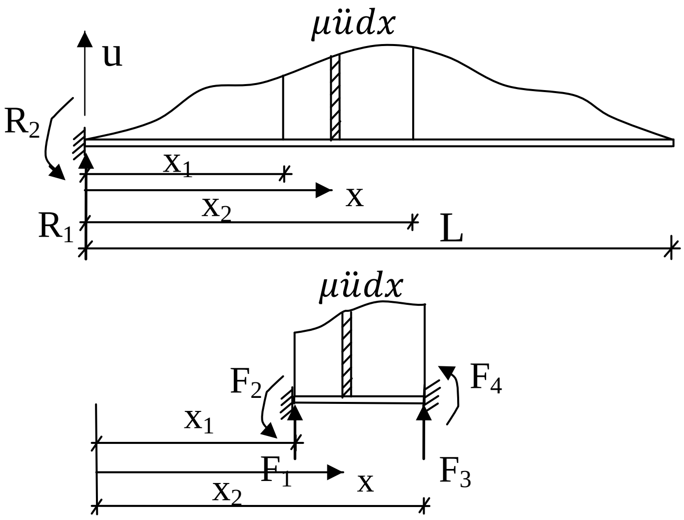
(3.275)

Equation (3.275) is the equation of the generalized mass of a fixed-free beam for the j^{th} mode of vibration.

Having derived the equations of the fixed-end forces for a segment of a fixed-free beam under free vibration, a summary of the equations are presented in Table 3.5. In the Table the distances x_1 and x_2 were normalized using equations (3.35) and (3.36) and square of the j^{th} natural frequency ω_j^2 was eliminated from the equations by substituting equation (3.173)

Table 3.5: Summary of Fixed-end forces on a Segment of a Fixed-free Beam under Free Lateral vibration

<table>
<tr><th>S/N</th><th>Description</th><th>Remarks</th></tr>
<tr><td>1</td><td>Fixed-Free Beam



For $0 \le \xi_1 < 1$ or $0 \le x_1 < L$

$0 < \xi_2 \le 1$ or $0 < x_2 \le L$

$\xi_2 > \xi_1$

$\xi_1 = {}^{x_1}/_{L}$

$\xi_2 = {}^{x_2}/_{L}$

$F_1 = -6\sum_{j=1}^{\infty} \frac{EIA_j}{L^3(\xi_2-\xi_1)^3} W_1$

Where

$W_1 = \beta_j^3 L^3 \left(\frac{(\xi_2+\xi_1)(\xi_2^2-\xi_1^2)}{2} - \frac{2(\xi_2^3-\xi_1^3)}{3}\right)\big(-\sinh\beta_j L\xi_1 -$

$d_{2j}\cosh\beta_j L\xi_1 + \sin\beta_j L\xi_1 + d_{4j}\cos\beta_j L\xi_1) + d_{2j}\big(\beta L(\xi_1 -$

$\xi_2)\cosh\beta_j L\xi_2 + 2\sinh\beta_j L\xi_2 - \beta_j L(\xi_2-\xi_1)\cosh\beta_j L\xi_1 -$

$2\sinh\beta_j L\xi_1) - d_{4j}\big(-\beta_j L(\xi_1-\xi_2)\cos\beta_j L\xi_2 - 2\sin\beta_j L\xi_2 +$</td><td>See Equation (3.263)</td></tr>
</table>

	$\beta_j L(\xi_2-\xi_1)\cos\beta_j L\xi_1 + 2\sin\beta_j L\xi_1) + \beta_j L(\xi_1-\xi_2)\sinh\beta_j L\xi_2 - \beta_j L(\xi_2-\xi_1)\sinh\beta_j L\xi_1 + \beta_j L(\xi_1-\xi_2)\sin\beta_j L\xi_2 - \beta_j L(\xi_2-\xi_1)\sin\beta_j L\xi_1 + 2\cosh\beta_j L\xi_2 - 2\cosh\beta_j L\xi_1 - 2\cos\beta_j L\xi_2 + 2\cos\beta_j L\xi_1$	
	$F_2 = -2\sum_{j=1}^{\infty}\frac{EIA_j}{L^2(\xi_2-\xi_1)^2}W_2$ Where $W_2 = \beta_j^3 L^3\left(\frac{(2\xi_2+\xi_1)(\xi_2^2-\xi_1^2)}{2} - \frac{\xi_1(\xi_2-\xi_1)^2}{2} - (\xi_2^3-\xi_1^3)\right)(-\sinh\beta_j L\xi_1 - d_{2j}\cosh\beta_j L\xi_1 + \sin\beta_j L\xi_1 + d_{4j}\cos\beta_j L\xi_1) - \left(\frac{\beta_j^2 L^2(\xi_2-\xi_1)^2}{2}\right)(\cosh\beta_j L\xi_1 + d_{2j}\sinh\beta_j L\xi_1 + \cos\beta_j L\xi_1 - d_{4j}\sin\beta_j L\xi_1) + \beta_j L(\xi_1-\xi_2)\sinh\beta_j L\xi_2 - 2\beta_j L(\xi_2-\xi_1)\sinh\beta_j L\xi_1 + \beta_j L(\xi_1-\xi_2)\sin\beta_j L\xi_2 - 2\beta_j L(\xi_2-\xi_1)\sin\beta_j L\xi_1 - 3\cos\beta_j L\xi_2 + 3\cos\beta_j L\xi_1 + 3\cosh\beta_j L\xi_2 - 3\cosh\beta_j L\xi_1 + d_{2j}(\beta_j L(\xi_1-\xi_2)\cosh\beta_j L\xi_2 - 2\beta_j L(\xi_2-\xi_1)\cosh\beta_j L\xi_1 + 3\sinh\beta_j L\xi_2 - 3\sinh\beta_j L\xi_1) - d_{4j}(-\beta_j L(\xi_1-\xi_2)\cos\beta_j L\xi_2 + 2\beta_j L(\xi_2-\xi_1)\cos\beta_j L\xi_1 - 3\sin\beta_j L\xi_2 + 3\sin\beta_j L\xi_1)$	See Equation (3.264)
	$F_3 = \sum_{j=1}^{\infty}\frac{EIA_j}{L^3(\xi_2-\xi_1)^3}\left[6W_1 + \beta_j^3 L^3(\xi_2-\xi_1)^3\left(\sinh\beta_j L\xi_2 - \sinh\beta_j L\xi_1 - \sin\beta_j L\xi_2 + \sin\beta_j L\xi_1 + d_{2j}(\cosh\beta_j L\xi_2 - \cosh\beta_j L\xi_1) - d_{4j}(\cos\beta_j L\xi_2 - \cos\beta_j L\xi_1)\right)\right]$	See Equation (3.265)

	$F_4 = \sum_{j=1}^{\infty} \frac{EIA_j}{L^2(\xi_2-\xi_1)^2} \Big[-6W_1 + 2W_2 - \beta_j^3 L^3 (\xi_2 - \xi_1)^3 \big(-\sinh\beta_j L\xi_1 - d_{2j}\cosh\beta_j L\xi_1 + \sin\beta_j L\xi_1 + d_{4j}\cos\beta_j L\xi_1\big) + \big(\cosh\beta_j L\xi_2 + d_{2j}\sinh\beta_j L\xi_2 + \cos\beta_j L\xi_2 - d_{4j}\sin\beta_j L\xi_2\big) - \big(\cosh\beta_j L\xi_1 + d_{2j}\sinh\beta_j L\xi_1 + \cos\beta_j L\xi_1 - d_{4j}\sin\beta_j L\xi_1\big)\Big]$	See Equation (3.266)
	$A_j = \frac{a\mu L^2}{M_j}\Big[\frac{-2(\sinh\beta_j L - \sin\beta L)}{\beta_j^3 L^3} + \frac{12(\cosh\beta_j L - \cos\beta_j L)}{\beta_j^4 L^4} - \frac{24(\sinh\beta_j L - \sin\beta_j L)}{\beta_j^5 L^5} + d_{2j}\Big(\frac{-2(\cosh\beta_j L - \cos\beta_j L)}{\beta_j^3 L^3} + \frac{12(\sinh\beta_j L - \sin\beta_j L)}{\beta_j^4 L^4} - \frac{24(\cosh\beta_j L + \cos\beta_j L - 2)}{\beta_j^5 L^5}\Big)\Big]$	See Equation (3.273)
	$M_j = \frac{\mu L}{2}\Big[2 + \frac{\sinh\beta L + \sin\beta L}{2\beta L} + d_{2j}^2\Big(\frac{\sinh\beta L}{2\beta L} - 1\Big) + d_{4j}^2\Big(1 - \frac{\sin\beta L}{2\beta L}\Big) - 2\Big(\frac{\cosh\beta L\sin\beta L + \sinh\beta L\cos\beta L}{\beta L}\Big) + 2d_{4j}\Big(\frac{1 - \cos\beta L\cosh\beta L + \sin\beta L\sinh\beta L - \sin^2\beta L}{\beta L}\Big) - 2d_{2j}\Big(\frac{\sinh\beta L\sin\beta L + \cosh\beta L\cos\beta L - \sinh^2\beta L - 1}{\beta L}\Big) + 2d_{2j}d_{4j}\Big(\frac{\cosh\beta L\sin\beta L - \sinh\beta L\cos\beta L}{\beta L}\Big)\Big]$	See Equation (3.275)
	$d_{2j} = \frac{-(\cosh\beta_j L + \cosh\beta_j L)}{\sinh\beta_j L - \sin\beta_j L}$	See Equation (3.261)
	$d_{4j} = \frac{\cosh\beta_j L + \cosh\beta_j L}{\sinh\beta_j L - \sin\beta_j L}$	See Equation (3.262)

	$\beta_1 L = 1.87510407$ $\beta_2 L = 4.6940912$, $\beta_3 L = 7.854757439$ $\beta_4 L = 10.9955407349$, $\beta_5 L = 14.137168392$ $\beta_6 L = 17.27875953209$ $\beta_7 L = 20.420352251042$ etc.	See equation (3.257)
Note: d is a dimensionless constant and can be taken as equal to unity		

3.2.5 **For a beam free at both ends (Free-free beam)**

If the beam is free at the near end (x = 0) and free at the far end (x = L), the boundary conditions are

$$\emptyset''(0) = 0$$

$$\emptyset'''(0) = 0$$

$$\emptyset''(L) = 0 \qquad (3.276)$$

$$\emptyset'''(L) = 0$$

By substituting equations (3.276) into (3.109) we obtain

$$C_1 - C_3 = 0 \qquad (3.277a)$$

$$C_2\beta - C_4\beta = 0 \qquad (3.277b)$$

$$C_1\beta^2 \cosh\beta L + C_2\beta^2 \sinh\beta L - C_3\beta^2 \cos\beta L - C_4\beta^2 \sin\beta L = 0 \qquad (3.277c)$$

$$C_1\beta^3 \sinh\beta L + C_2\beta^3 \cosh\beta L + C_3\beta^3 \sin\beta L - C_4\beta^3 \cos\beta L = 0 \qquad (3.277d)$$

Putting equations (3.277a) to (3.277b) in matrix form

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \beta & 0 & \beta \\ \beta^2 \cosh \beta L & \beta^2 \sinh \beta L & -\beta^2 \cos \beta L & -\beta^2 \sin \beta L \\ \beta^3 \sinh \beta L & \beta^3 \cosh \beta L & \beta^3 \sin \beta L & -\beta^3 \cos \beta L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0 \qquad (3.278)$$

For a non-trivial solution the determinant of the coefficients of the constants C_1, C_2, C_3 and C_4 must be zero .

$$\therefore \beta^2(1 - \cosh \beta L \cos \beta L) = 0 \qquad (3.279)$$

This is the same as the equation we obtained for the case of a fixed-fixed beam so therefore the free-free beam shares the same natural frequencies as a fixed-fixed beam (except that the lowest frequency is zero).

The first seven roots of equation (3.256) were obtained using the bisection method as

$$\beta_1 L = 0,\ \beta_2 L = 4.7300408, \beta_3 L = 7.8532047,\ \beta_4 L = 10.99560784,$$

$$\beta_5 L = 14.1371655, \beta_6 L = 17.278759658,$$

$$\beta_7 L = 20.42035224563 \qquad (3.280)$$

By substituting equations (3.280) into (3.114)

$$\omega_1 = 0\ ,\ \omega_2 = 22.37328597\sqrt{\frac{EI}{\mu L^4}},\ \omega_3 = 61.67282406\sqrt{\frac{EI}{\mu L^4}}$$

$$\omega_4 = 120.9033918\sqrt{\frac{EI}{\mu L^4}},\ \omega_5 = 199.8594484\sqrt{\frac{EI}{\mu L^4}}\ ,\ \omega_6 = 298.555535\sqrt{\frac{EI}{\mu L^4}}\ ,$$

$$\omega_7 = 416.9907856\sqrt{\frac{EI}{\mu L^4}} \qquad (3.281)$$

By taking C_1 to be equal to one, the other constants C_2, C_3 and C_4 can be obtained from equations (3.278) as follows

$$C_3 = 1 \tag{3.282a}$$

$$C_2 = C_3 = \frac{\cosh \beta L - \cos \beta L}{\sin \beta L - \sinh \beta L} \tag{3.282b}$$

By substituting equation (3.282a) – (3.282b) into equation (3.109) we obtain the equation of the jth mode shape of vibration

$$\emptyset_j(x) = \cosh \beta_j x + \left(\frac{\cosh \beta_j L - \cos \beta_j L}{\sin \beta_j L - \sinh \beta_j L}\right) \sinh \beta_j x + \cos \beta_j x + \left(\frac{\cosh \beta_j L - \cos \beta_j L}{\sin \beta_j L - \sinh \beta_j L}\right) \sin \beta_j x$$

$$\emptyset_j(x) = \cosh \beta_j x + e_{2j} \sinh \beta_j x + \cos \beta_j x + e_{4j} \sin \beta_j x \tag{3.283}$$

Where

$$e_{2j} = e_{4j} = \frac{\cosh \beta L - \cos \beta L}{\sin \beta L - \sinh \beta L} \tag{3.284}$$

Equation (3.283) is the equation of the jth mode of vibration of a fixed-free beam.

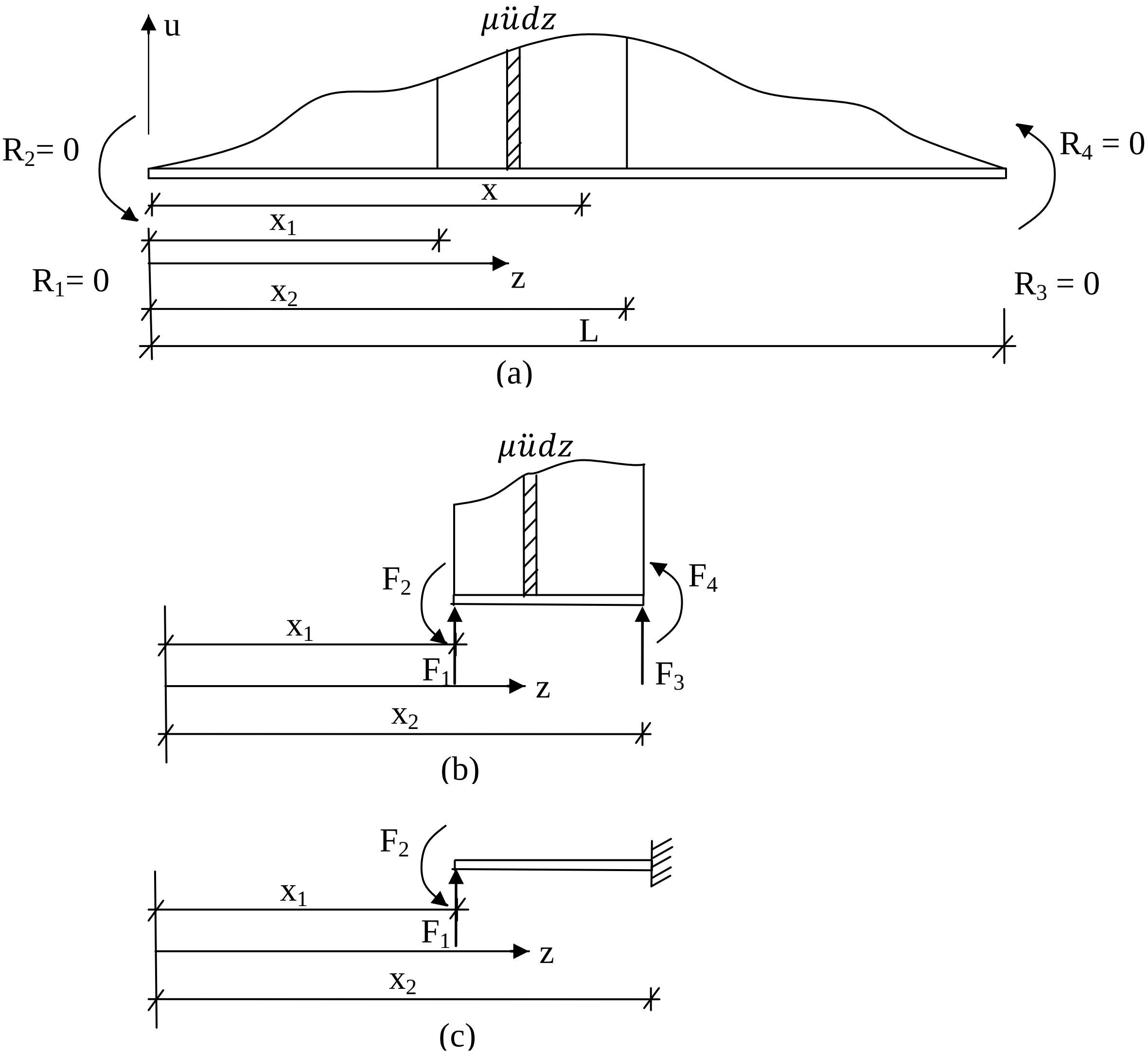


Figure 3.22

(a) A free-free beam under lateral vibration due to the inertial forces $\mu\ddot{u}$

(b) A segment of the beam under longitudinal vibration due to inertial forces $\mu\ddot{u}$

(c) The reduced/basic structure of an arbitrary element of the vibrating beam

Figure 3.22a shows a fixed-free beam under inertia forces. A segment of the beam showed is being restrained by the fixed end forces $F_1 - F_4$. The reduced structure of basic system is shown in Figure 3.22c.

The acceleration at any point in the vibrating beam as stated earlier (equation 3.119) is given by mode superposition as

$\ddot{u} = \sum_{j=1}^{\infty} -\omega_j^2 \emptyset_j(x)\left(A_j \cos\omega_j t + B_j \sin\omega_j t\right)$

For the segment of a free-free beam above (see figure 3.22b) the beam segment is largely the same as that of a fixed-fixed beam except that in the equation for mode shape Ø, a_{2j} and a_{4j} has been replaced it e_{2j} and e_{4j} and values of $\beta_j L$ defined earlier by equation (3.113) is now defined by equation (3.280). Compare equation (3.283) with (3.117).

Hence the equations for the fixed end forces (F_1 – F_4) for a segment of the fixed-fixed beam can be reproduced for this case of a fixed-free beam with a_{2j} and a_{4j} replaced by e_{2j} and e_{4j} and by negating every trigonometric function not pre-multiplied by e_{4j}.

From equation (3.132a)

$$F_1 = -6\sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^3} W_1 \qquad (3.285a)$$

Where

$$W_1 = \beta_j^3\left(\frac{(x_2+x_1)(x_2^2-x_1^2)}{2} - \frac{2(x_2^3-x_1^3)}{3}\right)\left(-\sinh\beta_j x_1 - e_{2j}\cosh\beta_j x_1 - \sin\beta_j x_1 + e_{4j}\cos\beta_j x_1\right) + e_{2j}\left(\beta(x_1-x_2)\cosh\beta_j x_2 + 2\sinh\beta_j x_2 - \beta_j(x_2-x_1)\cosh\beta_j x_1 - 2\sinh\beta_j x_1\right) - e_{4j}\left(-\beta(x_1-x_2)\cos\beta_j x_2 - 2\sin\beta_j x_2 + \beta_j(x_2-x_1)\cos\beta_j x_1 + 2\sin\beta_j x_1\right) + \beta(x_1-x_2)\sinh\beta_j x_2 - \beta(x_2-x_1)\sinh\beta_j x_1 - \beta(x_1-x_2)\sin\beta_j x_2 + \beta(x_2-x_1)\sin\beta_j x_1 + 2\cosh\beta_j x_2 - 2\cosh\beta_j x_1 + 2\cos\beta_j x_2 - 2\cos\beta_j x_1 \qquad (3.285b)$$

From equation (3.134a)

$$F_2 = -2\sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^2} W_2 \quad . \qquad . \qquad . \qquad . \qquad (3.286a)$$

Where

$$W_2 = \beta_j^3 \left(\frac{(2x_2 + x_1)(x_2^2 - x_1^2)}{2} - \frac{x_1(x_2 - x_1)^2}{2} - (x_2^3 - x_1^3) \right) (-\sinh \beta_j x_1 - d_{2j} \cosh \beta_j x_1 - \sin \beta_j x_1 + d_{4j} \cos \beta_j x_1) - \left(\frac{\beta_j^2 (x_2 - x_1)^2}{2} \right) (\cosh \beta_j x_1 + d_{2j} \sinh \beta_j x_1 - \cos \beta_j x_1 - d_{4j} \sin \beta_j x_1) + \beta_j (x_1 - x_2) \sinh \beta_j x_2 - 2\beta_j (x_2 - x_2) \sinh \beta_j x_1 - \beta_j (x_1 - x_2) \sin \beta_j x_2 + 2\beta_j (x_2 - x_1) \sin \beta_j x_1 + 3 \cos \beta_j x_2 - 3 \cos \beta_j x_1 + 3 \cosh \beta_j x_2 - 3 \cosh \beta_j x_1 + d_{2j} (\beta_j (x_1 - x_2) \cosh \beta_j x_2 - 2\beta_j (x_2 - x_1) \cosh \beta_j x_1 + 3 \sinh \beta_j x_2 - 3 \sinh \beta_j x_1) - d_{4j} (-\beta_j (x_1 - x_2) \cos \beta_j x_2 + 2\beta_j (x_2 - x_1) \cos \beta_j x_1 - 3 \sin \beta_j x_2 + 3 \sin \beta_j x_1) \qquad (3.286b)$$

From equation (3.138)

$$F_3 = \sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2 - x_1)^3} \left[6W_1 + \beta_j^3 (x_2 - x_1)^3 \left(\sinh \beta_j x_2 - \sinh \beta_j x_1 + \sin \beta_j x_2 - \sin \beta_j x_1 + d_{2j} (\cosh \beta_j x_2 - \cosh \beta_j x_1) - d_{4j} (\cos \beta_j x_2 - \cos \beta_j x_1) \right) \right] \qquad (3.287)$$

From equation (3.144)

$$F_4 = \sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2 - x_1)^2} [-6W_1 + 2W_2 - \beta_j^3 (x_2 - x_1)^3 (-\sinh \beta_j x_1 - d_{2j} \cosh \beta_j x_1 - \sin \beta_j x_1 + d_{4j} \cos \beta_j x_1) + (\cosh \beta_j x_2 + d_{2j} \sinh \beta_j x_2 - \cos \beta_j x_2 - d_{4j} \sin \beta_j x_2) - (\cosh \beta_j x_1 + d_{2j} \sinh \beta_j x_1 - \cos \beta_j x_1 - d_{4j} \sin \beta_j x_1)] \qquad (3.288)$$

Equations (3.285 – 3.288) are the equations of the fixed-end forces on a segment of a free-free beam under free lateral vibration.

In order to evaluate the equations for the fixed end forces F_1, F_2, F_3 and F_4, there is need to derive an expression for A_j for a free-free beam.

a) **Derivation of the expression for A_j for a free-free beam**

Consider a uniform free-free beam under the action of its self weight propped at its centre (the point through which its resultant weight acts)

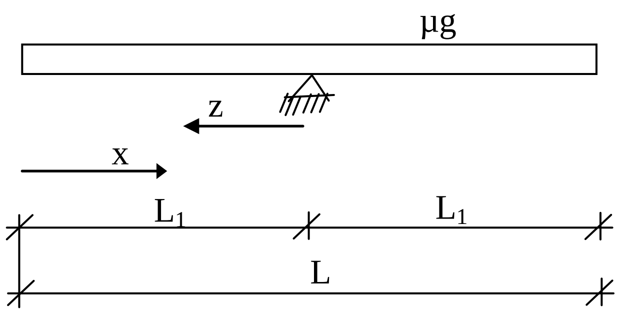


Figure 3.23: A Uniform fixed-free beam under the action of its self weight propped at the centre

From the equation for the deflection of a fixed-free beam the equation of the deflection of the beam can be written as

$$u(z,0) = dL\left(\frac{4z^3}{L_1^3} - \frac{z^4}{L_1^4} - \frac{6z^2}{L_1^2}\right) \qquad (3.289)$$

Where d is dimensionless constant equal to $\frac{\mu g\, L_1^3}{24EI}$

But $z = L_1 - x$ (3.290)

By substituting equation (2.290) into equation (3.289)

$$u(x,0) = \frac{d}{L_1^3}(-3L_1^4 + 4xL_1^3 - x^4) \qquad (3.291)$$

From equation (2.61a)

$$A_j = \frac{\mu}{M_j}\int_0^{L_1} u(x,0)\emptyset_j\, dx$$

$$= \frac{2\mu d}{M_j L_1^3} \int_0^{L_1} (-3L_1^4 + 4xL_1^3 - x^4)\left(\cosh \beta_j x + e_{2j} \sinh \beta_j x - \cos \beta_j x + e_{4j} \sin \beta_j x\right) dx \tag{3.292}$$

The integral of equation (3.292) was pre-multiplied by two to take care of the other half of the beam (L_1 is only half of the full length)

By evaluating the integrals and simplifying

$$A_j = \frac{2\mu d\, L_1^2}{M_j} \left[\frac{-12(\sinh \beta_j L_1 - \sin \beta_j L_1)}{\beta_j^3 L_1^3} + \frac{24(\cosh \beta_j L_1 + \cos \beta_j L_1)}{\beta_j^4 L_1^4} - \frac{24(\sinh \beta_j L_1 + \sin \beta_j L_1)}{\beta_j^5 L_1^3} - e_{2j} \left(\frac{12(\cosh \beta_j L_1 + \cos \beta_j L_1)}{\beta_j^3 L_1^3} - \frac{24(\sinh \beta_j L_1 + \sin \beta_j L_1)}{\beta_j^4 L_1^4} + \frac{24(\cosh \beta_j L_1 - \cos \beta_j L_1)}{\beta_j^5 L^5} \right) \right] \tag{3.293}$$

Please note that $L_1 = {}^{L}/_{2}$ (3.294)

Equation (3.293) is the equation for the arbitrary constant A_j for a free-free beam.

In order to evaluate equation (3.293) there is need to derive an expression for the generalized mass Mj.

b) **Derivation of the expression for the generalized mass M_j for a fixed-free beam**

By substituting the general modal equation (3.283) into equation (3.165)

$$M_j = \mu \int_0^L \left(\cosh \beta_j x + e_{2j} \sinh \beta_j x + \cos \beta_j x + e_{4j} \sin \beta_j x\right)^2 dx$$

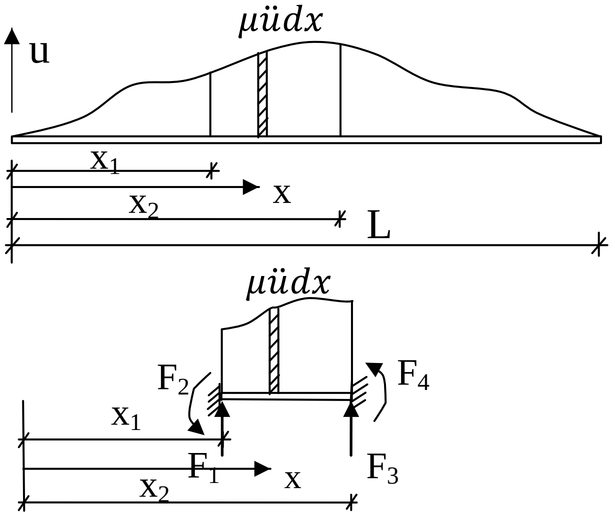
$$= \mu \int_0^L \left(\cosh^2 \beta_j x + 2e_{2j} \sinh \beta_j x \cosh \beta_j x + 2 \cosh \beta_j x \cos \beta_j x + 2e_{4j} \sin \beta_j x \cosh \beta_j x + e_{2j}^2 \sinh^2 \beta_j x + 2e_{2j} \sinh \beta_j x \cos \beta_j x + 2e_{2j} e_{4j} \sin \beta_j x \sinh \beta_j x + \cos^2 \beta_j x - 2e_{4j} \cos \beta_j x \sin \beta_j x + e_{4j}^2 \sin^2 \beta_j x\right) dx$$

$$
= \frac{\mu L}{2}\left[2 + \frac{\sinh\beta L - \sin\beta L}{2\beta L} + e_{2j}^2\left(\frac{\sinh\beta L}{2\beta L} - 1\right) + e_{4j}^2\left(1 - \frac{\sin\beta L}{2\beta L}\right) + 2\left(\frac{\cosh\beta L\sin\beta L + \sinh\beta L\cos\beta L}{\beta L}\right) + 2e_{4j}\left(\frac{1 - \cos\beta L\cosh\beta L + \sin\beta L\sinh\beta L - \sin^2\beta L}{\beta L}\right) - 2e_{2j}\left(\frac{\sinh\beta L\sin\beta L + \cosh\beta L\cos\beta L - \sinh^2\beta L - 1}{\beta L}\right) + 2e_{2j}e_{4j}\left(\frac{\cosh\beta L\sin\beta L - \sinh\beta L\cos\beta L}{\beta L}\right)\right] \tag{3.295}
$$

Equation (3.295) is the equation of the generalized mass of a free-free beam for the jth mode of vibration.

Having derived the equations of the fixed-end forces for a segment of a fixed-free beam under free vibration, a summary of the equations are presented in Table 3.5 below. In the Table the distances x_1 and x_2 were normalized using equations (3.35) and (3.36) and square of the jth natural frequency ω_j^2 was eliminated from the equations by substituting equation (3.173)

Table 3.6: Summary of Fixed-end forces on a Segment of a Free-free Beam under Free Lateral vibration

S/N	Description	Remarks
1	Free-Free Beam  For $0 \le \xi_1 < 1$ or $0 \le x_1 < L$ $0 < \xi_2 \le 1$ or $0 < x_2 \le L$ $\xi_2 > \xi_1$ $\xi_1 = {}^{x_1}/_{L}$ $\xi_2 = {}^{x_2}/_{L}$ $F_1 = -6\sum_{j=1}^{\infty}\frac{EIA_j}{L^3(\xi_2-\xi_1)^3}W_1$ Where $W_1 = \beta_j^3 L^3\left(\frac{(\xi_2+\xi_1)(\xi_2^2-\xi_1^2)}{2} - \frac{2(\xi_2^3-\xi_1^3)}{3}\right)(-\sinh\beta_j L\xi_1 -$ $e_{2j}\cosh\beta_j L\xi_1 - \sin\beta_j L\xi_1 + e_{4j}\cos\beta_j L\xi_1) + e_{2j}(\beta L(\xi_1 -$ $\xi_2)\cosh\beta_j L\xi_2 + 2\sinh\beta_j L\xi_2 - \beta_j L(\xi_2 - \xi_1)\cosh\beta_j L\xi_1 -$	See Equation (3.285a)

	$2\sinh\beta_j L\xi_1) - e_{4j}\left(-\beta_j L(\xi_1 - \xi_2)\cos\beta_j L\xi_2 - 2\sin\beta_j L\xi_2 +\right.$ $\beta_j L(\xi_2 - \xi_1)\cos\beta_j L\xi_1 + 2\sin\beta_j L\xi_1) + \beta_j L(\xi_1 -$ $\xi_2)\sinh\beta_j L\xi_2 - \beta_j L(\xi_2 - \xi_1)\sinh\beta_j L\xi_1 - \beta_j L(\xi_1 -$ $\xi_2)\sin\beta_j L\xi_2 + \beta_j L(\xi_2 - \xi_1)\sin\beta_j L\xi_1 + 2\cosh\beta_j L\xi_2 -$ $2\cosh\beta_j L\xi_1 + 2\cos\beta_j L\xi_2 - 2\cos\beta_j L\xi_1$ $F_2 = -2\sum_{j=1}^{\infty}\frac{EIA_j}{L^2(\xi_2-\xi_1)^2}W_2$ Where $W_2 =$ $\beta_j^3 L^3\left(\frac{(2\xi_2+\xi_1)(\xi_2^2-\xi_1^2)}{2} - \frac{\xi_1(\xi_2-\xi_1)^2}{2} - (\xi_2^3 - \xi_1^3)\right)\left(-\sinh\beta_j L\xi_1 -\right.$ $e_{2j}\cosh\beta_j L\xi_1 - \sin\beta_j L\xi_1 + e_{4j}\cos\beta_j L\xi_1) -$ $\left(\frac{\beta_j^2 L^2(\xi_2-\xi_1)^2}{2}\right)\left(\cosh\beta_j L\xi_1 + e_{2j}\sinh\beta_j L\xi_1 - \cos\beta_j L\xi_1 -\right.$ $e_{4j}\sin\beta_j L\xi_1) + \beta_j L(\xi_1 - \xi_2)\sinh\beta_j L\xi_2 - 2\beta_j L(\xi_2 -$ $\xi_1)\sinh\beta_j L\xi_1 - \beta_j L(\xi_1 - \xi_2)\sin\beta_j L\xi_2 + 2\beta_j L(\xi_2 -$ $\xi_1)\sin\beta_j L\xi_1 + 3\cos\beta_j L\xi_2 - 3\cos\beta_j L\xi_1 + 3\cosh\beta_j L\xi_2 -$ $3\cosh\beta_j L\xi_1 + e_{2j}\left(\beta_j L(\xi_1 - \xi_2)\cosh\beta_j L\xi_2 - 2\beta_j L(\xi_2 -\right.$ $\xi_1)\cosh\beta_j L\xi_1 + 3\sinh\beta_j L\xi_2 - 3\sinh\beta_j L\xi_1) -$ $e_{4j}\left(-\beta_j L(\xi_1 - \xi_2)\cos\beta_j L\xi_2 + 2\beta_j L(\xi_2 - \xi_1)\cos\beta_j L\xi_1 -\right.$ $3\sin\beta_j L\xi_2 + 3\sin\beta_j L\xi_1)$	See Equation (3.286a)
	$F_3 = \sum_{j=1}^{\infty}\frac{EIA_j}{L^3(\xi_2-\xi_1)^3}\left[6W_1 + \beta_j^3 L^3(\xi_2 - \xi_1)^3\left(\sinh\beta_j L\xi_2 -\right.\right.$ $\sinh\beta_j L\xi_1 + \sin\beta_j L\xi_2 - \sin\beta_j L\xi_1 + e_{2j}\left(\cosh\beta_j L\xi_2 -\right.$ $\left.\left.\cosh\beta_j L\xi_1) - e_{4j}\left(\cos\beta_j L\xi_2 - \cos\beta_j L\xi_1\right)\right)\right]$	See Equation (3.287)

	$F_4 = \sum_{j=1}^{\infty} \frac{EIA_j}{L^2(\xi_2-\xi_1)^2}\Big[-6W_1 + 2W_2 - \beta_j^3 L^3(\xi_2-\xi_1)^3\big(-\sinh\beta_j L\xi_1 - e_{2j}\cosh\beta_j L\xi_1 - \sin\beta_j L\xi_1 + e_{4j}\cos\beta_j L\xi_1\big) + \big(\cosh\beta_j L\xi_2 + e_{2j}\sinh\beta_j L\xi_2 - \cos\beta_j L\xi_2 - e_{4j}\sin\beta_j L\xi_2\big) - \big(\cosh\beta_j L\xi_1 + e_{2j}\sinh\beta_j L\xi_1 - \cos\beta_j L\xi_1 - e_{4j}\sin\beta_j L\xi_1\big)\Big]$	See Equation (3.288)
	$A_j = \frac{2\mu d L_1^2}{M_j}\Bigg[\frac{-12(\sinh\beta_j L_1 - \sin\beta_j L_1)}{\beta_j^3 L_1^3} + \frac{24(\cosh\beta_j L_1 + \cos\beta_j L_1)}{\beta_j^4 L_1^4} - \frac{24(\sinh\beta_j L_1 + \sin\beta_j L_1)}{\beta_j^5 L_1^3} - e_{2j}\left(\frac{12(\cosh\beta_j L_1 + \cos\beta_j L_1)}{\beta_j^3 L_1^3} - \frac{24(\sinh\beta_j L_1 + \sin\beta_j L_1)}{\beta_j^4 L_1^4} + \frac{24(\cosh\beta_j L_1 - \cos\beta_j L_1)}{\beta_j^5 L^5}\right)\Bigg]$	See Equation (3.293)
	$L_1 = {}^{L}/_{2}$	
	$M_j = \frac{\mu L}{2}\Bigg[2 + \frac{\sinh\beta L - \sin\beta L}{2\beta L} + e_{2j}^2\left(\frac{\sinh\beta L}{2\beta L} - 1\right) + e_{4j}^2\left(1 - \frac{\sin\beta L}{2\beta L}\right) + 2\left(\frac{\cosh\beta L\sin\beta L + \sinh\beta L\cos\beta L}{\beta L}\right) + 2e_{4j}\left(\frac{1-\cos\beta L\cosh\beta L + \sin\beta L\sinh\beta L - \sin^2\beta L}{\beta L}\right) - 2e_{2j}\left(\frac{\sinh\beta L\sin\beta L + \cosh\beta L\cos\beta L - \sinh^2\beta L - 1}{\beta L}\right) + 2e_{2j}e_{4j}\left(\frac{\cosh\beta L\sin\beta L - \sinh\beta L\cos\beta L}{\beta L}\right)\Bigg]$	See Equation (3.295)
	$e_{2j} = e_{4j} = \frac{\cosh\beta L - \cos\beta L}{\sin\beta L - \sinh\beta L}$	See Equation (3.284)

	$\beta_1 L = 0,\ \beta_2 L = 4.7300408, \beta_3 L = 7.8532047,\ \beta_4 L = 10.99560784,$ $\beta_5 L = 14.1371655, \beta_6 L = 17.278759658,$ $\beta_7 L = 20.42035224563$	See Equation (3.280)
Note: d is a dimensionless constant and can be taken as equal to unity		

3.2.6 **For a beam pinned at one end and free at the other (Pinned-free beam)**

If the beam is pinned at the near end (x = 0) and free at the far end (x = L), the boundary conditions are

$$\emptyset(0) = 0$$

$$\emptyset''(0) = 0$$

$$\emptyset''(L) = 0 \qquad (3.296)$$

$$\emptyset'''(L) = 0$$

By substituting equations (3.296) into (3.109) we obtain

$$C_1 + C_3 = 0 \qquad (3.297a)$$

$$C_1\beta - C_3\beta = 0 \qquad (3.297b)$$

$$C_1\beta^2 \cosh\beta L + C_2\beta^2 \sinh\beta L - C_3\beta^2 \cos\beta L - C_4\beta^2 \sin\beta L = 0 \qquad (3.297c)$$

$$C_1\beta^3\sinh\beta L + C_2\beta^3\cosh\beta L + C_3\beta^3\sin\beta L - C_4\beta^3\cos\beta L = 0 \qquad (3.297d)$$

Putting equations (3.297a) to (3.297b) in matrix form

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ \beta & 0 & -\beta & 0 \\ \beta^2\cosh\beta L & \beta^2\sinh\beta L & -\beta^2\cos\beta L & -\beta^2\sin\beta L \\ \beta^3\sinh\beta L & \beta^3\cosh\beta L & \beta^3\sin\beta L & -\beta^3\cos\beta L \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0 \qquad (3.298)$$

For a non-trivial solution the determinant of the coefficients of the constants C_1, C_2, C_3 and C_4 must be zero .

$$\therefore \tan\beta L - \tanh\beta L = 0 \qquad (3.299)$$

The first seven roots of equation (3.299) were obtained using the bisection method as

$$\beta_1 L = 0, \beta_2 L = 3.92660232,\ \beta_3 L = 7.06858275,$$

$$\beta_4 L = 10.210176123,$$

$$\beta_5 L = 13.35176878, \beta_6 L = 16.49336143135,$$

$$\beta_7 L = 19.634954084937 \qquad (3.300)$$

By substituting equations (3.300) into (3.114)

$$\omega_1 = 15.41820578\sqrt{\frac{EI}{\mu L^4}},\ \omega_2 = 49.96486209\sqrt{\frac{EI}{\mu L^4}},\ \omega_3 = 104.2476964\sqrt{\frac{EI}{\mu L^4}}$$

$$\omega_4 = 178.2697296\sqrt{\frac{EI}{\mu L^4}},\ \omega_5 = 272.0309713\sqrt{\frac{EI}{\mu L^4}},\ \omega_6 = 385.5314217\sqrt{\frac{EI}{\mu L^4}},$$

$$\omega_7 = 518.7710809\sqrt{\frac{EI}{\mu L^4}}, \qquad (3.301)$$

We observe that the natural frequencies for a pinned-free beam is the same as that of a fixed-free beam except that the lowest frequency for a pinned-free bar is zero. This is the frequency corresponding to a rigid body motion.

From equation (3.298) $C_1 = C_3 = 0$ and by taking C_2 to be equal to one, the other constant C4 can be obtained from equations as

$$C_4 = \frac{\sinh\beta L}{\sin\beta L} \tag{3.302}$$

By substituting the values of $C_1 - C_4$ into equation (3.109) we obtain the equation of the j^{th} mode shape of vibration

$$\emptyset_j(x) = \sinh\beta_j x + f_{4j}\sin\beta_j x \tag{3.303}$$

Where

$$f_{4j} = \frac{\sinh\beta L}{\sin\beta L} \tag{3.304}$$

Equation (3.303) is the equation of the j^{th} mode of vibration of a pinned-free beam.

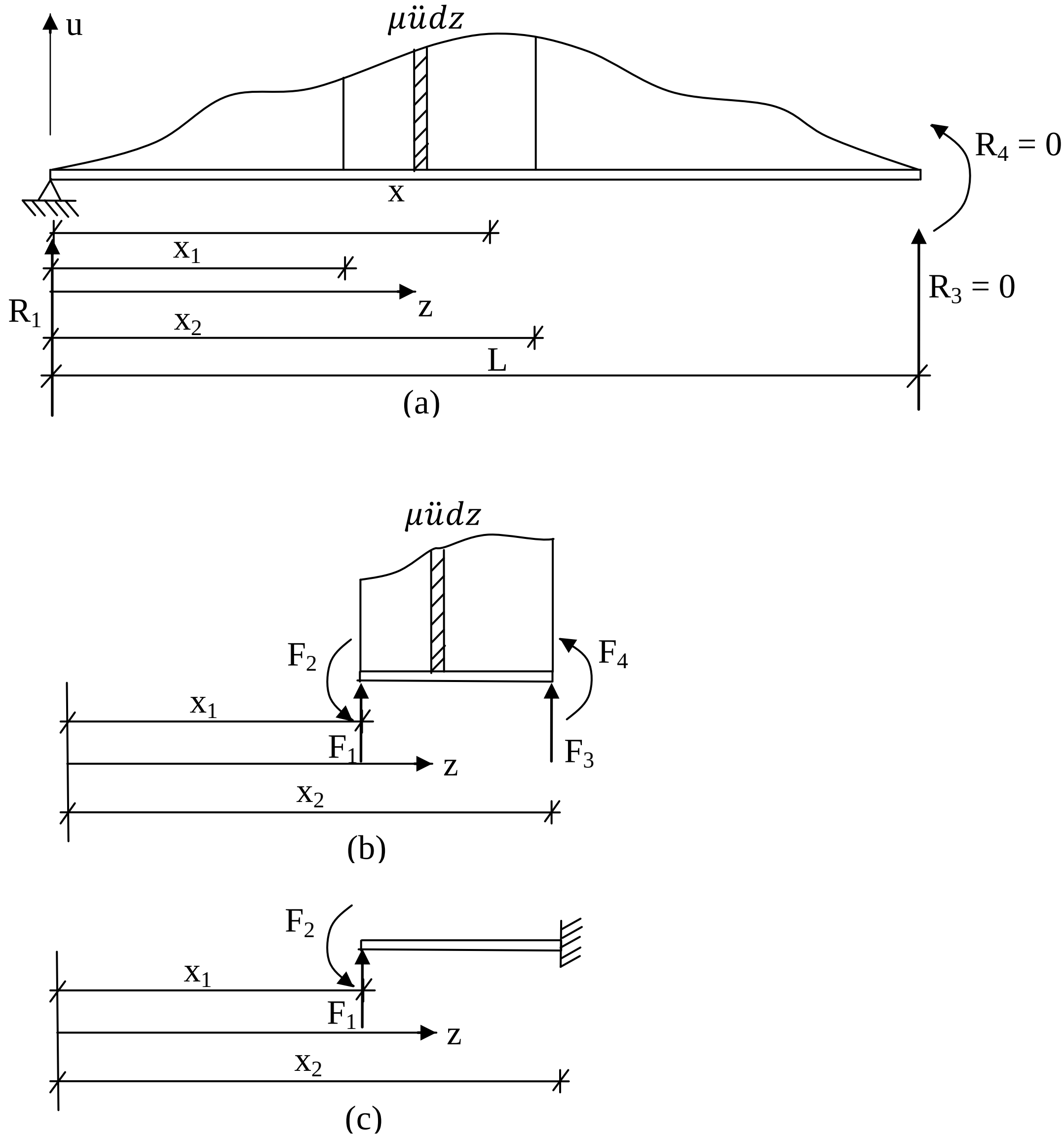


Figure 3.24

(a) A pinned-free beam under lateral vibration due to the inertial forces $\mu\ddot{u}$

(b) A segment of the beam under longitudinal vibration due to inertial forces $\mu\ddot{u}$

(c) The reduced/basic structure of an arbitrary element of the vibrating beam

Figure 3.24a shows a pinned-free beam under inertia forces. A segment of the beam presented in Figure 3.24b is restrained by the fixed end forces $F_1 - F_4$. The reduced structure of the basic system is shown in Figure 3.24c.

The equations for the fixed end forces ($F_1 - F_4$) for a segment of the fixed-fixed beam can be reproduced for this case of a pinned-free beam with a_{2j} and a_{4j} replaced by $f_{2j}=1$ and f_{4j} and all functions not multiplied by a_{2j} and a_{4j} expunged.

From equation (3.132a)

$$F_1 = -6\sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^3} W_1 \qquad (3.305a)$$

Where

$$W_1 = \beta_j^3\left(\frac{(x_2+x_1)(x_2^2-x_1^2)}{2} - \frac{2(x_2^3-x_1^3)}{3}\right)\left(-\cosh\beta_j x_1 + f_{4j}\cos\beta_j x_1\right) + \left(\beta(x_1 - x_2)\cosh\beta_j x_2 + 2\sinh\beta_j x_2 - \beta_j(x_2 - x_1)\cosh\beta_j x_1 - 2\sinh\beta_j x_1\right) - f_{4j}\left(-\beta(x_1 - x_2)\cos\beta_j x_2 - 2\sin\beta_j x_2 + \beta_j(x_2 - x_1)\cos\beta_j x_1 + 2\sin\beta_j x_1\right) \quad .. \quad . \quad . \quad (3.305b)$$

From equation (3.134a)

$$F_2 = -2\sum_{j=1}^{\infty}\frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2-x_1)^2} W_2 \qquad (3.306a)$$

Where

$$W_2 = \beta_j^3\left(\frac{(2x_2+x_1)(x_2^2-x_1^2)}{2} - \frac{x_1(x_2-x_1)^2}{2} - (x_2^3 - x_1^3)\right)\left(-\cosh\beta_j x_1 + f_{4j}\cos\beta_j x_1\right) - \left(\frac{\beta_j^2(x_2-x_1)^2}{2}\right)\left(\sinh\beta_j x_1 - f_{4j}\sin\beta_j x_1\right) + \left(\beta_j(x_1 - x_2)\cosh\beta_j x_2 - 2\beta_j(x_2 - x_1)\cosh\beta_j x_1 + 3\sinh\beta_j x_2 - 3\sinh\beta_j x_1\right) - f_{4j}\left(-\beta_j(x_1 - x_2)\cos\beta_j x_2 + 2\beta_j(x_2 - x_1)\cos\beta_j x_1 - 3\sin\beta_j x_2 + 3\sin\beta_j x_1\right) \qquad (3.306b)$$

From equation (3.138)

$$F_3 = \sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2 - x_1)^3} \Big[6W_1 + \beta_j^3 (x_2 - x_1)^3 \Big(\big(\cosh \beta_j x_2 - \cosh \beta_j x_1\big) -$$

$$f_{4j}\big(\cos \beta_j x_2 - \cos \beta_j x_1\big)\Big)\Big] \qquad (3.307)$$

From equation (3.144)

$$F_4 = \sum_{j=1}^{\infty} \frac{\omega_j^2 A_j \mu}{\beta_j^4 (x_2 - x_1)^2} \big[-6W_1 + 2W_2 - \beta_j^3 (x_2 - x_1)^3 \big(-\cosh \beta_j x_1 + f_{4j} \cos \beta_j x_1\big) +$$

$$\big(\sinh \beta_j x_2 - f_{4j} \sin \beta_j x_2\big) - \big(\sinh \beta_j x_1 - f_{4j} \sin \beta_j x_1\big)\big] \qquad (3.308)$$

Equations (3.305a – 3.308) are the equations of the fixed-end forces on a segment of a fixed-free beam under free lateral vibration.

In order to evaluate the equations for the fixed end forces $F_{1,}$ F_2, F_3 and F_4, there is need to derive an expression for A_j for a pinned-free beam.

a) **Derivation of the expression for A_j for a pinned-free beam**

Consider a uniform pinned-free beam under the action of its self weight propped at its centre of gravity.

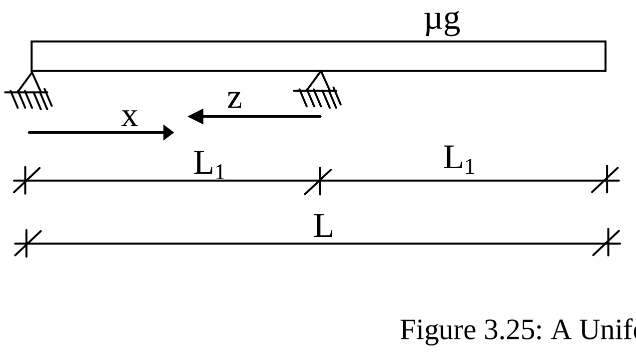


Figure 3.25: A Uniform pinned-free beam under the action of its self weight

For $0 \le x \le L_1$

The equation of the bending moment at a distance x from the left support is

$$M_x = -\frac{\mu g x^2}{2} \tag{3.309}$$

Where μ is the mass per unit length of the beam and g is the acceleration due to gravity.

By substituting equation (3.309) into the equation of elastic curve (equation 3.146) and solving for the deflection y

$$EIy' = \int_0^L M_x \, dx$$

$$= -\frac{\mu g x^3}{6} + c_1 \tag{3.310}$$

$$EIy = -\frac{\mu g x^4}{24} + c_1 x + c_2 \tag{3.311}$$

Consider the boundary conditions

At $x = 0$, $y = 0$ $\quad \therefore c_2 = 0$

At $x = L_1$, $y = 0$ $\quad \therefore c_1 = \frac{\mu g L_1^3}{24}$

Hence the equation for the static deformation or deflection of the uniform beam under its self weight for the region $(0 \leq x \leq L_1)$ considered is

$$EIy = -\frac{\mu g}{24}(x^4 - L_1^3 x) \tag{3.312}$$

For $L_1 \leq x \leq 2L_1$

$$M_x = \mu g\left(-\frac{x^2}{2} + L_1 x - L_1^2\right) \tag{3.313}$$

By substituting equation (3.313) into the equation of elastic curve and solving for the deflection y

$$EIy' = \int_0^L M_x \, dx$$

$$= \mu g \left(-\frac{x^3}{6} + \frac{L_1 x^2}{2} - L_1^2 x\right) + c_1 \tag{3.314}$$

$$EIy = \mu g \left(-\frac{x^4}{24} + \frac{L_1 x^3}{6} - \frac{L_1^2 x^2}{2}\right) + c_1 x + c_2 \tag{3.315}$$

Consider the boundary conditions

At $x = L_1$, $\; y' = -\frac{\mu g \, L_1^3}{8EI} \quad \therefore c_1 = \frac{13\mu g \, L_1^3}{24}$

At $x = L_1$, $\; y = 0 \quad \therefore c_2 = -\frac{\mu g \, L_1^4}{6}$

Hence the equation for the static deformation or deflection of the uniform beam under its self weight for the region $(L_1 \ll x \ll 2L_1)$ considered is

$$EIy = \mu g \left(-\frac{x^4}{24} + \frac{L_1 x^3}{6} - \frac{L_1^2 x^2}{2} + \frac{13 L_1^3 x}{24} - \frac{L_1^4}{6}\right) \tag{3.316}$$

From equations (3.312) and (3.316) let the initial deflection of the beam (at time t = 0) be

$$u(x,0) = \frac{d}{L_1^3}(L_1^3 x - x^4) \quad for \quad 0 \ll x \ll L_1$$

$$u(x,0) = \frac{d}{L_1^3}(-x^4 + 4L_1 x^3 - 12L_1^2 x^2 + 13L_1^3 x - 4L_1^4) \quad for \quad L_1 \ll x \ll 2L_1$$

(3.317)

Where d is a dimensionless constant equal to $\frac{\mu g \, L_1^3}{24EI}$

From equation (2.61a)

$$A_j = \frac{\mu}{M_j}\int_0^L u(x,0)\emptyset_j\, dx = \frac{\mu}{M_j}\int_0^{L_1} u(x,0)\emptyset_j\, dx + \frac{\mu}{M_j}\int_{L_1}^{2L_1} u(x,0)\emptyset_j\, dx \qquad (3.318)$$

$$\frac{\mu}{M_j}\int_0^{L_1} u(x,0)\emptyset_j\, dx = \frac{\mu}{M_j}\int_0^{L_1} \frac{d}{L_1^3}(L_1^3 x - x^4)\left(\sinh\beta_j x + f_{4j}\sin\beta_j x\right) dx$$

$$= \frac{d\mu L_1^2}{M_j}\left[\frac{3\sinh\beta_j L_1}{\beta_j^2 L_1^2} - \frac{12\cosh\beta_j L_1}{\beta_j^3 L_1^3} + \frac{24\sinh\beta_j L_1}{\beta_j^4 L_1^4} - \frac{24\cosh\beta_j L_1}{\beta_j^5 L_1^5} + \right.$$

$$\left. f_{4j}\left(\frac{3\sin\beta_j L_1}{\beta_j^2 L_1^2} - \frac{12\cos\beta_j L_1}{\beta_j^3 L_1^3} + \frac{24\sin\beta_j L_1}{\beta_j^4 L_1^4} - \frac{24(-\cos\beta_j L_1)}{\beta_j^5 L_1^5}\right)\right] \qquad (3.319)$$

$$\frac{\mu}{M_j}\int_{L_1}^{2L_1} u(x,0)\emptyset_j\, dx =$$

$$\frac{\mu}{M_j}\int_0^{2L_1} \frac{d}{L_1^3}(-x^4 + 4L_1x^3 - 12L_1^2x^2 + 13L_1^3x - 4L_1^4)\left(\sinh\beta_j x + f_{4j}\sin\beta_j x\right) dx$$

$$-\frac{\mu}{M_j}\int_0^{L_1} \frac{d}{L_1^3}(-x^4 + 4L_1x^3 - 12L_1^2x^2 + 13L_1^3x - 4L_1^4)\left(\sinh\beta_j x + f_{4j}\sin\beta_j x\right) dx$$

(3.320)

$$\frac{\mu}{M_j}\int_0^{L_1} \frac{d}{L_1^3}(-x^4 + 4L_1x^3 - 12L_1^2x^2 + 13L_1^3x - 4L_1^4)\left(\sinh\beta_j x + f_{4j}\sin\beta_j x\right) dx$$

$$= \frac{d\mu L_1^2}{M_j}\left[\frac{3\sinh\beta_j L_1}{\beta_j^2 L_1^2} - \frac{12(\cosh\beta_j L_1 - 2)}{\beta_j^3 L_1^3} - \frac{24\cosh\beta_j L_1}{\beta_j^5 L_1^5} + f_{4j}\left(\frac{-3\sin\beta_j L_1}{\beta_j^2 L_1^2} - \right.\right.$$

$$\left.\left.\frac{12(\cos\beta_j L_1 - 2)}{\beta_j^3 L_1^3} + \frac{24\cos\beta_j L_1}{\beta_j^5 L_1^5}\right)\right] \qquad (3.321)$$

$$\frac{\mu}{M_j}\int_0^{2L_1} \frac{d}{L_1^3}(-x^4 + 4L_1x^3 - 12L_1^2x^2 + 13L_1^3x - 4L_1^4)\left(\sinh\beta_j x + f_{4j}\sin\beta_j x\right) dx =$$

$$\frac{d\mu L_1^2}{M_j}\left[\frac{-10\cosh 2\beta_j L_1}{\beta_j L_1} + \frac{19\sinh 2\beta_j L_1}{\beta_j^2 L_1^2} - \frac{24\cosh 2\beta_j L_1}{\beta_j^3 L_1^3} + \frac{24\sinh 2\beta_j L_1}{\beta_j^4 L_1^4} - \right.$$

$$\left.\frac{24\cosh 2\beta_j L_1}{\beta_j^5 L_1^5} + f_{4j}\left(\frac{10\cos 2\beta_j L_1}{\beta_j L_1} - \frac{19\sin 2\beta_j L_1}{\beta_j^2 L_1^2} + \frac{24\cos 2\beta_j L_1}{\beta_j^3 L_1^3} + \frac{24\sin 2\beta_j L_1}{\beta_j^4 L_1^4} + \frac{24\cosh 2\beta_j L_1}{\beta_j^5 L_1^5}\right)\right]$$

(3.322)

By substituting equation (3.322) and (3.321) into equation (3.320) we obtain

$$\frac{\mu}{M_j}\int_{L_1}^{2L_1} u(x,0)\emptyset_j\, dx = \frac{d\mu L_1^2}{M_j}\left[\frac{-10\cosh\ 2\beta_j L_1}{\beta_j L_1} + \frac{19\sinh\ 2\beta_j L_1}{\beta_j^2 L_1^2} - \frac{24\cosh\ 2\beta_j L_1}{\beta_j^3 L_1^3} + \right.$$

$$\frac{24\sinh\ 2\beta_j L_1}{\beta_j^4 L_1^4} - \frac{24\cosh\ 2\beta_j L_1}{\beta_j^5 L_1^5} + f_{4j}\left(\frac{10\cos\ 2\beta_j L_1}{\beta_j L_1} - \frac{19\sin\ 2\beta_j L_1}{\beta_j^2 L_1^2} + \frac{24\cos\ 2\beta_j L_1}{\beta_j^3 L_1^3} + \frac{24\sin\ 2\beta_j L_1}{\beta_j^4 L_1^4} + \right.$$

$$\left.\left.\frac{24\cosh\ 2\beta_j L_1}{\beta_j^5 L_1^5}\right)\right] - \frac{d\mu L_1^2}{M_j}\left[\frac{3\sinh\ \beta_j L_1}{\beta_j^2 L_1^2} - \frac{12(\cosh\ \beta_j L_1 - 2)}{\beta_j^3 L_1^3} - \frac{24\cosh\ \beta_j L_1}{\beta_j^5 L_1^5} + f_{4j}\left(\frac{-3\sin\ \beta_j L_1}{\beta_j^2 L_1^2} - \right.\right.$$

$$\left.\left.\frac{12(\cos\ \beta_j L_1 - 2)}{\beta_j^3 L_1^3} + \frac{24\cos\ \beta_j L_1}{\beta_j^5 L_1^5}\right)\right] \tag{3.323}$$

By substituting equation 3.323 and (3.319) into (3.318) we obtain

$$A_j =$$

$$\frac{d\mu L_1^2}{M_j}\left[\frac{-10\cosh\ 2\beta_j L_1}{\beta_j L_1} + \frac{19\sinh\ 2\beta_j L_1}{\beta_j^2 L_1^2} - \frac{24\cosh\ 2\beta_j L_1}{\beta_j^3 L_1^3} + \frac{24\sinh\ 2\beta_j L_1}{\beta_j^4 L_1^4} - \right.$$

$$\left.\frac{24\cosh\ 2\beta_j L_1}{\beta_j^5 L_1^5} + f_{4j}\left(\frac{10\cos\ 2\beta_j L_1}{\beta_j L_1} - \frac{19\sin\ 2\beta_j L_1}{\beta_j^2 L_1^2} + \frac{24\cos\ 2\beta_j L_1}{\beta_j^3 L_1^3} + \frac{24\sin\ 2\beta_j L_1}{\beta_j^4 L_1^4} + \frac{24\cosh\ 2\beta_j L_1}{\beta_j^5 L_1^5}\right)\right] -$$

$$\frac{d\mu L_1^2}{M_j}\left[\frac{24}{\beta_j^3 L_1^3} + \frac{24\sinh\ \beta_j L_1}{\beta_j^4 L_1^4} + f_{4j}\left(\frac{-6\sin\ \beta_j L_1}{\beta_j^2 L_1^2} + \frac{24}{\beta_j^3 L_1^3} - \frac{24\sin\ \beta_j L_1}{\beta_j^4 L_1^4}\right)\right] \tag{3.324}$$

Equation (3.324) is the equation for the arbitrary constant A_j for a pinned-free beam.

b) **Derivation of the expression for the generalized mass M_j for a pinned-free beam**

By substituting the general modal equation (3.303) into equation (3.165)

$$M_j = \mu\int_0^L \left(\sinh\beta_j x + f_{4j}\sin\beta_j x\right)^2 dx$$

$$= \mu\int_0^L \left(2\sinh\beta_j x\cosh\beta_j x + 2f_{4j}\sin\beta_j x\cosh\beta_j x + \sinh^2\beta_j x - \right.$$

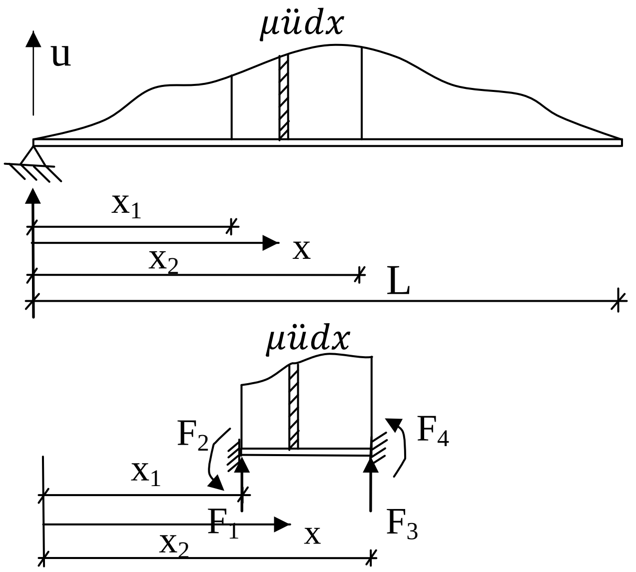
$$\left.2\sinh\beta_j x\cos\beta_j x + 2f_{4j}\sin\beta_j x\sinh\beta_j x - 2f_{4j}\cos\beta_j x\sin\beta_j x + f_{4j}^2\sin^2\beta_j x\right)dx$$

$$= \frac{\mu L}{2}\left[\left(\frac{\sinh\ \beta L}{2\beta L} - 1\right) + f_{4j}^2\left(1 - \frac{\sin\ \beta L}{2\beta L}\right) + 2f_{4j}\left(\frac{1 - \cos\ \beta L\ \cosh\ \beta L + \sin\ \beta L\ \sinh\ \beta L - \sin^2 \beta L}{\beta L}\right) - 2\left(\frac{\sinh\ \beta L\ \sin\ \beta L + \cosh\ \beta L\ \cos\ \beta L - \sinh^2 \beta L - 1}{\beta L}\right) + 2f_{4j}\left(\frac{\cosh\ \beta L\ \sin\ \beta L - \sinh\ \beta L\ \cos\ \beta L}{\beta L}\right)\right] \quad (3.325)$$

Equation (3.325) is the equation of the generalized mass of a pinned-free beam for the j^{th} mode of vibration.

Having derived the equations of the fixed-end forces for a segment of a pinned-free beam under free vibration, a summary of the equations are presented in Table 3.7. As in previous Tables the distances x_1 and x_2 were normalized using equations (3.35) and (3.36) and square of the j^{th} natural frequency ω_j^2 was eliminated from the equations by substituting equation (3.173)

Table 3.7: Summary of Fixed-end forces on a Segment of a Fixed-free Beam under Free Lateral vibration

S/N	**Description**	**Remarks**
1	Pinned-Free Beam  For $0 \leq \xi_1 < 1$ or $0 \leq x_1 < L$ $0 < \xi_2 \leq 1$ or $0 < x_2 \leq L$ $\xi_2 > \xi_1$ $\xi_1 = {}^{x_1}/_{L}$ $\xi_2 = {}^{x_2}/_{L}$ $F_1 = -6\sum_{j=1}^{\infty} \frac{EIA_j}{L^3(\xi_2-\xi_1)^3} W_1$ Where $W_1 = \beta_j^3 L^3 \left(\frac{(\xi_2+\xi_1)(\xi_2^2-\xi_1^2)}{2} - \frac{2(\xi_2^3-\xi_1^3)}{3}\right)\big(-\cosh\beta_j L\xi_1 +$ $f_{4j}\cos\beta_j L\xi_1\big) + \big(\beta L(\xi_1-\xi_2)\cosh\beta_j L\xi_2 + 2\sinh\beta_j L\xi_2 -$ $\beta_j L(\xi_2-\xi_1)\cosh\beta_j L\xi_1 - 2\sinh\beta_j L\xi_1\big) - f_{4j}\big(-\beta_j L(\xi_1 -$	See Equation (3.305a)

	$\xi_2) \cos \beta_j L\xi_2 - 2 \sin \beta_j L\xi_2 + \beta_j L(\xi_2 - \xi_1) \cos \beta_j L\xi_1 +$ $2 \sin \beta_j L\xi_1)$	
	$F_2 = -2 \sum_{j=1}^{\infty} \frac{EIA_j}{L^2(\xi_2-\xi_1)^2} W_2$ Where $W_2 =$ $\beta_j^3 L^3 \left(\frac{(2\xi_2+\xi_1)(\xi_2^2-\xi_1^2)}{2} - \frac{\xi_1(\xi_2-\xi_1)^2}{2} - \right.$ $\left. (\xi_2^3 - \xi_1^3) \right) \left(-\cosh \beta_j L\xi_1 + f_{4j} \cos \beta_j L\xi_1 \right) -$ $\left(\frac{\beta_j^2 L^2 (\xi_2-\xi_1)^2}{2} \right) \left(\sinh \beta_j L\xi_1 - f_{4j} \sin \beta_j L\xi_1 \right) + \left(\beta_j L(\xi_1 - \right.$ $\xi_2) \cosh \beta_j L\xi_2 - 2\beta_j L(\xi_2 - \xi_1) \cosh \beta_j L\xi_1 + 3 \sinh \beta_j L\xi_2 -$ $3 \sinh \beta_j L\xi_1) - f_{4j} \left(-\beta_j L(\xi_1 - \xi_2) \cos \beta_j L\xi_2 + 2\beta_j L(\xi_2 - \right.$ $\xi_1) \cos \beta_j L\xi_1 - 3 \sin \beta_j L\xi_2 + 3 \sin \beta_j L\xi_1)$	See Equation (3.306a)
	$F_3 = \sum_{j=1}^{\infty} \frac{EIA_j}{L^3(\xi_2-\xi_1)^3} \left[6W_1 + \beta_j^3 L^3 (\xi_2 - \xi_1)^3 \left(\left(\cosh \beta_j L\xi_2 - \right. \right. \right.$ $\left. \left. \left. \cosh \beta_j L\xi_1 \right) - f_{4j} \left(\cos \beta_j L\xi_2 - \cos \beta_j L\xi_1 \right) \right) \right]$	See Equation (3.307a)
	$F_4 =$ $\sum_{j=1}^{\infty} \frac{EIA_j}{L^2(\xi_2-\xi_1)^2} \left[-6W_1 + 2W_2 - \right.$ $\beta_j^3 L^3 (\xi_2 - \xi_1)^3 \left(-\cosh \beta_j L\xi_1 + f_{4j} \cos \beta_j L\xi_1 \right) + \left(\sinh \beta_j L\xi_2 - \right.$ $\left. f_{4j} \sin \beta_j L\xi_2 \right) - \left(\sinh \beta_j L\xi_1 - f_{4j} \sin \beta_j L\xi_1 \right) \Big]$	See Equation (3.308)
	$A_j = \frac{d\mu L_1^2}{M_j} \left[\frac{-10 \cosh 2\beta_j L_1}{\beta_j L_1} + \frac{19 \sinh 2\beta_j L_1}{\beta_j^2 L_1^2} - \frac{24 \cosh 2\beta_j L_1}{\beta_j^3 L_1^3} + \right.$	See Equation

	$\frac{24\sinh\ 2\beta_j L_1}{\beta_j^4 L_1^4} - \frac{24\cosh\ 2\beta_j L_1}{\beta_j^5 L_1^5} + f_{4j}\left(\frac{10\cos\ 2\beta_j L_1}{\beta_j L_1} - \frac{19\sin\ 2\beta_j L_1}{\beta_j^2 L_1^2} + \right.$ $\left.\frac{24\cos\ 2\beta_j L_1}{\beta_j^3 L_1^3} + \frac{24\sin\ 2\beta_j L_1}{\beta_j^4 L_1^4} + \frac{24\cosh\ 2\beta_j L_1}{\beta_j^5 L_1^5}\right)\Big] - \frac{d\mu L_1^2}{M_j}\Big[\frac{24}{\beta_j^3 L_1^3} +$ $\frac{24\sinh\ \beta_j L_1}{\beta_j^4 L_1^4} + f_{4j}\left(\frac{-6\sin\ \beta_j L_1}{\beta_j^2 L_1^2} + \frac{24}{\beta_j^3 L_1^3} - \frac{24\sin\ \beta_j L_1}{\beta_j^4 L_1^4}\right)\Big]$	(3.324)
	$M_j =$ $\frac{\mu L}{2}\Big[\left(\frac{\sinh\ \beta L}{2\beta L} - 1\right) + f_{4j}^2\left(1 - \frac{\sin\ \beta L}{2\beta L}\right) +$ $2f_{4j}\left(\frac{1-\cos\ \beta L\ \cosh\ \beta L + \sin\ \beta L\ \sinh\ \beta L - \sin^2\beta L}{\beta L}\right) -$ $2\left(\frac{\sinh\ \beta L\ \sin\ \beta L + \cosh\ \beta L\ \cos\ \beta L - \sinh^2\beta L - 1}{\beta L}\right) +$ $2f_{4j}\left(\frac{\cosh\ \beta L\ \sin\ \beta L - \sinh\ \beta L\ \cos\ \beta L}{\beta L}\right)\Big]$	See Equation (3.325)
	$f_{2j} = \frac{\sinh\ \beta_j L}{\sin\ \beta_j L}$	See Equation (3.304)
	$\beta_1 L = 0, \beta_2 L = 3.92660232,\ \beta_3 L = 7.06858275,$ $\beta_4 L = 10.210176123,$ $\beta_5 L = 13.35176878, \beta_6 L = 16.49336143135,$ $\beta_7 L = 19.634954084937$etc.	See equation (3.300)
Note: d is a dimensionless constant and can be taken as equal to unity		

Chapter Four

CALCULATION OF THE STIFFNESS MODIFICATION FACTORS FOR SEGMENTS OF A VIBRATING SYSTEM

The equations for calculating the fixed end forces for segments of a vibrating beam are presented in Tables 3.1 to 3.6. With these equations the force equilibrium equations for segments of a vibrating beam can be written and the inherent forces in the system that are causing motion calculated at the nodes/junctions of the element. An arbitrary segment of a vibrating element is identified by means of the normalized distances ξ_1 and ξ_2 of its nodes from an origin. ξ_1 and ξ_2 are numbers between 0 and 1.

4.1 Longitudinally vibrating bars

The equations of the fixed end forces F_1 and F_2 of segments of a longitudinally vibrating bar are presented in Tables 3.1 and 3.2. The force equilibrium equations for a segment of a longitudinally vibrating bar can be written as

$$\{F\} + [k]\{u\} = \{P\} \quad (4.1)$$

Where {F} is the fixed end forces, [k] is the stiffness of the segment under consideration and {u} is a vector of nodal displacements.

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad (4.2)$$

$$[k] = \begin{bmatrix} \frac{EA}{l} & -\frac{EA}{l} \\ -\frac{EA}{l} & \frac{EA}{l} \end{bmatrix} \quad (4.3)$$

$$\{u\} = \begin{Bmatrix} u(x_1,0) \\ u(x_2,0) \end{Bmatrix} = \begin{Bmatrix} u(\xi_1,0) \\ u(\xi_2,0) \end{Bmatrix}$$

$$= \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad (4.4)$$

E is the young's modulus of elasticity of the material of the segment and A is the cross sectional area of the bar. l is the length of the segment and is equal to $x_2 - x_1$ but since the length of the bar L has been normalized and is now equal to unity

$$l = \xi_2 - \xi_1 \tag{4.5}$$

u_1 is the total displacement at the position x_1 while u_2 is the total displacement at the position x_2. The total displacement is obtained by totaling the displacements due to all the modes of vibration.

{P} is the vector of nodal forces; they represent the forces acting on the nodes of the isolated segment.

$$\{P\} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \tag{4.6}$$

From equation (4.1) P_1 and P_2 can be expressed as

$$P_1 = F_1 + \frac{EA}{l}u_1 - \frac{EA}{l}u_2$$

$$= F_1 + \frac{EA}{\xi_2 - \xi_1}(u_1 - u_2) \tag{4.6a}$$

$$P_2 = F_2 - \frac{EA}{l}u_1 + \frac{EA}{l}u_2$$

$$= F_2 + \frac{EA}{\xi_2 - \xi_1}(-u_1 + u_2) \tag{4.6b}$$

Just like we do in the decomposition of structures, a segment of a vibrating bar can be isolated and will be in equilibrium with the application of the force vector {P}. The force {P} represents the effect of the removed adjourning elements on the isolated segment.

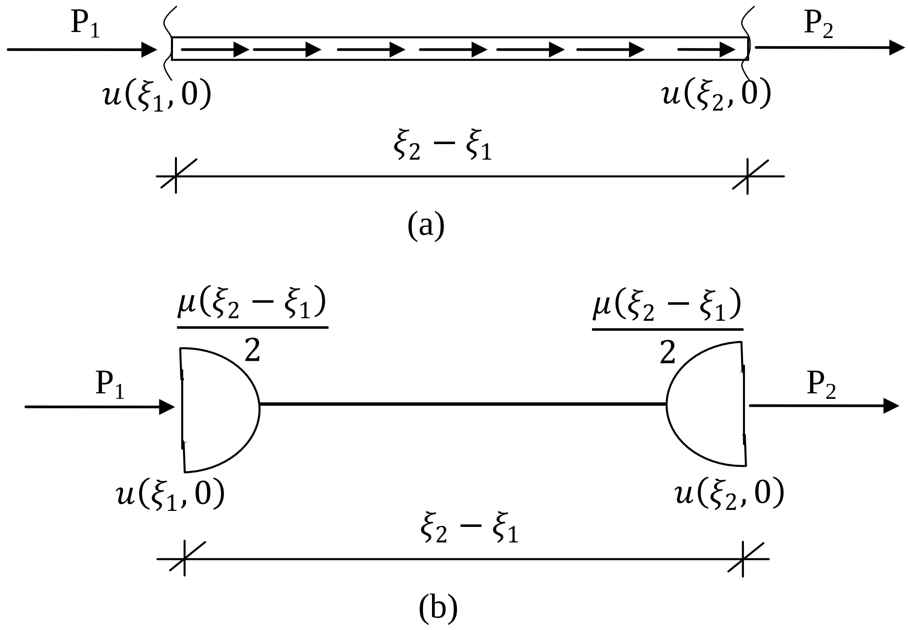


Figure 4.1

(a) An isolated segment of the longitudinally vibrating continuous bar showing the nodal forces P_1 and P_2

(b) An equivalent lumped massed segment showing the nodal forces

Figure 4.1a shows a segment of the vibrating continuous or real bar. The nodal forces on the bar P_1 and P_2 are calculated from the equilibrium equations (equation (4.1)). When the continuous bar is represented by a lumped massed bar (a bar that has its distributed masses lumped at selected nodes), the equivalent segment of the bar is shown in Figure 4.1b. Just like the real segment the equivalent segment is supported by the same nodal forces P_1 and P_2 and has the same nodal displacements as the real bar. This implies that for the lumped massed beam to be equivalent to the real beam they must share the same inherent forces and displacements at the nodes.

The equation of motion for the lumped massed bar is given as

$$[m]\{\ddot{u}\} + [k_d]\{u\} = \{P\} \tag{4.7}$$

Where [m] is the inertial matrix, {u} is a vector of nodal displacement and k_d is the stiffness of the lumped massed segment under consideration.

The proposed stiffness matrix for the lumped massed segment k_d is

$$[k_d] = \begin{bmatrix} \frac{EA}{l}\alpha_1 & -\frac{EA}{l}\alpha_2 \\ -\frac{EA}{l}\alpha_2 & \frac{EA}{l}\alpha_1 \end{bmatrix} \tag{4.8}$$

where α_1 and α_2 are the stiffness modification factors for longitudinal vibration. They help redistribute the stiffness of the lumped massed bar in such a way as to annul the effect of the discretization of the bar due to the lumping of its distributed mass on selected nodes.

$$[m] = \begin{bmatrix} \frac{\mu(\xi_2-\xi_1)}{2} & 0 \\ 0 & \frac{\mu(\xi_2-\xi_1)}{2} \end{bmatrix} \tag{4.9}$$

μ is the mass per unit length of the beam.

When treating the isolated segment of the vibrating beam alone the vector of nodal acceleration is written as

$$\{\ddot{u}\} = \begin{Bmatrix} \ddot{u}(\xi_1,0) \\ \ddot{u}(\xi_2,0) \end{Bmatrix} = \begin{Bmatrix} -\omega^2 u(\xi_1,0) \\ -\omega^2 u(\xi_2,0) \end{Bmatrix} = \begin{Bmatrix} -\omega^2 u_{11} \\ -\omega^2 u_{21} \end{Bmatrix} \tag{4.10}$$

ω is the fundamental frequency of the vibrating mass while u_{11} and u_{21} are the values of u_1 and u_2 for the first mode only.

By substituting equations (4.8) to (4.10) into equation (4.7) we obtain

$$\begin{bmatrix} \frac{\mu(\xi_2-\xi_1)}{2} & 0 \\ 0 & \frac{\mu(\xi_2-\xi_1)}{2} \end{bmatrix} \begin{Bmatrix} -\omega^2 u_{11} \\ -\omega^2 u_{21} \end{Bmatrix} + \begin{bmatrix} \frac{EA}{l}\alpha_1 & -\frac{EA}{l}\alpha_2 \\ -\frac{EA}{l}\alpha_2 & \frac{EA}{l}\alpha_1 \end{bmatrix} \begin{Bmatrix} u_{11} \\ u_{21} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \tag{4.11}$$

By substituting equation (4.5) into (4.11) and multiplying out the first row of equation (4.11)

$$-\frac{\mu(\xi_2-\xi_1)\omega^2 u_{11}}{2}+\frac{EA}{\xi_2-\xi_1}\alpha_1 u_{11}-\frac{EA}{\xi_2-\xi_1}\alpha_2 u_{21}=P_1$$

$$\alpha_1 u_{11}-\alpha_2 u_{21}=\frac{\xi_2-\xi_1}{EA}\left(P_1+\frac{\mu(\xi_2-\xi_1)\omega^2 u_{11}}{2}\right) \tag{4.12}$$

Similarly by multiplying out the second row of equation (4.11)

$$-\frac{\mu(\xi_2-\xi_1)\omega^2 u_{21}}{2}-\frac{EA}{\xi_2-\xi_1}\alpha_2 u_{11}+\frac{EA}{\xi_2-\xi_1}\alpha_1 u_{21}=P_2$$

$$\alpha_1 u_{21}-\alpha_2 u_{11}=\frac{\xi_2-\xi_1}{EA}\left(P_2+\frac{\mu(\xi_2-\xi_1)\omega^2 u_{21}}{2}\right) \tag{4.13}$$

Put in matrix form equation (4.12) and (4.13) can be written as

$$\begin{bmatrix} u_{11} & -u_{21} \\ u_{21} & -u_{11} \end{bmatrix}\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}=\begin{Bmatrix} \frac{\xi_2-\xi_1}{EA}\left(P_1+\frac{\mu(\xi_2-\xi_1)\omega^2 u_{11}}{2}\right) \\ \frac{\xi_2-\xi_1}{EA}\left(P_2+\frac{\mu(\xi_2-\xi_1)\omega^2 u_{21}}{2}\right) \end{Bmatrix} \tag{4.14}$$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}=\begin{bmatrix} u_{11} & -u_{21} \\ u_{21} & -u_{11} \end{bmatrix}^{-1}\begin{Bmatrix} \frac{\xi_2-\xi_1}{EA}\left(P_1+\frac{\mu(\xi_2-\xi_1)\omega^2 u_{11}}{2}\right) \\ \frac{\xi_2-\xi_1}{EA}\left(P_2+\frac{\mu(\xi_2-\xi_1)\omega^2 u_{21}}{2}\right) \end{Bmatrix}$$

$$=\frac{1}{(u_2^2-u_1^2)}\begin{bmatrix} -u_{11} & u_{21} \\ -u_{21} & u_{11} \end{bmatrix}\begin{Bmatrix} \frac{\xi_2-\xi_1}{EA}\left(P_1+\frac{\mu(\xi_2-\xi_1)\omega^2 u_{11}}{2}\right) \\ \frac{\xi_2-\xi_1}{EA}\left(P_2+\frac{\mu(\xi_2-\xi_1)\omega^2 u_{21}}{2}\right) \end{Bmatrix} \tag{4.14}$$

From equation (4.14) α_1 and α_2 can be expressed as

$$\alpha_1=\frac{-\frac{\xi_2-\xi_1}{EA}u_{11}\left(P_1+\frac{\mu(\xi_2-\xi_1)\omega^2 u_{11}}{2}\right)+\frac{\xi_2-\xi_1}{EA}u_{21}\left(P_2+\frac{\mu(\xi_2-\xi_1)\omega^2 u_{21}}{2}\right)}{u_{21}^2-u_{11}^2} \tag{4.15}$$

$$\alpha_2=\frac{\frac{\xi_2-\xi_1}{EA}u_{11}\left(P_2+\frac{\mu(\xi_2-\xi_1)\omega^2 u_{21}}{2}\right)-\frac{\xi_2-\xi_1}{EA}u_{21}\left(P_1+\frac{\mu(\xi_2-\xi_1)\omega^2 u_{11}}{2}\right)}{u_{21}^2-u_{11}^2} \tag{4.16}$$

4.1.1 Fixed-fixed, fixed-pinned and pinned-pinned bars

These are bars restrained from longitudinal vibration at the ends. From section 3.1.1 the first natural frequency w for such a bar is given as

$$\omega = \pi\sqrt{\frac{EA}{\mu L^2}} \tag{4.17}$$

In normalized coordinates, the length of the bar L = 1 hence

$$\omega = \pi\sqrt{\frac{EA}{\mu}} \tag{4.17a}$$

Likewise

$$u_1 = u(x_1, 0) = \textstyle\sum_{j=1}^{\infty} A_j \sin\frac{j\pi x_1}{L}$$

$$= \textstyle\sum_{j=1}^{\infty} A_j \sin j\pi\xi_1 \tag{4.18}$$

$$u_2 = u(x_2, 0) = \textstyle\sum_{j=1}^{\infty} A_j \sin\frac{j\pi x_2}{L}$$

$$= \textstyle\sum_{j=1}^{\infty} A_j \sin j\pi\xi_2 \tag{4.19}$$

From Table 3.1

$$A_j = 2eL\left(\frac{2-2(-1)^j}{j^3\pi^3}\right) = 2\left(\frac{2-2(-1)^j}{j^3\pi^3}\right) \tag{4.20}$$

$$j = 1,2,3,4,5, \dots, \infty$$

Even though equations (4.18) and (4.19) are infinite series, only the first few terms needs to be evaluated to get values of very good precision. This is so because the series converge very quickly.

By substituting equation (4.17a) into equations (4.15) and (4.16) and taking EA = 1 we obtain

$$\alpha_1 = \frac{-(\xi_2-\xi_1)u_{11}\left(P_1+\frac{(\xi_2-\xi_1)\pi^2 u_{11}}{2}\right)+(\xi_2-\xi_1)u_{21}\left(P_2+\frac{(\xi_2-\xi_1)\pi^2 u_{21}}{2}\right)}{u_{21}^2-u_{11}^2} \quad (4.21)$$

$$\alpha_2 = \frac{(\xi_2-\xi_1)u_{11}\left(P_2+\frac{(\xi_2-\xi_1)\pi^2 u_{21}}{2}\right)-(\xi_2-\xi_1)u_{21}\left(P_1+\frac{(\xi_2-\xi_1)\pi^2 u_{11}}{2}\right)}{u_{21}^2-u_{11}^2} \quad (4.22)$$

Equations (4.21) and (4.22) can be used to evaluate the stiffness modification factor for longitudinal vibration of a segment of a fixed-fixed or fixed-pinned bar located between ξ_1 and ξ_2 of the bar's total length. Some numerical demonstrations of their use are presented below. For ease of presentation the calculations will be presented in a tabular form.

Example 1: when $\xi_1 = 0$, $\xi_2 = 0.3$

Table 4.1: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0$, $\xi_2 = 0.3$ on a fixed-fixed bar under longitudinal vibration

Using the equations presented in Table 3.1

$\xi_1 = 0,\ \ \xi_2 = 0.3$					
j	Aj	F_{1j}	F_{2j}	u_{1j}	u_{2j}
1	0.25801227546560	0.11478175044198	0.21934693877846	0	0.20873631560902
2	0	0	0	0	0
3	0.00955601020243	0.08022004251217	0.09549849578482	0	0.00295296955097
4	0	0	0	0	0
5	0.00206409820372	0.03930310611130	-0.0068803273457	0	-0.0020640982037
6	0	0	0	0	0
7	0.00075222237745	0.01576740240345	-0.0149577678401	0	0.00023244949818
8	0	0	0	0	0

9	0.00035392630379	0.00905258916811	0.00683642625235	0	0.00028633239453
Total		**0.25912489063701**	**0.29984376562978**	**0**	**0.21014396884897**

$u_{11} = 0$

$u_{21} = 0.20873631560902$

From equations 4.6a and 4.6b

$P_1 = -0.95960478680024$

$P_2 = 0.40063613053345$

From equations 4.21 and 4.22 the stiffness modification factors for longitudinal vibration of the element are

$\alpha_1 = 1.01993444314165$

$\alpha_2 = 1.37916315711590$

j is the mode number, j = 1 stands for the first mode, j = 2 for the second mode and so on. The values of the paramaters A_j, F_{1j}, F_{2j}, u_{1j} and u_{2j} are evaluated for modes 1 to 9 and summed using the equations present in Table 3.1 to obtain end forces F_1 and F_2 and the end displacements u_1 and u_2. The same procedure is applied to the other examples that follow.

Table 4.2: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.3$, $\xi_2 = 0.5$ on a fixed-fixed bar under longitudinal vibration

Using the equations presented in Table 3.1

$\xi_1 = 0.3$, $\xi_2 = 0.5$					
j	Aj	F_{1j}	F_{2j}	u_{1j}	u_{2j}
1	0.25801227546560	0.23006098063538	0.24637979928289	0.20873631560902	0.25801227546560
2	0	0	0	0	0
3	0.00955601020243	-0.0231103651813	-0.0625448987670	0.00295296955097	-0.0095560102024

4	0	0	0	0	0
5	0.00206409820372	-0.0206409820373	0.02064098203725	-0.0020640982037	0.0020640982037
6	0	0	0	0	0
7	0.00075222237745	0.02065595887886	-0.0049233593782	0.00023244949818	-0.0007522223775
8	0	0	0	0	0
9	0.00035392630379	-0.0062199544836	0.00033796954634	0.00028633239453	0.00035392630379
Total		**0.20074563781213**	**0.19989049272132**	**0.21014396884897**	**0.25012206739323**

u_{11} = 0.20873631560902

u_{21} = 0.25801227546560

From equations 4.6a and 4.6b

P_1 = -0.40063613053345

P_2 = 0

From equations 4.21 and 4.22 the stiffness modification factors for longitudinal vibration of the element are

α_1 = 0.92460237451664

α_2 = 0.89888134804473

Table 4.3: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.5$, $\xi_2 = 1.0$ on a fixed-fixed bar under longitudinal vibration

Using the equations presented in Table 3.1

$\xi_1 = 0.5,\ \ \xi_2 = 1$					
j	Aj	F_{1j}	F_{2j}	u_{1j}	u_{2j}
1	0.25801227546560	0.51602455093119	0.29454491820751	0.25801227546560	0
2	0	0	0	0	0
3	0.00955601020243	-0.0191120204049	0.10917529475360	-0.0095560102024	0
4	0	0	0	0	0
5	0.00206409820372	0.00412819640745	0.02829458235810	0.0020640982037	0

6	0	0	0	0	0
7	0.00075222237745	-0.0015044447549	0.01804667881896	-0.0007522223775	0
8	0	0	0	0	0
9	0.00035392630379	0.00070785260759	0.00929917787561	0.00035392630379	0
Total		**0.50024413478647**	**0.45936065201378**	**0.25012206739323**	**0**

u_{11} = 0.25801227546560

u_{21} = 0

From equations 4.6a and 4.6b

P_1 = 0

P_2 = -0.95960478680024

From equations 4.21 and 4.22 the stiffness modification factors for longitudinal vibration of the element are

α_1 = 1.23370055013617

α_2 = 1.85961072020428

Table 4.4: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0$, $\xi_2 = 0.2$ on a fixed-fixed bar under longitudinal vibration

Using the equations presented in Table 3.1

$\xi_1 = 0, \ \ \xi_2 = 0.2$					
j	Aj	F_{1j}	F_{2j}	u_{1j}	u_{2j}
1	0.25801227546560	0.05229041699320	0.10251457649081	0	0.15165581042910
2	0	0	0	0	0
3	0.00955601020243	0.04462174548473	0.07327261120683	0	0.00908830577280
4	0	0	0	0	0
5	0.00206409820372	0.03242277876555	0.03242277876555	0	0
6	0	0	0	0	0
7	0.00075222237745	0.02011926403294	0.00153480148183	0	-0.0007154059938

8	0	0	0	0	0
9	0.00035392630379	0.01104719379204	-0.0091360210330	0	-0.0002080326618
Total		**0.16050139906845**	**0.20060874691205**	**0**	**0.15982067754636**

$u_{11} = 0$

$u_{21} = 0.15165581042910$

From equations 4.6a and 4.6b

$P_1 = -0.95960478680024$

$P_2 = 0.59849464081974$

From equations 4.21 and 4.22 the stiffness modification factors for longitudinal vibration of the element are

$\alpha_1 = 0.98667228655336$

$\alpha_2 = 1.26550348988951$

Table 4.5: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.2$, $\xi_2 = 0.6$ on a fixed-fixed bar under longitudinal vibration

Using the equations presented in Table 3.1

$\xi_1 = 0.2, \quad \xi_2 = 0.6$					
j	Aj	F_{1j}	F_{2j}	u_{1j}	u_{2j}
1	0.25801227546560	0.42144336206320	0.48480085467683	0.15165581042910	0.24538425586570
2	0	0	0	0	0
3	0.00955601020243	0.00893188675855	-0.1096256886186	0.00908830577280	-0.0056168818677
4	0	0	0	0	0
5	0.00206409820372	0.03242277876555	0.03242277876555	0	0
6	0	0	0	0	0
7	0.00075222237745	-0.0080057094849	-0.0104890704485	-0.0007154059938	0.00044214521991
8	0	0	0	0	0
9	0.00035392630379	0.00841728586349	0.00277091434318	-0.0002080326618	-0.0003366039175

Total	**0.39836404643476**	**0.39987978871846**	**0.15982067754636**	**0.23987291530035**

u_{11} = 0.15165581042910

u_{21} = 0.24538425586570

From equations 4.6a and 4.6b

P_1 = -0.59849464081974

P_2 = -0.19974919433349

From equations 4.21 and 4.22 the stiffness modification factors for longitudinal vibration of the element are

α_1 = 1.23832355716995

α_2 = 1.25294994804916

Table 4.6: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.6$, $\xi_2 = 1$ on a fixed-fixed bar under longitudinal vibration

Using the equations presented in Table 3.1

$\xi_1 = 0.6, \quad \xi_2 = 1$					
j	Aj	F_{1j}	F_{2j}	u_{1j}	u_{2j}
1	0.25801227546560	0.36298089857890	0.19710882947446	0.24538425586570	0
2	0	0	0	0	0
3	0.00955601020243	0.05882051484783	0.10410547901811	-0.0056168818677	0
4	0	0	0	0	0
5	0.00206409820372	-0.0324227787656	0.03242277876555	0	0
6	0	0	0	0	0
7	0.00075222237745	0.01448831153252	0.01543687101428	0.00044214521991	0
8	0	0	0	0	0
9	0.00035392630379	-0.0039338522763	0.01084854027697	-0.0003366039175	0
Total		**0.39993309391739**	**0.35992249854937**	**0.23987291530035**	**0**

u_{11} = 0.24538425586570

u_{21} = 0

From equations 4.6a and 4.6b

P_1 = 0.19974919433348

P_2 = -0.95960478680024

From equations 4.21 and 4.22 the stiffness modification factors for longitudinal vibration of the element are

α_1 = 1.11517880110114

α_2 = 1.56424833926666

Tables 4.1 to 4.6 are illustrations on how the inherent nodal forces P_1 and P_2 and the stiffness modification factors α_1 and α_2 are calculated. The nodal forces P_1 and P_2 are the forces acting at the selected nodal point if the beam segment under consideration is decomposed. These nodal forces represent the effect of the removed adjacent beam segment on the beam segment under consideration. Using the methods presented in Table 4.1 to 4.6 the values of stiffness modification factors at different values of ξ_1 and ξ_2 for the longitudinal vibration of a fixed-fixed bar are presented in Table 4.7. A sample matlab program for the calculation of the stiffness modification factors for a segment of a beam restrained at both end can be found in Appendix B.

Table 4.7: Stiffness modification factors for the longitudinal vibration of a fixed-fixed/fixed-pinned/pinned-pinned bar

			ξ_2						
			0	0.05	0.10	0.15	0.20	0.25	0.30
ξ_1	0	α_1	-	1.143151	1.058495	0.997834	0.986672	1.000866	1.019934
		α_2	-	1.188747	1.203565	1.228843	1.265503	1.314943	1.379163
	0.05	α_1	1.143151	-	1.080427	1.024910	1.006729	1.012629	1.026512
		α_2	1.188747	-	1.113157	1.115460	1.141255	1.181044	1.232144
	0.10	α_1	1.058495	1.080427	-	0.998704	0.980187	0.981916	0.991600
		α_2	1.203565	1.113157	-	1.014836	1.019911	1.042110	1.074280
	0.15	α_1	0.997834	1.024910	0.998704	-	0.938539	0.935178	0.938885
		α_2	1.228843	1.115460	1.014836	-	0.943689	0.948312	0.962200
	0.20	α_1	0.986672	1.006729	0.980187	0.938539	-	0.901715	0.897092
		α_2	1.265503	1.141255	1.019911	0.943689	-	0.903322	0.902645
	0.25	α_1	1.000866	1.012629	0.981916	0.935178	0.901715	-	0.867017
		α_2	1.314943	1.181044	1.042110	0.948312	0.903322	-	0.868061
	0.30	α_1	1.019934	1.026512	0.991599	0.938885	0.897092	0.867017	-
		α_2	1.379163	1.232144	1.074280	0.962200	0.902645	0.868061	-
	0.35	α_1	1.054605	1.057992	1.018826	0.959383	0.908659	0.867950	0.829199
		α_2	1.460963	1.299001	1.121355	0.990718	0.916692	0.869455	0.828832
	0.40	α_1	1.115179	1.117462	1.074506	1.008167	0.948618	0.897608	0.846355
		α_2	1.564248	1.386386	1.188496	1.039554	0.952085	0.893711	0.841244
	0.45	α_1	1.182917	1.189762	1.146079	1.074437	1.007395	0.947269	0.884015
		α_2	1.694512	1.496212	1.274497	1.104955	1.003103	0.933439	0.868971
	0.50	α_1	1.233701	1.254602	1.215722	1.141712	1.069081	1.000866	0.924602
		α_2	1.859611	1.632023	1.379063	1.183281	1.063227	0.979259	0.898881
	0.55	α_1	1.268354	1.315400	1.288508	1.216097	1.140810	1.066697	0.976738
		α_2	2.071070	1.803974	1.511273	1.283034	1.141149	1.040586	0.940450
	0.60	α_1	1.288113	1.378511	1.374214	1.310002	1.238324	1.165702	1.068050
		α_2	2.346373	2.027226	1.685114	1.418270	1.252950	1.137999	1.020543
	0.65	α_1	1.249068	1.411994	1.447670	1.401772	1.343039	1.284923	1.195354
		α_2	2.713217	2.318450	1.909302	1.591796	1.398286	1.272354	1.146426
	0.70	α_1	1.074514	1.359921	1.462504	1.446515	1.407191	1.374004	1.310103*
		α_2	3.218047	2.702364	2.192812	1.798719	1.559662	1.416893	1.288271*
	0.75	α_1	0.698504	1.194162	1.408082	1.436868	1.416483	1.406445*	1.374004
		α_2	3.944830	3.23164	2.570783	2.059370	1.740833	1.557625*	1.416893
	0.80	α_1	0.001153	0.883447	1.298618	1.409046	1.412313*	1.416483	1.407191
		α_2	5.062014	4.010719	3.125278	2.446031	2.003625*	1.740833	1.559662
	0.85	α_1	-1.459811	0.229120	1.047829	1.340164*	1.409046	1.436868	1.446515
		α_2	6.963446	5.225115	3.975516	3.060516*	2.446031	2.059370	1.798719
	0.90	α_1	-5.085133	-1.343961	0.348648*	1.047829	1.298618	1.408082	1.462504
		α_2	10.832088	7.246460	5.263551*	3.975516	3.125258	2.570783	2.192812
	0.95	α_1	-17.031813	-5.197933*	-1.343961	0.229120	0.883447	1.194162	1.359921
		α_2	22.586202	11.159537*	7.246460	5.225115	4.010719	3.231645	2.702364
	1.00	α_1	-	-17.031813	-5.085133	-1.459811	0.001153	0.698504	1.074515
		α_2	-	22.586202	10.832088	6.963446	5.062014	3.944830	3.218047
			ξ_2						

			0.35	0.40	0.45	0.50	0.55	0.60	0.65
ξ_1	0	α_1	1.054605	1.115179	1.182917	1.233701	1.268354	1.288113	1.249068
		α_2	1.460963	1.564248	1.694512	1.859611	2.071070	2.346373	2.713217
	0.05	α_1	1.057992	1.117462	1.117462	1.254602	1.315400	1.378511	1.411994
		α_2	1.299001	1.386386	1.386386	1.632023	1.803974	2.027226	2.318450
	0.10	α_1	1.018826	1.074506	1.074506	1.215722	1.288508	1.374214	1.447670
		α_2	1.121355	1.188496	1.188496	1.379063	1.511273	1.685114	1.909302
	0.15	α_1	0.959383	1.008167	1.008167	1.141712	1.216097	1.310002	1.401772
		α_2	0.990718	1.039554	1.039554	1.183281	1.283034	1.418270	1.591796
	0.20	α_1	0.908659	0.948618	0.948618	1.069081	1.140810	1.238324	1.343039
		α_2	0.916692	0.952085	0.952085	1.063227	1.141149	1.252950	1.398286
	0.25	α_1	0.867949	0.897608	0.897608	1.000866	1.066697	1.165702	1.284923
		α_2	0.869455	0.893711	0.893711	0.979259	1.040586	1.137999	1.272354
	0.30	α_1	0.829199	0.846355	0.846355	0.924602	0.976738	1.068050	1.195354
		α_2	0.828832	0.841244	0.841244	0.898881	0.940450	1.020543	1.146426
	0.35	α_1	-	0.806627	0.806627	0.854040	0.883238	0.951720	1.069273*
		α_2	-	0.804377	0.804377	0.833896	0.850730	0.903873	1.010934*
	0.40	α_1	0.806627	-	0.810954	0.820405	0.825416	0.861619*	0.951720
		α_2	0.804377	-	0.808189	0.810737	0.805463	0.827032*	0.903873
	0.45	α_1	0.830575	0.810954	-	0.825638	0.816815*	0.825416	0.883238
		α_2	0.820765	0.808189	-	0.823439	0.808272*	0.805463	0.850730
	0.50	α_1	0.854040	0.820405	0.820405	-	0.825638	0.820405	0.854040
		α_2	0.833896	0.810737	0.810737	-	0.823439	0.810737	0.833896
	0.55	α_1	0.883238	0.825416	0.816815*	0.825638	-	0.810954	0.830575
		α_2	0.850730	0.805463	0.808272*	0.823439	-	0.808189	0.820765
	0.60	α_1	0.951720	0.861619*	0.825416	0.820405	0.810954	-	0.806627
		α_2	0.903873	0.827032*	0.805463	0.810737	0.808189	-	0.804377
	0.65	α_1	1.069273*	0.951720	0.883238	0.854040	0.830575	0.806627	-
		α_2	1.010934*	0.903873	0.850730	0.833896	0.820765	0.804377	-
	0.70	α_1	1.195354	1.068050	0.976738	0.924602	0.884015	0.846355	0.829199
		α_2	1.146426	1.020543	0.940450	0.898881	0.868971	0.841244	0.828832
	0.75	α_1	1.284923	1.165702	1.066697	1.000866	0.947269	0.897608	0.867949
		α_2	1.272354	1.137999	1.040586	0.979259	0.933439	0.893711	0.869455
	0.80	α_1	1.343039	1.238324	1.140810	1.069081	1.007395	0.948618	0.908659
		α_2	1.398286	1.252950	1.141149	1.063227	1.003103	0.952085	0.916692
	0.85	α_1	1.401772	1.310002	1.216097	1.141712	1.074437	1.008167	0.959383
		α_2	1.591796	1.418270	1.283034	1.183281	1.104955	1.039554	0.990718
	0.90	α_1	1.447670	1.374214	1.288508	1.215722	1.146079	1.074506	1.018826
		α_2	1.909302	1.685114	1.511273	1.379063	1.274497	1.188496	1.121355
	0.95	α_1	1.411994	1.378511	1.315400	1.254602	1.189762	1.117462	1.057992
		α_2	2.318450	2.027226	1.803974	1632023	1.496212	1.386386	1.299001
	1.00	α_1	1.249068	1.288113	1.268354	1.233701	1.182917	1.115179	1.054605
		α_2	2.713217	2.346373	2.071070	1.859611	1.694512	1.564248	1.460963

			ξ_2						
			0.70	0.75	0.80	0.85	0.90	0.95	1.00
ξ_1	0	α_1	1.074514	0.698504	0.001153	-1.459811	-5.085133	-17.031812	-
		α_2	3.218047	3.944830	5.062014	6.963446	10.832088	22.586202	-
	0.05	α_1	1.359921	1.194162	0.883447	0.229120	-1.343961	-5.197933*	-17.031812
		α_2	2.702364	3.231645	4.010719	5.225115	7.246460	11.159537*	22.586202
	0.10	α_1	1.462504	1.408082	1.298618	1.047829	0.348648*	-1.343961	-5.085133
		α_2	2.192812	2.570783	3.125258	3.975516	5.263551*	7.246460	10.832088
	0.15	α_1	1.446515	1.436868	1.409046	1.340164*	1.047829	0.229120	-1.459811
		α_2	1.798719	2.059370	2.446031	3.060516*	3.975516	5.225115	6.963446
	0.20	α_1	1.407191	1.416483	1.412313*	1.409046	1.298618	0.883447	0.001153
		α_2	1.559662	1.740833	2.003625*	2.446031	3.125258	4.010719	5.062014
	0.25	α_1	1.374004	1.406445*	1.416483	1.436868	1.408082	1.194162	0.698504
		α_2	1.416893	1.557625*	1.740833	2.059370	2.570783	3.231645	3.944801
	0.30	α_1	1.310103*	1.374004	1.407191	1.446515	1.462504	1.359921	1.074515
		α_2	1.288271*	1.416893	1.559662	1.798719	2.192812	2.702364	3.218047
	0.35	α_1	1.195354	1.284923	1.343039	1.401772	1.447670	1.411994	1.249068
		α_2	1.146426	1.272354	1.398286	1.591796	1.909302	2.318450	2.713217
	0.40	α_1	1.068050	1.165702	1.238324	1.310002	1.374214	1.378511	1.288113
		α_2	1.020543	1.137999	1.252950	1.418270	1.685114	2.027226	2.346373
	0.45	α_1	0.976738	1.066696	1.140810	1.216097	1.288508	1.315400	1.268354
		α_2	0.940450	1.040586	1.141149	1.283034	1.511273	1.803974	2.071070
	0.50	α_1	0.924602	1.000866	1.069081	1.141712	1.215722	1.254602	1.233701
		α_2	0.898881	0.979259	1.063227	1.183281	1.379063	1.632023	1.859611
	0.55	α_1	0.884015	0.947269	1.007395	1.074437	1.146079	1.189762	1.182917
		α_2	0.868971	0.933439	1.003103	1.104955	1.274497	`1.496212	1.694512
	0.60	α_1	0.846355	0.897608	0.948618	1.008167	1.074506	1.117462	1.115179
		α_2	0.841244	0.893711	0.952085	1.039554	1.188496	1.386386	1.564248
	0.65	α_1	0.829199	0.867949	0.908659	0.959383	1.018826	1.057992	1.054605
		α_2	0.828832	0.869455	0.916692	0.990718	1.121355	1.299001	1.460963
	0.70	α_1	-	0.867017	0.897092	0.938885	0.991599	1.026512	1.019934
		α_2	-	0.868061	0.902645	0.962200	1.074280	1.232144	1.379163
	0.75	α_1	0.867017	-	0.901715	0.935178	0.981916	1.012629	1.000866
		α_2	0.868061	-	0.903322	0.948312	1.042110	1.181044	1.314943
	0.80	α_1	0.897092	0.901715	-	0.938539	0.980187	1.006729	0.986672
		α_2	0.902645	0.903322	-	0.943689	1.019911	1.141255	1.265503
	0.85	α_1	0.938885	0.935178	0.938539	-	0.998704	1.024910	0.997834
		α_2	0.962200	0.948312	0.943689	-	1.014836	1.115460	1.228845
	0.90	α_1	0.991598	0.981916	0.980187	0.998704	-	1.080427	1.058495
		α_2	1.074280	1.042110	1.019911	1.014836	-	1.113157	1.203565
	0.95	α_1	1.026512	1.012629	1.006729	1.024910	1.080427	-	1.143151
		α_2	1.232144	1.181044	1.141255	1.115460	1.113157	-	1.188747
	1.00	α_1	1.019934	1.000866	0.986672	0.997834	1.058495	1.143151	-
		α_2	1.379163	1.314943	1.265503	1.228843	1.203565	1.188747	-

4.1.2 Fixed-free and pinned-free bars

These are bars restrained from longitudinal vibration at only one end. From section 3.1.2 the first natural frequency ω for such a bar is given as

$$\omega = \frac{\pi}{2}\sqrt{\frac{EA}{\mu L^2}} \tag{4.23}$$

In normalized coordinates, the length of the bar L = 1 hence

$$\omega = \frac{\pi}{2}\sqrt{\frac{EA}{\mu}} \tag{4.23a}$$

Likewise

$$u_1 = u(x_1, 0) = \sum_{j=1}^{\infty} A_j \sin\frac{j\pi x_1}{2L}$$

$$= \sum_{j=1}^{\infty} A_j \sin\frac{j\pi \xi_1}{2} \tag{4.24}$$

$$u_2 = u(x_2, 0) = \sum_{j=1}^{\infty} A_j \sin\frac{j\pi x_2}{2L}$$

$$= \sum_{j=1}^{\infty} A_j \sin\frac{j\pi \xi_2}{2} \tag{4.25}$$

From Table 3.2

$$A_j = \frac{16eL}{i^3\pi^3} \tag{4.26}$$

$j = 1,2,3,4,5, \dots, \infty$

$i = 1,3,5,7,9, \dots, \infty$

Just like in the previous section, only the first few terms of equations (4.24 – 4.26) needs to be evaluated to get values of very good precision. By substituting equation (4.24a) into equations (4.15) and (4.16) and taking EA = 1 we obtain

$$\alpha_1 = \frac{-(\xi_2-\xi_1)u_{11}\left(P_1+\frac{(\xi_2-\xi_1)\pi^2 u_{11}}{8}\right)+(\xi_2-\xi_1)u_{21}\left(P_2+\frac{(\xi_2-\xi_1)\pi^2 u_{21}}{8}\right)}{u_{21}^2-u_{11}^2} \tag{4.27}$$

$$\alpha_2 = \frac{(\xi_2-\xi_1)u_{11}\left(P_2+\frac{(\xi_2-\xi_1)\pi^2 u_{21}}{8}\right)-(\xi_2-\xi_1)u_{21}\left(P_1+\frac{(\xi_2-\xi_1)\pi^2 u_{11}}{8}\right)}{u_{21}^2-u_{11}^2} \tag{4.28}$$

Equations (4.27) and (4.28) can be used to evaluate the stiffness modification factor for longitudinal vibration of a segment of a fixed-free or pinned-free bar located between ξ_1 and ξ_2 of the bar's total length. Some numerical demonstrations of their use are presented below. For ease of presentation the calculations will also be presented in a tabular form.

Example 1: when $\xi_1 = 0$, $\xi_2 = 0.3$

Table 4.8: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0$, $\xi_2 = 1.0$ on a fixed-free bar under longitudinal vibration

From Table 3.1

$\xi_1 = 0,\ \ \xi_2 = 1.0$					
j	Aj	F_{1j}	F_{2j}	u_{1j}	u_{2j}
1	0.51602455093119	0.29454491820751	0.51602455093119	0	0.51602455093119
2	0.01911202040486	0.10917529475360	-0.0191120204049	0	-0.0191120204049
3	0.00412819640745	0.02829458235810	0.00412819640745	0	0.00412819640745
4	0.00150444475490	0.01804667881896	-0.0015044447549	0	-0.0015044447549
5	0.00070785260759	0.00929917787561	0.00070785260759	0	0.00070785260759
6	0.00038769688274	0.00708661811529	-0.0003876968827	0	-0.0003876968827
7	0.00023487690074	0.00456139214742	0.00023487690074	0	0.00023487690074
8	0.00015289616324	0.00375542713719	-0.0001528961632	0	-0.0001528961632
9	0.00010503247526	0.00269970617228	0.00010503247526	0	0.00010503247526
Total		**0.47746379558596**	**0.50004345111648**	**0**	**0.50004345111648**
$u_{11} = 0$					

u_{21} = 0.51602455093119 From equations 4.6a and 4.6b P_1 = -0.97750724670244 P_2 = 0 From equations 4.27 and 4.28 the stiffness modification factors for longitudinal vibration of the element are α_1 = 1.23370055013617 α_2 = 1.89430375926587

j is the mode number, j = 1 stands for the first mode, j = 2 for the second mode and so on. The values of the paramaters A_j, F_{1j}, F_{2j}, u_{1j} and u_{2j} are evaluated for modes 1 to 9 and summed using the equations present in Table 3.2 to obtain end forces F_1 and F_2 and the end displacements u_1 and u_2. The same procedure is applied to the other examples that follow.

Table 4.9: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0$, $\xi_2 = 0.5$ on a fixed-free bar under longitudinal vibration

From Table 3.1

$\xi_1 = 0, \ \xi_2 = 0.5$					
j	Aj	F_{1j}	F_{2j}	u_{1j}	u_{2j}
1	0.51602455093119	0.08080055069432	0.15660975019362	0	0.36488445922219
2	0.01911202040486	0.06303479588784	0.09071283048876	0	0.01351423923045
3	0.00412819640745	0.03826093011310	0.01708821538248	0	-0.0029190756738
4	0.00150444475490	0.01866984024028	0.01708821538248	0	-0.0010638030881
5	0.00070785260759	0.00900597572538	-0.0138247320589	0	0.00050052737891
6	0.00038769688274	0.00615063504289	-0.0060749843564	0	0.00027414309483

7	0.00023487690074	0.00512843514668	0.00528513881984	0	-0.0001660830493
8	0.00015289616324	0.00381875880164	0.00305930826983	0	-0.0001081139138
9	0.00010503247526	0.00265620029654	-0.0027636019088	0	0.00007426917550
Total		**0.22752612194867**	**0.24825721346430**	**0**	**0.37499056237688**

$u_{11} = 0$

$u_{21} = 0.36488445922219$

From equations 4.6a and 4.6b

$P_1 = -0.97750724670244$

$P_2 = 0.50172391128947$

From equations 4.27 and 4.28 the stiffness modification factors for longitudinal vibration of the element are

$\alpha_1 = 0.99593579824973$

$\alpha_2 = 1.33947503380407$

Table 4.10: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.5, \xi_2 = 1.0$ on a fixed-free bar under longitudinal vibration

From Table 3.1

$\xi_1 = 0.5,\ \ \xi_2 = 1$					
j	Aj	F_{1j}	F_{2j}	u_{1j}	u_{2j}
1	0.51602455093119	0.27087898483275	0.30228018341801	0.36488445922219	0.51602455093119
2	0.01911202040486	0.00156816724276	-0.0652525192706	0.01351423923045	-0.0191120204049
3	0.00412819640745	-0.0370209108925	0.01409454416245	-0.0029190756738	0.00412819640745
4	0.00150444475490	0.01278409216525	-0.0008812833336	-0.0010638030881	-0.0015044447549
5	0.00070785260759	0.00666138865684	0.00041465045736	0.00050052737891	0.00070785260759
6	0.00038769688274	-0.0034131726750	-0.0013236799552	0.00027414309483	-0.0003876968827
7	0.00023487690074	-0.0041933942684	0.00080191990000	-0.0001660830493	0.00023487690074

8	0.00015289616324	0.00263693857990	-0.0000895644988	-0.0001081139138	-0.0001528961632
9	0.00010503247526	0.00192172311762	0.00006152659952	0.00007426917550	0.00010503247526
Total		**0.25161813381027**	**0.25010577747920**	**0.37499056237688**	**0.50004345111648**

u_{11} = 0.36488445922219

u_{21} = 0.51602455093119

From equations 4.6a and 4.6b

P_1 = -0.50172391128947

P_2 = 0

From equations 4.27 and 4.28 the stiffness modification factors for longitudinal vibration of the element are

α_1 = 0.99593579824973

α_2 = 0.97228690066021

Table 4.11: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.5, \xi_2 = 0.8$ on a fixed-free bar under longitudinal vibration

From Table 3.1

$\xi_1 = 0.5, \ \xi_2 = 0.8$					
j	Aj	F_{1j}	F_{2j}	u_{1j}	u_{2j}
1	0.51602455093119	0.15354563988675	0.16913376727867	0.36488445922219	0.49076851173139
2	0.01911202040486	0.018808991119194	-0.0096306237026	0.01351423923045	-0.0112337637355
3	0.00412819640745	-0.03265661897596	-0.0226925265196	-0.0029190756738	0
4	0.00150444475490	0.00520348078956	0.01987659357585	-0.0010638030881	0.00088429043982
5	0.00070785260759	0.01098848982730	-0.0070047931956	0.00050052737891	-0.0006732078350
6	0.00038769688274	-0.00505211446966	-0.0017548186654	0.00027414309483	0.00036872164668
7	0.00023487690074	-0.00348489393801	0.00397368273922	-0.0001660830493	-0.0001380571784
8	0.00015289616324	0.00218699436830	-0.0032421512611	-0.0001081139138	0
9	0.00010503247526	0.00202502516890	0.00222730577887	0.00007426917550	0.00006173653997

Total		**0.15156501384913**	**0.15088643602826**	**0.37499056237688**	**0.48003823160899**

u_{11} = 0.36488445922219

u_{21} = 0.49076851173139

From equations 4.6a and 4.6b

P_1 = -0.50172391128947

P_2 = 0.19927246141208

From equations 4.27 and 4.28 the stiffness modification factors for longitudinal vibration of the element are

α_1 = 1.22639917154470

α_2 = 0.88830930379005

Table 4.12: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.8$, $\xi_2 = 1.0$ on a fixed-free bar under longitudinal vibration

From Table 3.1

$\xi_1 = 0.8$, $\xi_2 = 1.0$					
j	Aj	F_{1j}	F_{2j}	u_{1j}	u_{2j}
1	0.51602455093119	0.12419954508633	0.12628019599901	0.49076851173139	0.51602455093119
2	0.01911202040486	-0.0334714361703	-0.0393912833469	-0.0112337637355	-0.0191120204049
3	0.00412819640745	0.01178179672830	0.02064098203725	0	0.00412819640745
4	0.00150444475490	-0.0014392725091	-0.0119436759736	0.00088429043982	-0.0015044447549
5	0.00070785260759	-0.0038129597305	0.00690530221305	-0.0006732078350	0.00070785260759
6	0.00038769688274	0.00585217315195	-0.0037820926471	0.00036872164668	-0.0003876968827
7	0.00023487690074	-0.0057449335651	0.00186467039552	-0.0001380571784	0.00023487690074
8	0.00015289616324	0.00436701179014	-0.0007644808162	0	-0.0001528961632
9	0.00010503247526	-0.0024855609071	0.00021647967644	0.00006173653997	0.00010503247526
Total		**0.09924636387458**	**0.10002609753749**	**0.48003823160899**	**0.50004345111648**

u_{11} = 0.49076851173139

u_{21} = 0.51602455093119

From equations 4.6a and 4.6b

P_1 = -0.19927246141208

P_2 = 0

From equations 4.27 and 4.28 the stiffness modification factors for longitudinal vibration of the element are

α_1 = 0.81856438428735

α_2 = 0.80880194720539

Tables 4.8 to 4.12 are illustrations on how the inherent nodal forces P_1 and P_2 and the stiffness modification factors α_1 and α_2 for a fixed-free bar under longitudinal vibration are calculated. The nodal forces P_1 and P_2 are the forces acting at the selected nodal point if the beam segment under consideration is decomposed. These nodal forces represent the effect of the removed adjacent beam segment on the beam segment under consideration. Using the methods presented in Table 4.8 to 4.12 the values of stiffness modification factors at different values of ξ_1 and ξ_2 for the longitudinal vibration of a fixed-free bar are presented in Table 4.13 below. A sample matlab program for the calculation of the stiffness modification factors for a segment of a beam restrained at one end can be found in Appendix C.

Table 4.13: Stiffness modification factors for the longitudinal vibration of a fixed-free/pinned-free bar

			ξ_2						
			0	0.05	0.10	0.15	0.20	0.25	0.30
ξ_1	0	α_1	-	1.183968	1.132904	1.082305	1.049036	1.028611	1.010495
		α_2	-	1.207192	1.210925	1.217182	1.226019	1.237514	1.251769
	0.05	α_1	1.183968	-	1.148560	1.105646	1.073301	1.051217	1.031511
		α_2	1.207192	-	1.166776	1.161223	1.164937	1.173842	1.185349
	0.10	α_1	1.132904	1.148560	-	1.097758	1.070537	1.050429	1.031727
		α_2	1.210925	1.166776	-	1.109006	1.102963	1.105153	1.110747
	0.15	α_1	1.082305	1.105646	1.097758	-	1.052532	1.035185	1.018205
		α_2	1.217182	1.161223	1.109006	-	1.058351	1.052824	1.051551
	0.20	α_1	1.049036	1.073301	1.070537	1.052532	-	1.018377	1.002485
		α_2	1.226019	1.164937	1.102963	1.058351	-	1.021661	1.014191
	0.25	α_1	1.028611	1.051217	1.050429	1.035185	1.018377	-	0.988096
		α_2	1.237514	1.173842	1.105153	1.052824	1.021661	-	0.990709
	0.30	α_1	1.010495	1.031511	1.031727	1.018205	1.002485	0.988096	-
		α_2	1.251769	1.185349	1.110747	1.051551	1.014191	0.990709	-
	0.35	α_1	0.994012	1.013619	1.014289	1.001798	0.986647	0.972243	0.956345
		α_2	1.268915	1.199472	1.119325	1.053751	1.01535	0.981802	0.958528
	0.40	α_1	0.985971	1.003608	1.003820	0.991424	0.976089	0.961088	0.944345
		α_2	1.289113	1.217183	1.132300	1.061173	1.012784	0.979345	0.951433
	0.45	α_1	0.988424	1.003569	1.002547	0.989434	0.973234	0.957085	0.939016
		α_2	1.312556	1.238683	1.149784	1.073861	1.020974	0.983413	0.951386
	0.50	α_1	0.995936	1.008776	1.006373	0.992253	0.974911	0.957376	0.937743
		α_2	1.339475	1.263467	1.170592	1.090040	1.032848	0.991349	0.955378
	0.55	α_1	1.004505	1.015592	1.012025	0.996925	0.978416	0.959454	0.938178
		α_2	1.370146	1.291408	1.194182	1.108782	1.047147	1.001606	0.961602
	0.60	α_1	1.016998	1.026639	1.021980	1.005838	0.986075	0.965597	0.942562
		α_2	1.404893	1.323124	1.221360	1.131033	1.064944	1.015379	0.971368
	0.65	α_1	1.037287	1.045494	1.039679	1.022343	1.001172	0.979044	0.954112
		α_2	1.444100	1.359301	1.253032	1.157883	1.087505	1.034112	0.986292
	0.70	α_1	1.062890	1.070165	1.063363	1.044883	1.022319	0.998546	0.971697
		α_2	1.488219	1.400138	1.289150	1.189058	1.114365	1.057173	1.005593
	0.75	α_1	1.088804	1.095812	1.088531	1.069239	1.045528	1.020300	0.991651
		α_2	1.537785	1.445710	1.329355	1.223801	1.144407	1.083120	1.027518
	0.80	α_1	1.114402	1.121952	1.114768	1.095064	1.070505	1.044057	1.013765
		α_2	1.593431	1.496594	1.374144	1.262525	1.177976	1.112240	1.052300
	0.85	α_1	1.143503	1.152264	1.145657	1.125868	1.100720	1.073272	1.041498
		α_2	1.655911	1.553805	1.424742	1.306646	1.216679	1.146335	1.081934
	0.90	α_1	1.176530	1.187281	1.181830	1.162373	1.136980	1.108834	1.075822
		α_2	1.726125	1.618209	1.481971	1.356951	1.261292	1.186192	1.117231
	0.95	α_1	1.208020	1.221961	1.218586	1.200152	1.175071	1.146697	1.112807
		α_2	1.805153	1.690400	1.545967	1.413160	1.311167	1.230829	1.156885
	1.00	α_1	1.233701	1.252387	1.252289	1.235779	1.211728	1.183714	1.149381
		α_2	1.894304	1.771274	1.617235	1.475420	1.366137	1.279797	1.200175
			ξ_2						

			0.35	0.40	0.45	0.50	0.55	0.60	0.65
	0	α_1	0.994012	0.985971	0.988424	0.995936	1.004505	1.016998	1.037209
		α_2	1.268915	1.289113	1.312556	1.339475	1.370146	1.404893	1.444100
	0.05	α_1	1.013619	1.003608	1.003569	1.008776	1..015592	1.026639	1.045494
		α_2	1.199472	1.217183	1.238683	1.263467	1.291408	1.323124	1.359301
	0.10	α_1	1.014289	1.003820	1.002547	1.006373	1.012025	1.021980	1.039680
		α_2	1.119325	1.132299	1.149784	1.170592	1.194182	1.221360	1.253032
	0.15	α_1	1.001798	0.991424	0.989434	0.992253	0.996924	1.005838	1.022344
		α_2	1.053752	1.061173	1.073861	1.090040	1.108782	1.131033	1.157883
	0.20	α_1	0.986647	0.976089	0.973234	0.974911	0.978416	0.986075	1.001172
		α_2	1.010535	1.012784	1.020974	1.032848	1.047147	1.064944	1.087505
	0.25	α_1	0.972243	0.961088	0.957085	0.957376	0.959454	0.965597	0.979044
		α_2	0.981802	0.979345	0.983413	0.991349	1.001606	1.015379	1.034112
	0.30	α_1	0.956345	0.944545	0.939016	0.937743	0.938178	0.942562	0.954113
		α_2	0.958528	0.951433	0.951386	0.955378	0.961602	0.971368	0.986292
	0.35	α_1	-	0.926411	0.919559	0.916524	0.915080	0.917445	0.926832
		α_2	-	0.927808	0.923634	0.923657	0.925826	0.931564	0.942654
	0.40	α_1	0.926411	-	0.903190	0.898132	0.894541	0.894626	0.901612
		α_2	0.927808	-	0.903857	0.900198	0.898599	0.900599	0.908154
	0.45	α_1	0.919559	0.903190	-	0.885113	0.879145	0.876737	0.881146
		α_2	0.923634	0.903857	-	0.885481	0.880507	0.879168	0.883610
ξ_1	0.50	α_1	0.916524	0.898132	0.885113	-	0.866839	0.861753	0.863414
		α_2	0.923657	0.900198	0.885481	-	0.867144	0.862598	0.864070
	0.55	α_1	0.915080	0.894541	0.879145	0.866839	-	0.847616	0.846324
		α_2	0.925826	0.898599	0.880507	0.867144	-	0.847731	0.846137
	0.60	α_1	0.917445	0.894626	0.876737	0.861753	0.847616	-	0.832200
		α_2	0.931564	0.900599	0.879168	0.862598	0.847731	-	0.831992
	0.65	α_1	0.926832	0.901612	0.881146	0.863414	0.846324	0.832200	-
		α_2	0.942653	0.908154	0.883610	0.864070	0.846137	0.831992	-
	0.70	α_1	0.942205	0.914540	0.891480	0.870988	0.850911	0.833642	0.822958
		α_2	0.958177	0.920244	0.892680	0.870302	0.849437	0.832421	0.822552
	0.75	α_1	0.959998	0.929909	0.904267	0.880998	0.857861	0.837329	0.823390
		α_2	0.976081	0.934463	0.903746	0.878371	0.854400	0.834331	0.821773
	0.80	α_1	0.980031	0.947556	0.919357	0.893300	0.867020	0.843089	0.825747
		α_2	0.996541	0.950995	0.916887	0.888309	0.861001	0.837633	0.822161
	0.85	α_1	1.005784	0.971013	0.940362	0.911614	0.882246	0.854950	0.834285
		α_2	1.021751	0.972221	0.934731	0.902995	0.872386	0.845765	0.827498
	0.90	α_1	1.038313	1.001420	0.968523	0.937299	0.905026	0.874551	0.850842
		α_2	1.052548	0.999025	0.958237	0.923478	0.889710	0.860016	0.839257
	0.95	α_1	1.073758	1.034947	1.000009	0.966482	0.931388	0.897758	0.871082
		α_2	1.087311	1.029475	0.985165	0.947202	0.910070	0.877090	0.853728
	1.00	α_1	1.109068	1.068526	1.031696	0.995936	0.957926	0.920893	0.890893
		α_2	1.125036	1.062282	1.013941	0.972287	0.931215	0.894297	0.867711

			ξ_2						
			0.70	0.75	0.80	0.85	0.90	0.95	1.00
ξ_1	0	α_1	1.062890	1.088804	1.114402	1.143503	1.176530	1.208020	1.233701
		α_2	1.488219	1.537785	1.593431	1.655911	1.726125	1.805153	1.894303
	0.05	α_1	1.070165	1.095812	1.121952	1.152264	1.187281	1.221961	1.252387
		α_2	1.400138	1.445710	1.496594	1.553805	0.618209	1.690400	1.771274
	0.10	α_1	1.063363	1.088530	1.114768	1.145658	1.181830	1.218586	1.252289
		α_2	1.289150	1.329355	1.374144	1.424742	1.481971	1.545967	1.617235
	0.15	α_1	1.044883	1.069239	1.095064	1.125868	1.162373	1.200152	1.235779
		α_2	1.189058	1.223801	1.262525	1.306646	1.356951	1.413160	1.475420
	0.20	α_1	1.022319	1.045528	1.070505	1.100720	1.136980	1.175071	1.211728
		α_2	1.114365	1.144407	1.177976	1.216679	1.261292	1.311167	1.366137
	0.25	α_1	0.998563	1.020300	1.044057	1.073272	1.108834	1.146697	1.183714
		α_2	1.057173	1.083120	1.112240	1.146335	1.186192	1.230829	1.279797
	0.30	α_1	0.971697	0.991651	1.013765	1.041498	1.075822	1.112807	1.149382
		α_2	1.005593	1.027518	1.052300	1.081934	1.117231	1.156885	1.200175
	0.35	α_1	0.942205	0.959998	0.980031	1.005784	1.038313	1.075758	1.109068
		α_2	0.958177	0.976081	0.996541	1.021751	1.052548	1.087311	1.125036
	0.40	α_1	0.914540	0.929909	0.947556	0.971013	1.001420	1.034947	1.068526
		α_2	0.920224	0.934463	0.950995	0.972221	0.999025	1.029475	1.062282
	0.45	α_1	0.891480	0.904267	0919357	0.940362	0.968523	1.000009	1.031696
		α_2	0.892679	0.903746	0.916887	0.934731	0.958237	0.985165	1.031941
	0.50	α_1	0.870988	0.880998	0.893300	0.911614	0.937299	0.966481	0.995936
		α_2	0.870302	0.878371	0.888309	0.902994	0.923477	0.947202	0.972287
	0.55	α_1	0.850911	0.857861	0.867020	0.882246	0.905026	0.931388	0.957926
		α_2	0.849437	0.854400	0.861001	0.872386	0.889710	0.910070	0.931215
	0.60	α_1	0.833642	0.837329	0.843089	0.854950	0.874551	0.897758	0.920893
		α_2	0.832421	0.834331	0.837633	0.845765	0.860016	0.877090	0.894297
	0.65	α_1	0.822958	0.823390	0.825747	0.834285	0.850842	0.871082	0.890973
		α_2	0.822552	0.821773	0.822161	0.827498	0.839257	0.853728	0.867711
	0.70	α_1	-	0.815363	0.814326	0.819601	0.833297	0.850832	0.867716
		α_2	-	0.814953	0.812699	0.815590	0.825356	0.837841	0.849245
	0.75	α_1	0.815363	-	0.804410	0.806085	0.816614	0.831061	0.844219
		α_2	0.814953	-	0.803998	0.804186	0.811760	0.822021	0.830317
	0.80	α_1	0.814326	0.804410	-	0.793129	0.799969	0.810603	0.818564
		α_2	0.812699	0.803998	-	0.792585	0.797518	0.804936	0.808802
	0.85	α_1	0.819601	0.806085	0.793129	-	0.790081	0.797298	0.799783
		α_2	0.815590	0.804186	0.792585	-	0.789393	0.794567	0.793962
	0.90	α_1	0.833297	0.816614	0.799969	0.790081	-	0.797687	0.797349
		α_2	0.825356	0.811760	0.797518	0.789394	-	0.797002	0.794797
	0.95	α_1	0.850832	0.831061	0.810603	0.797298	0.797687	-	0.806968
		α_2	0.837841	0.822021	0.804936	0.794567	0.797002	-	0.806370
	1.00	α_1	0.867716	0.844219	0.818564	0.799783	0.797349	0.806968	-
		α_2	0.849245	0.830317	0.808802	0.793962	0.794797	0.806370	-

4.1.3 Free-free bars

These are bars not restrained from longitudinal vibration at both ends. From section 3.1.1 the first or lowest natural frequency w for such a bar is given as

$$\omega = 0 \tag{4.29}$$

Likewise

$$u_1 = u(x_1, 0) = \sum_{j=1}^{\infty} A_j \cos\frac{j\pi x_1}{2L}$$

$$= \sum_{j=1}^{\infty} A_j \cos\frac{j\pi \xi_1}{2} \tag{4.30}$$

$$u_2 = u(x_2, 0) = \sum_{j=1}^{\infty} A_j \cos\frac{j\pi x_2}{2L}$$

$$= \sum_{j=1}^{\infty} A_j \cos\frac{j\pi \xi_2}{2} \tag{4.31}$$

From Table 3.2

$$A_j = 4fL\left(\frac{-\cos\frac{j\pi}{2}}{j^2\pi^2} + \frac{2\sin\frac{j\pi}{2}}{j^3\pi^3}\right) \tag{4.32}$$

$$j = 0,1,2,3,4,5, \dots, \infty$$

By substituting equation (4.29) into equations (4.15) and (4.16) and taking EA = 1 we obtain

$$\alpha_1 = \frac{-(\xi_2-\xi_1)u_{11}P_1+(\xi_2-\xi_1)u_{21}P_2}{u_{21}^2-u_{11}^2} \tag{4.33a}$$

$$\alpha_2 = \frac{(\xi_2-\xi_1)u_{11}P_2-(\xi_2-\xi_1)u_{21}P_1}{u_{21}^2-u_{11}^2} \tag{4.33b}$$

Equations (4.33a) and (4.33b) are the equations for evaluating the stiffness modification factor for longitudinal vibration of a segment of a free-free bar located between ξ_1 and ξ_2 of the bar's total length. From the equations it would be observed that α_1 and α_2 are equal.

Recall that u_{11} and u_{21} are the values of u_1 and u_2 for the first mode (ie when j = 0).

From equations (4.30) and (4.31) it would be observed that at j = 0

$$u_{11} = u_{21} = \sum_{j=1}^{\infty} A_j \tag{4.34}$$

Hence $\alpha_1 = \alpha_2 = \infty$ (4.35)

` Equation (4.35) implies that for segments of a free-free bar under longitudinal vibration their modification factors are numerically of infinite values.

4.2 Laterally vibrating beams

The equations of the fixed end forces F_1 to F_4 of segments of a laterally vibrating beam of different end constraints are presented in Tables 3.2 to 3.8. The force equilibrium equations for a segment of a laterally vibrating beam can be written as

$$\{F\} + [k]\{u\} = \{P\} \tag{4.36}$$

Where {F} is the vector of fixed end forces, [k] is the stiffness matrix of the segment under consideration and {u} is a vector of nodal displacements.

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} \tag{4.37}$$

$$[k] = \begin{bmatrix} \frac{12EI}{l^3} & \frac{6EI}{l^2} & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{4EI}{l} & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{2EI}{l} & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix} \tag{4.38}$$

$$\{u\} = \begin{Bmatrix} u(x_1,0) \\ u'(x_1,0) \\ u(x_2,0) \\ u'(x_2,0) \end{Bmatrix} = \begin{Bmatrix} u(\xi_1,0) \\ u'(\xi_1,0) \\ u(\xi_2,0) \\ u'(\xi_2,0) \end{Bmatrix}$$

$$= \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{4.39}$$

E is the young's modulus of elasticity of the material of the segment and A is the cross sectional area of the bar. l is the length of the segment and is equal to $x_2 - x_1$ but since the length of the bar L has been normalized and is now equal to unity

$$l = \xi_2 - \xi_1 \tag{4.40}$$

u_1 is the total transverse displacement at the position x_1, u_2 is the total rotational displacement at the position x_1 while u_3 is the total transverse displacement at the position x_2 and u_4 is the total rotational displacement at the position x_2. The total displacement is obtained by totaling the displacements due to all the modes of vibration.

{P} is the vector of nodal forces; they represent the forces acting on the nodes of the isolated segment of the vibrating beam.

$$\{P\} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} \tag{4.41}$$

From equation (4.36) P_1, P_2, P_3 and P_4 can be expressed as

$$P_1 = F_1 + \frac{12EI}{l^3}u_1 + \frac{6EI}{l^2}u_2 - \frac{12EI}{l^3}u_3 + \frac{6EI}{l^2}u_4 \tag{4.42a}$$

$$P_2 = F_1 + \frac{6EI}{l^2}u_1 + \frac{4EI}{l}u_2 - \frac{6EI}{l^2}u_3 + \frac{2EI}{l}u_4 \tag{4.42b}$$

$$P_3 = F_3 - \frac{12EI}{l^3}u_1 - \frac{6EI}{l^2}u_2 + \frac{12EI}{l^3}u_3 - \frac{6EI}{l^2}u_4 \tag{4.42c}$$

$$P_4 = F_4 + \frac{6EI}{l^2}u_1 + \frac{2EI}{l}u_2 - \frac{6EI}{l^2}u_3 + \frac{4EI}{l}u_4 \tag{4.42d}$$

If a segment of a vibrating beam is isolated it will be in equilibrium with the application of the force vector {P}. The force vector {P} represents the effect of the removed adjourning elements on the isolated segment.

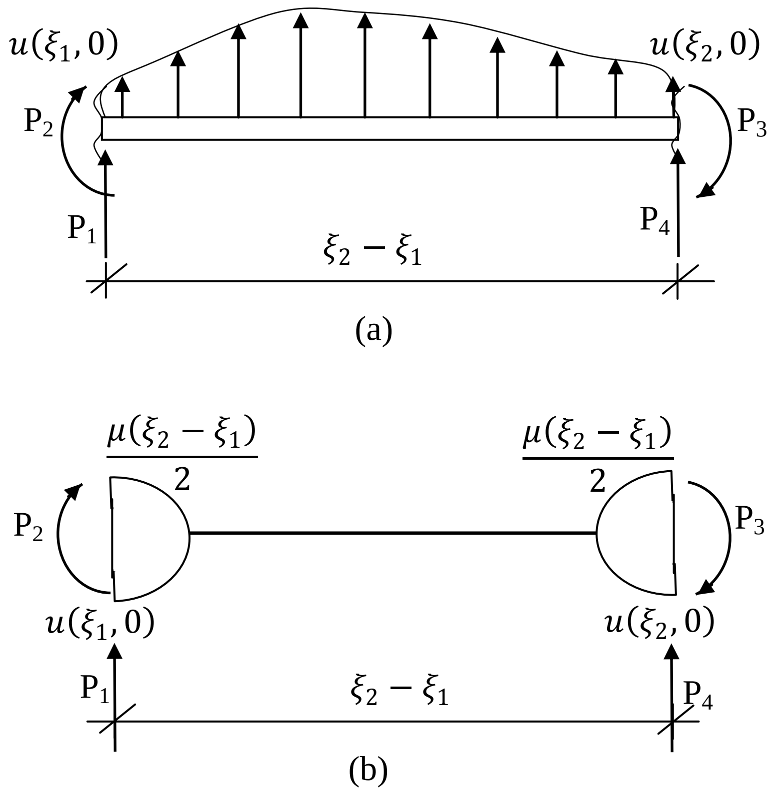


Figure 4.2

(a) An isolated segment of the laterally vibrating continuous beam showing the nodal forces P_1, P_2, P_3 and P_4

(b) An equivalent lumped massed segment showing the nodal forces

Figure 4.2a shows a segment of the vibrating continuous beam. The nodal forces on the bar P_1, P_2 P_3 and P_4 are calculated from the equilibrium equations (equation (4.36)) or from equations (4.42a) – (4.42d). When the continuous bar is represented by a lumped massed bar (a bar that has its distributed masses lumped at selected nodes), the equivalent segment of the bar is shown in Figure 4.1b. Just like the real segment the equivalent segment is supported by the same nodal forces P_1, P_2 P_3 and P_4 and has the same nodal displacements as the continuous/real bar. This implies that for the lumped massed beam to be equivalent to the real or continuous beam they must have the same inherent forces and displacements at the nodes.

The equation of motion for the lumped massed vibrating beam is given as

$$[m]\{\ddot{u}\} + [k_d]\{u\} = \{P\} \tag{4.43}$$

Where [m] is the inertial matrix, {u} is a vector of nodal displacement and k_d is the stiffness of the lumped massed segment under consideration.

The proposed stiffness matrix for the lumped massed segment k_d is

$$[k_d] = \begin{bmatrix} \frac{12EI}{l^3}\phi_1 & \frac{6EI}{l^2}\phi_2 & -\frac{12EI}{l^3}\phi_3 & \frac{6EI}{l^2}\phi_4 \\ \frac{6EI}{l^2}\phi_2 & \frac{4EI}{l}\phi_1 & -\frac{6EI}{l^2}\phi_4 & \frac{2EI}{l}\phi_3 \\ -\frac{12EI}{l^3}\phi_3 & -\frac{6EI}{l^2}\phi_4 & \frac{12EI}{l^3}\phi_1 & -\frac{6EI}{l^2}\phi_2 \\ \frac{6EI}{l^2}\phi_4 & \frac{2EI}{l}\phi_3 & -\frac{6EI}{l^2}\phi_2 & \frac{4EI}{l}\phi_1 \end{bmatrix} \tag{4.44}$$

where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are the stiffness modification factors for lateral vibration. They help redistribute the stiffness of the lumped massed segment in such a way as to annul the effect of the discretization of the bar due to the lumping of its distributed mass on selected nodes. By ignoring the rotational inertia of the lumped masses, the inertia matrix of a lumped massed segment can be expressed as

$$[m] = \begin{bmatrix} \frac{\mu(\xi_2-\xi_1)}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu(\xi_2-\xi_1)}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.45}$$

μ is the mass per unit length of the beam.

The vector of nodal acceleration is written as

$$\{\ddot{u}\} = \begin{Bmatrix} \ddot{u}(\xi_1,0) \\ \ddot{u}'(\xi_1,0) \\ \ddot{u}(\xi_2,0) \\ \ddot{u}'(\xi_2,0) \end{Bmatrix} = \begin{Bmatrix} -\omega^2 u(x_1,0) \\ -\omega^2 u'(x_1,0) \\ -\omega^2 u(x_2,0) \\ -\omega^2 u'(x_2,0) \end{Bmatrix} = \begin{Bmatrix} -\omega^2 u_{11} \\ -\omega^2 u_{21} \\ -\omega^2 u_{31} \\ -\omega^2 u_{41} \end{Bmatrix} \tag{4.46}$$

Where ω is the fundamental frequency of the vibrating mass, u_{11}, $u_{21,}$ u_{31}, u_{41} are the values of nodal displacements $u_{1,}$ $u_{2,}$ u_3 and u_4 for the first mode of vibration only.

By substituting equations (4.44) to (4.46) into equation (4.43) we obtain

$$\begin{bmatrix} \frac{\mu(\xi_2-\xi_1)}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu(\xi_2-\xi_1)}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} -\omega^2 u_{11} \\ -\omega^2 u_{21} \\ -\omega^2 u_{31} \\ -\omega^2 u_{41} \end{Bmatrix} +$$

$$\begin{bmatrix} \frac{12EI}{l^3}\phi_1 & \frac{6EI}{l^2}\phi_2 & -\frac{12EI}{l^3}\phi_3 & \frac{6EI}{l^2}\phi_4 \\ \frac{6EI}{l^2}\phi_2 & \frac{4EI}{l}\phi_1 & -\frac{6EI}{l^2}\phi_4 & \frac{2EI}{l}\phi_3 \\ -\frac{12EI}{l^3}\phi_3 & -\frac{6EI}{l^2}\phi_4 & \frac{12EI}{l^3}\phi_1 & -\frac{6EI}{l^2}\phi_2 \\ \frac{6EI}{l^2}\phi_4 & \frac{2EI}{l}\phi_3 & -\frac{6EI}{l^2}\phi_2 & \frac{4EI}{l}\phi_1 \end{bmatrix} \begin{Bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} \tag{4.47}$$

By multiplying out the first row of equation (4.43) we obtain

$$-\frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{11} + \frac{12EI}{l^3}\phi_1 u_{11} + \frac{6EI}{l^2}\phi_2 u_{21} - \frac{12EI}{l^3}\phi_3 u_{31} + \frac{6EI}{l^2}\phi_4 u_{41} = P_1$$

$$\frac{12EI}{l^3}\phi_1 u_{11} + \frac{6EI}{l^2}\phi_2 u_{21} - \frac{12EI}{l^3}\phi_3 u_{31} + \frac{6EI}{l^2}\phi_4 u_{41} = P_1 + \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{11} \tag{4.48a}$$

By multiplying out the second row of equation (4.43) we obtain

$$0 + \frac{6EI}{l^2}\phi_2 u_{11} + \frac{4EI}{l}\phi_1 u_{21} - \frac{6EI}{l^2}\phi_4 u_{31} + \frac{2EI}{l}\phi_3 u_{41} = P_2$$

$$\frac{4EI}{l}\phi_1 u_{21} + \frac{6EI}{l^2}\phi_2 u_{11} + \frac{2EI}{l}\phi_3 u_{41} - \frac{6EI}{l^2}\phi_4 u_{31} = P_2 \tag{4.48b}$$

By multiplying out the third row of equation (4.43) we obtain

$$-\frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{31} - \frac{12EI}{l^3}\phi_3 u_{11} - \frac{6EI}{l^2}\phi_4 u_{21} + \frac{12EI}{l^3}\phi_1 u_{31} - \frac{6EI}{l^2}\phi_2 u_{41} = P_3$$

$$\frac{12EI}{l^3}\phi_1 u_{31} - \frac{6EI}{l^2}\phi_2 u_{41} - \frac{12EI}{l^3}\phi_3 u_{11} - \frac{6EI}{l^2}\phi_4 u_{21} = P_3 - \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{31} \quad (4.48c)$$

By multiplying out the fourth row of equation (4.43) we obtain

$$0 + \frac{6EI}{l^2}\phi_4 u_{11} + \frac{2EI}{l}\phi_3 u_{21} - \frac{6EI}{l^2}\phi_3 u_{31} + \frac{4EI}{l}\phi_1 u_{41} = P_4$$

$$\frac{4EI}{l}\phi_1 u_{41} - \frac{6EI}{l^2}\phi_2 u_{31} + \frac{2EI}{l}\phi_3 u_{21} + \frac{6EI}{l^2}\phi_4 u_{11} = P_4 \quad (4.48d)$$

Putting equations (4.48a) – (4.48d) in matrix form

$$\begin{bmatrix} \frac{12EI}{l^3}u_{11} & \frac{6EI}{l^2}u_{21} & -\frac{12EI}{l^3}u_{31} & \frac{6EI}{l^2}u_{41} \\ \frac{4EI}{l}u_{21} & \frac{6EI}{l^2}u_{11} & \frac{2EI}{l}u_{41} & -\frac{6EI}{l^2}u_{31} \\ \frac{12EI}{l^3}u_{31} & -\frac{6EI}{l^2}u_{41} & -\frac{12EI}{l^3}u_{11} & -\frac{6EI}{l^2}u_{21} \\ \frac{4EI}{l}u_{41} & -\frac{6EI}{l^2}u_{31} & \frac{2EI}{l}u_{21} & \frac{6EI}{l^2}u_{11} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} P_1 + \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{11} \\ P_2 \\ P_3 + \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{31} \\ P_4 \end{Bmatrix}$$

(4.49)

By rearranging equation (4.49) we obtain

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{bmatrix} \frac{12EI}{l^3}u_{11} & \frac{6EI}{l^2}u_{21} & -\frac{12EI}{l^3}u_{31} & \frac{6EI}{l^2}u_{41} \\ \frac{4EI}{l}u_{21} & \frac{6EI}{l^2}u_{11} & \frac{2EI}{l}u_{41} & -\frac{6EI}{l^2}u_{3} \\ \frac{12EI}{l^3}u_{31} & -\frac{6EI}{l^2}u_{41} & -\frac{12EI}{l^3}u_{11} & -\frac{6EI}{l^2}u_{21} \\ \frac{4EI}{l}u_{41} & -\frac{6EI}{l^2}u_{31} & \frac{2EI}{l}u_{21} & \frac{6EI}{l^2}u_{11} \end{bmatrix}^{-1} \begin{Bmatrix} P_1 + \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{11} \\ P_2 \\ P_3 + \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{31} \\ P_4 \end{Bmatrix}$$

(4.50)

Equation (4.50) is a mathematical expression for calculating the four stiffness modification factors for a beam under lateral vibration.

4.2.1 **Fixed-fixed beams**

These are beams rigidly restrained from lateral vibration at both ends. From section 3.2.1 the first natural frequency w for such a bar is given as

$$\omega_1 = 22.37328597\sqrt{\frac{EI}{\mu L^4}} \tag{4.51}$$

In normalized coordinates, the length of the bar L = 1 hence

$$\omega_1 = 22.37328597\sqrt{\frac{EI}{\mu}}\,. \tag{4.51a}$$

Likewise

$$u_1 = u(x_1,0) = \sum_{j=1}^{\infty} A_j\left(\cosh\beta_j x_1 + a_{2j}\sinh\beta_j x_1 - \cos\beta_j x_1 + a_{4j}\sin\beta_j x_1\right)$$

$$= \sum_{j=1}^{\infty} A_j\left(\cosh\beta_j L\xi_1 + a_{2j}\sinh\beta_j L\xi_1 - \cos\beta_j L\xi_1 + a_{4j}\sin\beta_j L\xi_1\right)$$

(4.52a)

$$u_2 = u'(x_1,0) = \sum_{j=1}^{\infty} A_j\beta_j L\left(\sinh\beta_j x_1 + a_{2j}\cosh\beta_j x_1 + \sin\beta_j x_1 + a_{4j}\cos\beta_j x_1\right)$$

$$= \sum_{j=1}^{\infty} A_j\beta L\left(\sinh\beta_j L\xi_1 + a_{2j}\cosh\beta_j L\xi_1 + \sin\beta_j L\xi_1 + a_{4j}\cos\beta_j L\xi_1\right)$$

(4.52b)

$$u_3 = u(x_2,0) = \sum_{j=1}^{\infty} A_j\left(\cosh\beta_j x_2 + a_{2j}\sinh\beta_j x_2 - \cos\beta_j x_2 + a_{4j}\sin\beta_j x_2\right)$$

$$= \sum_{j=1}^{\infty} A_j\left(\cosh\beta_j L\xi_2 + a_{2j}\sinh\beta_j L\xi_2 - \cos\beta_j L\xi_2 + a_{4j}\sin\beta_j L\xi_2\right)$$

(4.52c)

$$u_4 = u'(x_2,0) = \sum_{j=1}^{\infty} A_j\beta L\left(\sinh\beta_j x_2 + a_{2j}\cosh\beta_j x_2 + \sin\beta_j x_2 + a_{4j}\cos\beta_j x_2\right)$$

$$= \sum_{j=1}^{\infty} A_j \beta_j L\left(\sinh \beta_j L\xi_2 + a_{2j} \cosh \beta_j L\xi_2 + \sin \beta_j L\xi_2 + a_{4j} \cos \beta_j L\xi_2\right)$$

(4.52d)

Equations (4.52a – 4.52d) are used to evaluate the total displacements u_1 – u_4 at the nodal points of a segment of the vibrating fixed-fixed beam. Though the equations represent the summation of an infinite series, an evaluation of the first few terms provide values of very good precision.

From Table 3.2

$$A_j = \frac{a\mu L^2}{M_j}\left[\frac{-2(\sinh \beta_j L - \sin \beta L)}{\beta_j^3 L^3} + \frac{12(\cosh \beta_j L - \cos \beta_j L)}{\beta_j^4 L^4} - \frac{24(\sinh \beta_j L - \sin \beta_j L)}{\beta_j^5 L^5} + a_{2j}\left(\frac{-2(\cosh \beta_j L - \cos \beta_j L)}{\beta_j^3 L^3} + \frac{12(\sinh \beta_j L - \sin \beta_j L)}{\beta_j^4 L^4} - \frac{24(\cosh \beta_j L + \cos \beta_j L - 2)}{\beta_j^5 L^5}\right)\right] \quad (4.53)$$

The values of $\beta_j L$, a_{2j}, and a_{4j} for j = 1, 2, 3, 4, 5, 6, 7 can be obtained from Table 3.2. The values of the fixed end forces F_1, F_2, F_3 and F_4 are evaluated using the equations provided in Table 3.2 while the values of the nodal displacements u_1, u_2, u_3 and u_4 are calculated from equations (4.52a) – (4.52b). These are substituted into the equations for nodal forces (equations 4.35) – (4.38) from which the nodal forces P_1, P_2, P_3 and P_4 are obtained.

Equations (4.52a) – (4.52d) are evaluated for the first mode, j = 1 to obtain the nodal displacements u_{11}, u_{21}, u_{31} and u_{41} due to the first mode. These together with the calculated nodal forces P_1 to P_4 are substituted into equation (4.49) in order to obtain the stiffness modification factors ϕ_1, ϕ_2, ϕ_3 and ϕ_4. Some numerical demonstrations of these steps are presented below. For clarity the calculations will be presented in a tabular form.

Example 1: when $\xi_1 = 0$, $\xi_2 = 0.5$

Table 4.14: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0$, $\xi_2 = 0.5$ on a fixed-fixed bar under lateral vibration

$\xi_1 = 0,\ \ \xi_2 = 0.5$					
j	**Aj**	**F_{1j}**	**F_{2j}**	**F_{3j}**	**F_{4j}**
1	-0.03983642568324	-6.07354293466814	-0.4349161689350	-2.2104276673494	0.26415775020641
2	0.00000000062688	-0.00000058939095	-0.0000000572726	-0.000000435889	0.00000004868688
3	-0.00059725591824	0.08061512479706	-0.0825567447022	-1.6685533888640	0.16457431427370
4	0.00000000001256	0.00000005627723	0.00000000200684	-0.0000000770088	0.00000000602481
5	-0.00006240862307	-0.00847499630378	0.02422490351705	-0.6354158651679	0.03514613064230
6	0	0	0	0	0
7	-0.00000002582156	0.00000350561250	-0.0000193968981	-0.0006790375569	0.00002954686938
	Total	**-6.00139983367608**	**-0.4932674622840**	**-4.5150764718360**	**0.46390779670347**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	-0.03983642568324	-0.06326607054284	0.00000000631491	0	-0.0000000207788
2	0.00000000062688	0	-0.0000000071591	0	0
3	-0.00059725591824	0.00083974090383	-0.0000000000093	0	-0.0000000000263
4	0.00000000001256	0	0.00000000025086	0	0
5	-0.00006240862307	-0.00008828121062	0	0	-0.0000000000033
6	0	0	0	0	0
7	-0.00000002582156	0.00000003651680	0	0	0
	Total	**-0.06251457433283**	**-0.0000000006026**	**0**	**-0.0000000208084**

u_{11} = -0.06326607054284 u_{31} = 0

u_{21} = 0.00000000631491 u_{41} = -0.00000002080837

From Table 3. 2

F_1 = 6.00139983367608 F_3 = 4.51507647183599

F_2 = 0.49326746228404 F_4 = -0.46390779670347

From equations (4.35) – (4.38)

P_1 = 1.838594375614049 x 10^{-5} P_3 = 10.51647612165263

P_2 = -1.00708240975866 P_4 = -1.96425774956910

From equation (4.49) taking EI = 1

ϕ_1 = 1.30355176828885 ϕ_3 = 1.73152250367623

ϕ_2 = 0.66325857542082 ϕ_4 = 1.29364863330555

Table 4.15: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0, \xi_2 = 0.3$ on a fixed-fixed bar under lateral vibration

$\xi_1 = 0, \ \xi_2 = 0.3$					
j	**Aj**	**F_{1j}**	**F_{2j}**	**F_{3j}**	**F_{4j}**
1	-0.03983642568324	-0.60422929864885	-0.0443311538700	-2.0275325645405	0.08085046175642
2	0.00000000062688	0.00000015570810	0.00000001115058	0.00000042202589	-0.0000000182381
3	-0.00059725591824	-0.8404133833101	-0.0581470356258	-1.6268933696662	0.08063557854974
4	0.00000000001256	0.00000005725286	0.00000000376584	0.00000006096232	-0.0000000039983
5	-0.00006240862307	-0.64922194590421	-0.0396555237773	-0.1335606852543	0.02595648146587
6	0	0	0	0	0
7	-0.00000002582156	-0.00073284865591	-0.0000344016524	0.00053527751290	-0.0000114516373
	Total	**-2.09467521857903**	**-0.1421681000090**	**-3.7874508589600**	**0.18743104789833**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	-0.03983642568324	0	0	-0.0436607183239	-0.1758753331736
2	0.00000000062688	0	0	0.00000000094377	-0.0000000004882
3	-0.00059725591824	0	0	-0.0005187962004	0.00775306253322
4	0.00000000001256	0	0	-0.0000000000053	-0.0000000002414

5	-0.00006240862307	0	0	0.00008358909649	0.00047729435847
6	0	0	0	0	0
7	-0.00000002582156	0	0	-0.0000000000220	-0.0000008598982
	Total	**0**	**0**	**0.04409592451136**	**-0.1676458369097**

$u_{11} = 0$ $u_{31} = -0.0436607183239$

$u_{21} = 0$ $u_{41} = -0.1758753331736$

From Table 3. 2

$F_1 = 2.09467521857903$ $F_3 = 3.78745085895998$

$F_2 = 0.14216810000903$ $F_4 = -0.18743104789833$

From equations (4.35) – (4.38)

$P_1 = 10.51647476298262$ $P_3 = -4.63434868544362$

$P_2 = 1.96425748803524$ $P_4 = 0.51701942739655$

From equation (4.49) taking EI = 1

$\phi_1 = 1.00368699132460$ $\phi_3 = 1.25523649330347$

$\phi_2 = 0.98624232179889$ $\phi_4 = 1.18047475960633$

Table 4.16: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.3, \xi_2 = 0.6$ on a fixed-fixed bar under lateral vibration

$\xi_1 = 0.3,\ \xi_2 = 0.6$					
j	**Aj**	$\mathbf{F_{1j}}$	$\mathbf{F_{2j}}$	$\mathbf{F_{3j}}$	$\mathbf{F_{4j}}$
1	-0.0398364256832	-4.1388906590437	-0.2180613342475	-4.5911143839929	0.22959746361135
2	0.00000000062688	0.00000035962860	0.00000001390725	-0.0000000417563	-0.0000000035487
3	-0.0005972559182	0.44248899356144	0.04544769491502	1.42875324175632	-0.0718625825940
4	0.00000000001256	-0.00000007088701	-0.0000000033493	0.000000001567609	0.00000000089973

5	-0.0000624086231	0.30210042255651	-0.0041101918850	-0.61321338253410	0.03204886005669
6	0	0	0	0	0
7	-0.0000000258216	-0.00056438352709	-0.0000138137699	0.00042416171754	-0.0000264593382
	Total	**-3.39486533771123**	**-0.1767824336067**	**-3.77515038917997**	**0.18975727908682**

j	**Aj**	$\mathbf{u_{1j}}$	$\mathbf{u_{2j}}$	$\mathbf{u_{3j}}$	$\mathbf{u_{4j}}$
1	-0.0398364256832	-0.0436607183239	-0.1758753331736	-0.05797992579367	0.10311372400732
2	0.00000000062688	0.0000000094377	-0.0000000004882	-0.00000000064855	-0.0000000051825
3	-0.0005972559182	-0.0005187962004	0.00775306253322	0.00037529284311	-0.0083468306820
4	0.00000000001256	-0.0000000000053	-0.0000000002414	0.00000000001751	0.00000000003863
5	-0.0000624086231	0.00008358909649	0.00047729435847	0.00001374263093	0.00150519231258
6	0	0	0	0	0
7	-0.0000000258216	-0.0000000000220	-0.0000008598982	-0.00000002582366	-0.0000006084548
	Total	**0**	**-0.1676458369097**	**0.04409592451136**	**0.09627147203923**

u_{11} = -0.0436607183239 u_{31} = -0.05797992579367

u_{21} = -0.1758753331736 u_{41} = -0.10311372400732

From Table 3. 2

F_1 = 3.39486533771123 F_3 = 3.77515038917997

F_2 = 0.17678243360667 F_4 = -0.18975727908682

From equations (4.35) – (4.38)

P_1 = 4.63434868544351 P_3 = 2.53566704144769

P_2 = -0.51701942739658 P_4 = 0.87588958623612

From equation (4.49) taking EI = 1

ϕ_1 = 0.96557551547973 ϕ_3 = 0.92644783424547

ϕ_2 = 0.87888971884435 ϕ_4 = 0.94910262055559

Table 4.17: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.6, \xi_2 = 1.0$ on a fixed-fixed bar under lateral vibration

$\xi_1 = 0.6,\ \ \xi_2 = 1.0$					
j	**Aj**	**F_{1j}**	**F_{2j}**	**F_{3j}**	**F_{4j}**
1	-0.0398364256832	-3.92667609561488	-0.2157462811042	-1.27949857135126	0.12386477919121
2	0.00000000062688	-0.00000060384580	-0.0000000387961	-0.00000029176047	0.00000002709014
3	-0.0005972559182	-1.23443453370169	-0.1167421711815	-1.34529951947107	0.11675986077328
4	0.00000000001256	0.00000001271502	-0.0000000024724	-0.00000007571927	0.00000000587101
5	-0.0000624086231	0.50902618582038	0.01142603596911	-0.70291231644793	0.04530618994493
6	0	0	0	0	0
7	-0.0000000258216	-0.00036539791839	0.00000728113439	-0.00064787284348	0.00002465979710
	Total	**-4.65245043254537**	**-0.3210551764508**	**-3.32835864759348**	**0.28595552266766**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	-0.0398364256832	-0.0579799257937	0.10311372400732	0	-0.0000000207788
2	0.00000000062688	-0.0000000006486	-0.0000000051825	0	-0
3	-0.0005972559182	0.00037529284311	-0.0083468306820	0	-0.0000000000263
4	0.00000000001256	0.00000000001751	0.00000000003863	0	0
5	-0.0000624086231	0.00001374263093	0.00150519231258	0	-0.0000000000033
6	0	0	0	0	0
7	-0.0000000258216	-0.0000000258237	-0.0000006084548	0	0
	Total	**-0.0575909167743**	**0.09627147203923**	**0**	**-0.0000000208084**

u_{11} = -0.0579799257937 u_{31} = 0

u_{21} = 0.10311372400732 u_{41} = -0.0000000207788

From Table 3. 2

F_1 = 4.65245043254537 F_3 = 3.32835864759348

F_2 = 0.32105517645079 F_4 = -0.28595552266766

From equations (4.35) – (4.38)

P_1 = -2.53566704148522	P_3 = 10.51647612162406
P_2 = -0.87588958623633	P_4 = -1.96425774959277
From equation (4.49) taking EI = 1	
ϕ_1 = 1.09521423566082	ϕ_3 = 1.40740638741001
ϕ_2 = 0.92225281851241	ϕ_4 = 1.23715098708709

Tables 4.14 to 4.17 are illustrations on how the inherent nodal forces P_1 - P_4 and the stiffness modification factors $\phi_1 - \phi_4$ for a element of a fixed-fixed beam under lateral vibration are calculated. It would be observed that the values of the of fixed end forces calculated where negated, this is to take care of the sign convention use in the development of the equations of Table 3.2. Using the methods presented in Table 4.14 to 4.17 the values of stiffness modification factors at different values of ξ_1 and ξ_2 for the lateral vibration of a fixed-fixed beam are presented in Table 4.18 below. A sample matlab program for the calculation of the stiffness modification factors for a segment of a fixed-fixed beam can be found in Appendix D.

Table 4.18: Stiffness modification factors for the lateral vibration of a fixed-fixed beam

			ξ_2						
			0	0.05	0.10	0.15	0.20	0.25	0.30
ξ_1	0	$\emptyset_1$	-	1.013215	0.760820	2.008155	1.153819	1.030637	1.003687
		$\emptyset_2$	-	1.002412	0.715577	2.162977	1.183432	1.039888	0.986242
		$\emptyset_3$	-	1.120140	1.137169	1.155718	1.179020	1.210749	1.255236
		$\emptyset_4$	-	1.113574	1.123447	1.133304	1.145018	1.160295	1.180475
	0.05	$\emptyset_1$	1.013215	-	1.251848	1.202652	1.133046	1.074884	1.048824
		$\emptyset_2$	1.002412	-	1.262441	1.220267	1.151721	1.089912	1.050724
		$\emptyset_3$	1.120140	-	1.248770	1.198178	1.147220	1.126195	1.142708
		$\emptyset_4$	1.113574	-	1.238714	1.181370	1.124812	1.093274	1.091974
	0.10	$\emptyset_1$	0.760820	1.251848	-	1.154825	1.096929	1.048733	1.026611
		$\emptyset_2$	0.715577	1.262441	-	1.158004	1.103855	1.057057	1.030556
		$\emptyset_3$	1.137169	1.248770	-	1.154955	1.100505	1.064020	1.060915
		$\emptyset_4$	1.123447	1.238714	-	1.151738	1.092409	1.048831	1.034234
	0.15	$\emptyset_1$	2.008155	1.202652	1.154824	-	1.042871	0.995485	0.973431
		$\emptyset_2$	2.162977	1.220267	1.1580043	-	1.044072	0.998423	0.975930
		$\emptyset_3$	1.155718	1.198178	1.154955	-	1.043087	0.998055	0.982075
		$\emptyset_4$	1.133304	1.181370	1.151738	-	1.041796	0.993597	0.971856
	0.20	$\emptyset_1$	1.153819	1.133046	1.096929	1.042871	-	0.932602	0.908153
		$\emptyset_2$	1.183432	1.151721	1.103855	1.044072	-	0.933239	0.909649
		$\emptyset_3$	1.179020	1.147220	1.100505	1.043087	-	0.932719	0.909284
		$\emptyset_4$	1.145018	1.124812	1.092409	1.041796	-	0.931983	0.906252
	0.25	$\emptyset_1$	1.030637	1.074884	1.048733	0.995485	0.932602	-	0.851998
		$\emptyset_2$	1.039888	1.089912	1.057057	0.998423	0.933239	-	0.852511
		$\emptyset_3$	1.210749	1.126195	1.064020	0.998055	0.932719	-	0.852034
		$\emptyset_4$	1.160295	1.093274	1.048831	0.993597	0.931983	-	0.851430
	0.30	$\emptyset_1$	1.003687	1.048824	1.026611	0.973431	0.908153	0.851998	-
		$\emptyset_2$	0.986242	1.050724	1.030556	0.975930	0.909649	0.852511	-
		$\emptyset_3$	1.255236	1.142708	1.060915	0.982075	0.909284	0.852034	-
		$\emptyset_4$	1.180475	1.091974	1.034234	0.971856	0.906252	0.851430	-
	0.35	$\emptyset_1$	1.032039	1.060324	1.036211	0.981090	0.912221	0.851512	0.814977
		$\emptyset_2$	0.960158	1.029813	1.023055	0.976047	0.911846	0.852561	0.815448
		$\emptyset_3$	1.318078	1.192309	1.091797	0.997997	0.915252	0.851688	0.814969
		$\emptyset_4$	1.206264	1.115534	1.048680	0.979188	0.908268	0.849301	0.814425
	0.40	$\emptyset_1$	1.095214	1.102706	1.073097	1.013832	0.939288	0.872083	0.829757
		$\emptyset_2$	0.922253	1.008075	1.020542	0.986785	0.929251	0.870775	0.830650
		$\emptyset_3$	1.407406	1.268613	1.149004	1.038429	0.943551	0.871841	0.829501
		$\emptyset_4$	1.237151	1.155559	1.084817	1.009308	0.931920	0.867081	0.827616
	0.45	$\emptyset_1$	1.181749	1.167309	1.130434	1.064753	0.981210	0.903958	0.852890
		$\emptyset_2$	0.836944	0.960688	1.003241	0.991084	0.946658	0.893002	0.851725
		$\emptyset_3$	1.536654	1.370256	1.225495	1.094095	0.983959	0.901426	0.851417
		$\emptyset_4$	1.269846	1.205903	1.136106	1.055329	0.969180	0.895178	0.848297
	0.50	$\emptyset_1$	1.303552	1.258836	1.212614	1.136291	1.036919	0.942833	0.877477
		$\emptyset_2$	0.663259	0.859073	0.948358	0.971087	0.950447	0.908051	0.868205
		$\emptyset_3$	1.731522	1.502877	1.318201	1.159271	1.030471	0.933818	0.872717
		$\emptyset_4$	1.293649	1.266247	1.205491	1.119627	1.019405	0.929761	0.869922

			ξ_2						
			0.35	0.40	0.45	0.50	0.55	0.60	0.65
ξ_1	0	$\boldsymbol{\emptyset_1}$	1.032039	1.095214	1.181749	1.303552	1.536978	2.308365	-11.58787
		$\boldsymbol{\emptyset_2}$	0.960158	0.922253	0.836944	0.663259	0.305475	-0.735707	15.060751
		$\boldsymbol{\emptyset_3}$	1.318078	1.407406	1.536654	1.731522	2.051911	2.691037	4.962535
		$\boldsymbol{\emptyset_4}$	1.206264	1.237151	1.269846	1.293649	1.274222	1.075525	-0.340917
	0.05	$\boldsymbol{\emptyset_1}$	1.060324	1.102706	1.167309	1.258836	1.426476	2.097818	1.004257
		$\boldsymbol{\emptyset_2}$	1.029813	1.008075	0.960688	0.859073	0.642329	-0.152837	0.804209
		$\boldsymbol{\emptyset_3}$	1.192309	1.268613	1.370256	1.502877	1.666256	1.567218	3.701355
		$\boldsymbol{\emptyset_4}$	1.115534	1.155559	1.205903	1.266247	1.355130	1.780706	0.262422
	0.10	$\boldsymbol{\emptyset_1}$	1.036211	1.073097	1.130434	1.212614	1.368495	2.376144	1.071407
		$\boldsymbol{\emptyset_2}$	1.023055	1.020542	1.003241	0.948358	0.804189	-0.174134	1.015598
		$\boldsymbol{\emptyset_3}$	1.091797	1.149004	1.225495	1.318201	1.406677	0.782915	2.700375
		$\boldsymbol{\emptyset_4}$	1.048680	1.084817	1.136106	1.205491	1.331175	2.285273	0.707693
	0.15	$\boldsymbol{\emptyset_1}$	0.981090	1.013832	1.064753	1.136291	1.264526	1.776638	0.734132
		$\boldsymbol{\emptyset_2}$	0.976047	0.986785	0.991084	0.971087	0.893589	0.480194	1.399571
		$\boldsymbol{\emptyset_3}$	0.997997	1.038429	1.094095	1.159271	1.219184	1.010929	2.411399
		$\boldsymbol{\emptyset_4}$	0.979188	1.009308	1.055329	1.119627	1.234772	1.736258	0.493322
	0.20	$\boldsymbol{\emptyset_1}$	0.912221	0.939288	0.981210	1.036919	1.126283	1.347719	-17.946756
		$\boldsymbol{\emptyset_2}$	0.911846	0.929251	0.946658	0.950447	0.924737	0.800052	16.314821
		$\boldsymbol{\emptyset_3}$	0.915252	0.943551	0.983959	1.030471	1.079146	1.077408	18.373465
		$\boldsymbol{\emptyset_4}$	0.908268	0.931920	0.969180	1.019405	1.101217	1.312523	-19.968175
	0.25	$\boldsymbol{\emptyset_1}$	0.851512	0.872083	0.903958	0.942833	0.998627	1.109463	1.455462
		$\boldsymbol{\emptyset_2}$	0.852561	0.870775	0.893002	0.908051	0.909315	0.880622	0.694531
		$\boldsymbol{\emptyset_3}$	0.851688	0.871841	0.901426	0.933818	0.969977	1.006804	0.931350
		$\boldsymbol{\emptyset_4}$	0.849301	0.867081	0.895178	0.929760	0.980360	1.083170	1.426430
	0.30	$\boldsymbol{\emptyset_1}$	0.814977	0.829757	0.852890	0.877477	0.908159	0.965576	1.097027
		$\boldsymbol{\emptyset_2}$	0.815448	0.830650	0.851725	0.868205	0.877316	0.878890	0.854051
		$\boldsymbol{\emptyset_3}$	0.814969	0.829501	0.851417	0.872717	0.894934	0.926448	0.964062
		$\boldsymbol{\emptyset_4}$	0.814425	0.827616	0.848297	0.869922	0.897094	0.949103	1.071548
	0.35	$\boldsymbol{\emptyset_1}$	-	0.816504	0.8335505	0.848000	0.861230	0.887520	-
		$\boldsymbol{\emptyset_2}$	-	0.816904	0.834321	0.847354	0.854150	0.860226	-
		$\boldsymbol{\emptyset_3}$	-	0.816482	0.833219	0.846490	0.856510	0.873619	-
		$\boldsymbol{\emptyset_4}$	-	0.816035	0.831840	0.844533	0.855543	0.878394	-
	0.40	$\boldsymbol{\emptyset_1}$	0.816504	-	0.837497	0.846144	0.848573	-	0.887519
		$\boldsymbol{\emptyset_2}$	0.816904	-	0.837801	0.846803	0.848319	-	0.860226
		$\boldsymbol{\emptyset_3}$	0.816482	-	0.837478	0.845881	0.847322	-	0.873619
		$\boldsymbol{\emptyset_4}$	0.816035	-	0.837149	0.844913	0.845989	-	0.878394
	0.45	$\boldsymbol{\emptyset_1}$	0.833551	0.837497	-	0.855399	-	0.848572	0.861230
		$\boldsymbol{\emptyset_2}$	0.834321	0.837801	-	0.855635	-	0.848318	0.854150
		$\boldsymbol{\emptyset_3}$	0.833219	0.837478	-	0.855385	-	0.847321	0.856510
		$\boldsymbol{\emptyset_4}$	0.831840	0.837149	-	0.855138	-	0.845988	0.855543
	0.50	$\boldsymbol{\emptyset_1}$	0.848000	0.846144	0.855399	-	0.855398	0.846143	0.847999
		$\boldsymbol{\emptyset_2}$	0.847354	0.846803	0.855635	-	0.855634	0.846802	0.847354
		$\boldsymbol{\emptyset_3}$	0.846490	0.845881	0.855385	-	0.855384	0.845881	0.846489
		$\boldsymbol{\emptyset_4}$	0.844533	0.844913	0.855138	-	0.855137	0.844913	0.844533

			ξ_2						
			0.70	0.75	0.80	0.85	0.90	0.95	1.00
ξ_1	0	$\emptyset_1$	-0.369799	0.170855	0.251525	0.009195	-0.807094	-3.568891	-
		$\emptyset_2$	1.662239	0.219907	-1.351017	-3.718070	-7.901304	-18.741665	-
		$\emptyset_3$	-5.242594	0.256588	1.515910	2.696528	4.658887	10.235807	-
		$\emptyset_4$	8.601731	4.996296	5.332107	6.689763	9.800599	19.518472	-
	0.05	$\emptyset_1$	2.531415	-3.570186	-0.202681	0.104046	-0.106133	-0.935524	-3.568891
		$\emptyset_2$	-1.230657	6.113724	0.950537	-0.992074	-3.402389	-2.007178	-18.71663
		$\emptyset_3$	6.158617	-8.383279	-0.017202	1.591581	2.918805	4.901361	10.235807
		$\emptyset_4$	-1.131689	10.926486	5.260006	5.307910	6.429347	14.695172	19.518476
	0.10	$\emptyset_1$	1.618052	2.710606	-10.017119	-0.400145	-1.387680	-0.106133	-0.807094
		$\emptyset_2$	0.316248	-1.298827	14.850474	1.334026	4.908252	-3.402387	-7.901303
		$\emptyset_3$	3.344160	5.657693	-21.493753	-0.369144	-1.386908	2.918804	4.658887
		$\emptyset_4$	0.670560	-0.566326	20.877762	5.592012	11.376029	6.429349	9.800600
	0.15	$\emptyset_1$	1.287215	1.737565	2.760835	-	-0.400147	0.104046	0.009195
		$\emptyset_2$	0.883529	0.293027	-1.268890	-	1.334030	-0.992073	-3.718069
		$\emptyset_3$	2.581048	3.370308	5.566265	-	-0.369147	1.591580	2.696528
		$\emptyset_4$	0.834612	0.703115	-0.412044	-	5.592016	5.307911	6.689764
	0.20	$\emptyset_1$	0.943272	1.338288	3.185895	2.760837	-	-0.202681	0.251525
		$\emptyset_2$	1.232249	0.881176	-0.727963	-1.268891	14.850480	0.950538	-1.351017
		$\emptyset_3$	2.212601	2.582974	6.750157	5.566269	-	-0.017203	1.515910
		$\emptyset_4$	0.666466	0.834243	-1.149167	-0.412046	20.877769	5.260007	5.332107
	0.25	$\emptyset_1$	-0.347414	-	1.338288	1.737565	2.710607	-3.570187	0.170855
		$\emptyset_2$	2.165812	-	0.881175	0.293027	-1.298827	6.113724	0.219907
		$\emptyset_3$	2.803123	-	2.582975	3.370309	5.657694	-8.383280	0.256587
		$\emptyset_4$	-0.645941	-	0.834242	0.703114	-0.566327	10.926487	4.996297
	0.30	$\emptyset_1$	0.588312	-0.347414	0.943273	1.287215	1.618052	2.531416	-0.369798
		$\emptyset_2$	0.126178	2.165812	1.232249	0.883529	0.316248	-1.230657	1.662238
		$\emptyset_3$	0.644735	2.803123	2.212601	2.581048	3.344160	6.158618	-5.242598
		$\emptyset_4$	0.096074	-0.645941	0.666467	0.834611	0.670559	-1.131690	8.601735
	0.35	$\emptyset_1$	1.097026	1.455462	-17.946841	0.734132	1.071407	1.044257	-11.587900
		$\emptyset_2$	0.854051	0.694531	16.314889	1.399571	1.015598	0.804210	15.060783
		$\emptyset_3$	0.964062	0.931350	18.373540	2.411399	2.700376	3.701356	4.962526
		$\emptyset_4$	1.071548	1.426430	19.968268	0.493322	0.707693	0.262422	-0.340909
	0.40	$\emptyset_1$	0.965575	1.109462	1.347718	1.776638	2.376142	2.097817	2.308364
		$\emptyset_2$	0.878889	0.880622	0.800052	0.480194	-0.174132	-0.152836	-0.735707
		$\emptyset_3$	0.926448	1.006804	1.077408	1.010930	0.782916	1.567219	2.691037
		$\emptyset_4$	0.949102	1.083169	1.312523	1.736257	2.285272	1.780705	1.075525
	0.45	$\emptyset_1$	0.908159	0.998627	1.126283	1.264526	1.368495	1.426476	1.536978
		$\emptyset_2$	0.877316	0.909315	0.924737	0.893589	0.804189	0.642329	0.305475
		$\emptyset_3$	0.894934	0.969977	1.079146	1.219185	1.406677	1.666257	2.051911
		$\emptyset_4$	0.897094	0.980360	1.101217	1.234772	1.331175	1.355130	1.274222
	0.50	$\emptyset_1$	0.877476	0.942833	1.036919	1.136291	1.212614	1.258836	1.303552
		$\emptyset_2$	0.868205	0.908052	0.950447	0.971087	0.948358	0.859073	0.663259
		$\emptyset_3$	0.872717	0.933818	1.030471	1.159272	1.318201	1.502877	1.731523
		$\emptyset_4$	0.869921	0.929761	1.019405	1.119627	1.205491	1.266247	1.293649

			ξ_2						
			0	0.05	0.10	0.15	0.20	0.25	0.30
ξ_1	0.55	$\boldsymbol{\emptyset_1}$	1.536978	1.426476	1.368495	1.264526	1.126283	0.998627	0.908159
		$\boldsymbol{\emptyset_2}$	0.305475	0.642329	0.804189	0.893589	0.924737	0.909315	0.877316
		$\boldsymbol{\emptyset_3}$	2.051911	1.666256	1.406677	1.219184	1.079146	0.969977	0.894934
		$\boldsymbol{\emptyset_4}$	1.274222	1.355130	1.331175	1.234772	1.101217	0.980360	0.897094
	0.60	$\boldsymbol{\emptyset_1}$	2.308365	2.097818	2.376144	1.776638	1.347719	1.109463	0.965576
		$\boldsymbol{\emptyset_2}$	-0.735707	-0.152837	-0.174134	0.480194	0.800052	0.880622	0.878890
		$\boldsymbol{\emptyset_3}$	2.691037	1.567218	0.782915	1.010929	1.077408	1.006804	0.926448
		$\boldsymbol{\emptyset_4}$	1.075525	1.780706	2.285273	1.736258	1.312523	1.083170	0.949103
	0.65	$\boldsymbol{\emptyset_1}$	-11.587872	1.044257	1.071407	0.734132	-17.94676	1.455462	1.097027
		$\boldsymbol{\emptyset_2}$	15.060751	0.804209	1.015598	1.399571	16.314821	0.694531	0.854051
		$\boldsymbol{\emptyset_3}$	4.962535	3.701355	2.700375	2.411399	18.373465	0.931350	0.964062
		$\boldsymbol{\emptyset_4}$	-0.340917	0.262422	0.707693	0.493322	-19.96818	1.426430	1.071548
	0.70	$\boldsymbol{\emptyset_1}$	-0.369799	2.531415	1.618052	1.287215	0.943272	-0.347414	1.097026
		$\boldsymbol{\emptyset_2}$	1.662239	-1.230657	0.316248	0.883529	1.232249	2.165812	0.854051
		$\boldsymbol{\emptyset_3}$	-5.242594	6.158617	3.344160	2.581048	2.212601	2.803123	0.964062
		$\boldsymbol{\emptyset_4}$	8.601731	-1.131689	0.670560	0.834612	0.666466	-0.645941	1.071548
	0.75	$\boldsymbol{\emptyset_1}$	0.170855	-3.570186	2.710606	1.737565	1.338288	-	-0.347414
		$\boldsymbol{\emptyset_2}$	0.219907	6.113724	-1.298827	0.293027	0.881176	-	2.165812
		$\boldsymbol{\emptyset_3}$	0.256588	-8.383279	5.65769	3.370308	2.582974	-	2.803123
		$\boldsymbol{\emptyset_4}$	4.996296	10.926486	-0.566326	0.703115	0.834243	-	-0.645941
	0.80	$\boldsymbol{\emptyset_1}$	0.251525	-0.202681	-10.017119	2.760835	-	1.338288	0.943273
		$\boldsymbol{\emptyset_2}$	-1.351017	0.950537	14.850474	-1.268890	-	0.881175	1.232249
		$\boldsymbol{\emptyset_3}$	1.515910	-0.017202	-21.493753	5.566265	-	2.582975	2.212601
		$\boldsymbol{\emptyset_4}$	5.33211	5.260006	20.877762	-0.412044	-	0.834242	0.666466
	0.85	$\boldsymbol{\emptyset_1}$	0.009195	0.104046	-0.400145	-	2.760837	1.737565	1.287215
		$\boldsymbol{\emptyset_2}$	-3.718070	-0.992074	1.334026	-	-1.268891	0.293027	0.883529
		$\boldsymbol{\emptyset_3}$	2.696528	1.591581	-0.369144	-	5.566269	3.370309	2.581048
		$\boldsymbol{\emptyset_4}$	6.689763	5.307910	5.592012	-	-0.412046	0.703114	0.834611
	0.90	$\boldsymbol{\emptyset_1}$	-0.807094	-0.106133	-	-0.400147	-10.01712	2.710607	1.618052
		$\boldsymbol{\emptyset_2}$	-7.901304	-3.402389	-	1.334030	14.850480	-1.298827	0.316248
		$\boldsymbol{\emptyset_3}$	4.658887	2.918805	-	-0.369147	-21.49376	5.657694	3.344160
		$\boldsymbol{\emptyset_4}$	9.800599	6.429347	-	5.592016	20.877769	-0.566327	0.670560
	0.95	$\boldsymbol{\emptyset_1}$	-3.568891	-	-0.106133	0.104046	-0.202681	-3.570187	2.531416
		$\boldsymbol{\emptyset_2}$	-18.741665	-	-3.402387	-0.992073	0.950538	6.113725	-1.230657
		$\boldsymbol{\emptyset_3}$	10.235807	-	2.918804	1.591580	-0.017203	-8.383280	6.158618
		$\boldsymbol{\emptyset_4}$	19.518472	-	6.429349	5.307911	5.260007	10.926487	-1.131690
	1.00	$\boldsymbol{\emptyset_1}$	-	-3.568891	-0.807094	0.009195	0.251525	0.170855	-0.369798
		$\boldsymbol{\emptyset_2}$	-	-18.741663	-7.901303	-3.718069	-1.351017	0.219907	1.662238
		$\boldsymbol{\emptyset_3}$	-	10.235807	4.658887	2.696528	1.515910	0.256587	-5.242598
		$\boldsymbol{\emptyset_4}$	-	19.518476	9.800600	6.689764	5.332107	4.996297	8.601735

			ξ_2						
			0.35	0.40	0.45	0.50	0.55	0.60	0.65
ξ_1	0.55	$\emptyset_1$	0.861230	0.848573	-	0.855398	-	0.837497	0.833550
		$\emptyset_2$	0.854150	0.848319	-	0.855634	-	0.837801	0.834321
		$\emptyset_3$	0.856510	0.847322	-	0.855384	-	0.837477	0.833219
		$\emptyset_4$	0.855543	0.845989	-	0.855137	-	0.837149	0.831840
	0.60	$\emptyset_1$	0.887520	-	0.848572	0.846143	0.837497	-	0.816504
		$\emptyset_2$	0.860226	-	0.848318	0.846802	0.837801	-	0.816903
		$\emptyset_3$	0.873619	-	0.847321	0.845881	0.837477	-	0.816482
		$\emptyset_4$	0.878394	-	0.845988	0.844913	0.837149	-	0.816035
	0.65	$\emptyset_1$	-	0.887519	0.861230	0.847999	0.833550	0.816504	-
		$\emptyset_2$	-	0.860226	0.854150	0.847354	0.834321	0.816903	-
		$\emptyset_3$	-	0.873619	0.856510	0.846489	0.833219	0.816482	-
		$\emptyset_4$	-	0.878394	0.855543	0.844533	0.831840	0.816035	-
	0.70	$\emptyset_1$	1.097026	0.965575	0.908159	0.877476	0.852889	0.829757	0.814977
		$\emptyset_2$	0.854051	0.878889	0.877316	0.868205	0.851725	0.830650	0.815448
		$\emptyset_3$	0.964062	0.926448	0.894934	0.872717	0.851417	0.829501	0.814968
		$\emptyset_4$	1.071548	0.949102	0.897094	0.869921	0.848297	0.827616	0.814425
	0.75	$\emptyset_1$	1.455462	1.109462	0.998627	0.942833	0.903958	0.872083	0.851512
		$\emptyset_2$	0.694531	0.880622	0.909315	0.908051	0.893002	0.870775	0.852561
		$\emptyset_3$	0.931350	1.006804	0.969977	0.933818	0.901425	0.871841	0.851688
		$\emptyset_4$	1.426430	1.083169	0.980360	0.929761	0.895178	0.867081	0.849301
	0.80	$\emptyset_1$	-17.946841	1.347718	1.126283	1.036919	0.981210	0.939288	0.912221
		$\emptyset_2$	16.314889	0.800052	0.924737	0.950447	0.946658	0.929251	0.911846
		$\emptyset_3$	18.373540	1.077408	1.079146	1.030472	0.983959	0.943551	0.915252
		$\emptyset_4$	-19.968268	1.312523	1.101217	1.019405	0.969180	0.931920	0.908268
	0.85	$\emptyset_1$	0.734132	1.776638	1.264526	1.136291	1.064753	1.013832	0.981090
		$\emptyset_2$	1.399571	0.480194	0.893589	0.971087	0.991084	0.986785	0.976047
		$\emptyset_3$	2.411399	1.010930	1.219185	1.159272	1.094095	1.038429	0.997997
		$\emptyset_4$	0.493322	1.736257	1.234772	1.119627	1.055329	1.009307	0.979188
	0.90	$\emptyset_1$	1.071407	2.376142	1.368495	1.212614	1.130434	1.073097	1.036211
		$\emptyset_2$	1.015598	-0.174132	0.804189	0.948358	1.003241	1.020542	1.023055
		$\emptyset_3$	2.700376	0.782916	1.406677	1.318201	1.225495	1.149004	1.091798
		$\emptyset_4$	0.707693	2.285272	1.331175	1.205491	1.136106	1.084817	1.048680
	0.95	$\emptyset_1$	1.044257	2.097817	1.426476	1.258836	1.167309	1.102706	1.060324
		$\emptyset_2$	0.804210	-0.152836	0.642329	0.859073	0.960688	1.008075	1.029813
		$\emptyset_3$	3.701356	1.567219	1.666257	1.502877	1.370256	1.268613	1.192309
		$\emptyset_4$	0.262422	1.780705	1.355130	1.266247	1.205903	1.155559	1.115534
	1.00	$\emptyset_1$	-11.587900	2.308364	1.536978	1.303552	1.181749	1.095214	1.032039
		$\emptyset_2$	15.060783	-0.735707	0.305475	0.663259	0.836944	0.922253	0.960158
		$\emptyset_3$	4.962526	2.691037	2.051911	1.731523	1.536654	1.407406	1.318078
		$\emptyset_4$	-0.340909	1.075525	1.274222	1.293649	1.269846	1.237151	1.206264

			ξ_2						
			0.70	0.75	0.80	0.85	0.90	0.95	1.00
ξ_1	0.55	$Ø_1$	0.852889	0.903958	0.981210	1.064753	1.130434	1.167309	1.181749
		$Ø_2$	0.851725	0.893002	0.946658	0.991084	1.003241	0.960688	0.836944
		$Ø_3$	0.851417	0.901425	0.983959	1.094095	1.225495	1.370256	1.536654
		$Ø_4$	0.848297	0.895178	0.969180	1.055329	1.136106	1.205903	1.269846
	0.60	$Ø_1$	0.829757	0.872083	0.939288	1.013832	1.073097	1.102706	1.095214
		$Ø_2$	0.830650	0.870775	0.929251	0.986785	1.020542	1.008075	0.922253
		$Ø_3$	0.829501	0.871841	0.943551	1.038429	1.149004	1.268613	1.407406
		$Ø_4$	0.827616	0.867081	0.931920	1.009308	1.084818	1.155559	1.237151
	0.65	$Ø_1$	0.814977	0.851511	0.912221	0.981090	1.036211	1.060324	1.032039
		$Ø_2$	0.815448	0.852561	0.911846	0.976047	1.023055	1.029813	0.960158
		$Ø_3$	0.814968	0.851688	0.915252	0.997997	1.091797	1.192309	1.318078
		$Ø_4$	0.814425	0.849301	0.908268	0.979188	1.048680	1.115534	1.206264
	0.70	$Ø_1$	-	0.851998	0.908154	0.973431	1.026612	1.048824	1.003687
		$Ø_2$	-	0.852511	0.909649	0.975930	1.030556	1.050724	0.986242
		$Ø_3$	-	0.852034	0.909284	0.982075	1.060915	1.142708	1.255237
		$Ø_4$	-	0.851430	0.906252	0.971856	1.034234	1.091975	1.180475
	0.75	$Ø_1$	0.851998	-	0.932602	0.995485	1.048733	1.074884	1.030637
		$Ø_2$	0.852511	-	0.933239	0.998423	1.057057	1.089912	1.039888
		$Ø_3$	0.852034	-	0.932719	0.998055	1.064020	1.126195	1.210749
		$Ø_4$	0.851430	-	0.931983	0.993597	1.048832	1.093227	1.160295
	0.80	$Ø_1$	0.908154	0.932602	-	1.042871	1.096929	1.133046	1.153820
		$Ø_2$	0.909649	0.933239	-	1.044073	1.103856	1.151722	1.183433
		$Ø_3$	0.909284	0.932719	-	1.043087	1.100505	1.147220	1.179020
		$Ø_4$	0.906252	0.931983	-	1.041796	1.092409	1.124812	1.145018
	0.85	$Ø_1$	0.973431	0.995485	1.042871	-	1.154825	1.202652	2.008161
		$Ø_2$	0.975930	0.998423	1.044073	-	1.158004	1.220267	2.162983
		$Ø_3$	0.982075	0.998055	1.043087	-	1.154955	1.198178	1.155718
		$Ø_4$	0.971856	0.993597	1.041796	-	1.151739	1.181370	1.133304
	0.90	$Ø_1$	1.026612	1.048733	1.096929	1.154825	-	1.251848	0.760820
		$Ø_2$	1.030556	1.057057	1.103856	1.158004	-	1.262441	0.715578
		$Ø_3$	1.060915	1.064020	1.100505	1.154955	-	1.248770	1.137170
		$Ø_4$	1.034234	1.048832	1.092409	1.151739	-	1.238714	1.123447
	0.95	$Ø_1$	1.048824	1.074884	1.133046	1.202652	1.251848	-	1.013216
		$Ø_2$	1.050724	1.089912	1.151722	1.220267	1.262441	-	1.002412
		$Ø_3$	1.142708	1.126195	1.147220	1.198178	1.248770	-	1.120140
		$Ø_4$	1.091975	1.093275	1.124812	1.181370	1.238714	-	1.113575
	1.00	$Ø_1$	1.003687	1.030637	1.153820	2.008160	0.760820	1.013216	-
		$Ø_2$	0.986242	1.039888	1.183433	2.162983	0.715578	1.002412	-
		$Ø_3$	1.255237	1.210749	1.179020	1.155718	1.137170	1.120140	-
		$Ø_4$	1.180475	1.160295	1.145018	1.133304	1.123447	1.113575	-

Because our method of solution involves the computation of nodal forces, for segments that have zero nodal forces their modification factors could not be computed. This occurs for segments when $\xi_1 = 1 - \xi_2$

4.2.2 Fixed-pinned beams

These are beams rigidly restrained at one end and pinned at the other end. From section 3.2.2 the first natural frequency w for such a beam is given as

$$\omega_1 = 15.41820578\sqrt{\frac{EI}{\mu L^4}} \tag{4.54}$$

In normalized coordinates, the length of the bar L = 1 hence

$$\omega_1 = 15.41820578\sqrt{\frac{EI}{\mu}} \tag{4.55}$$

Likewise

$$u_1 = u(x_1, 0) = \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j x_1 + b_{2j} \sinh \beta_j x_1 - \cos \beta_j x_1 + b_{4j} \sin \beta_j x_1\right)$$

$$= \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j L\xi_1 + b_{2j} \sinh \beta_j L\xi_1 - \cos \beta_j L\xi_1 + b_{4j} \sin \beta_j L\xi_1\right)$$

(4.56a)

$$u_2 = u'(x_1, 0) = \sum_{j=1}^{\infty} A_j \beta_j L\left(\sinh \beta_j x_1 + b_{2j} \cosh \beta_j x_1 + \sin \beta_j x_1 + b_{4j} \cos \beta_j x_1\right)$$

$$= \sum_{j=1}^{\infty} A_j \beta L\left(\sinh \beta_j L\xi_1 + b_{2j} \cosh \beta_j L\xi_1 + \sin \beta_j L\xi_1 + b_{4j} \cos \beta_j L\xi_1\right)$$

(4.56b)

$$u_3 = u(x_2, 0) = \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j x_2 + b_{2j} \sinh \beta_j x_2 - \cos \beta_j x_2 + b_{4j} \sin \beta_j x_2\right)$$

$$= \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j L\xi_2 + b_{2j} \sinh \beta_j L\xi_2 - \cos \beta_j L\xi_2 + b_{4j} \sin \beta_j L\xi_2\right)$$

(4.56c)

$$u_4 = u'(x_2, 0) = \sum_{j=1}^{\infty} A_j \beta L\left(\sinh \beta_j x_2 + b_{2j} \cosh \beta_j x_2 + \sin \beta_j x_2 + b_{4j} \cos \beta_j x_2\right)$$

$$= \sum_{j=1}^{\infty} A_j \beta_j L\left(\sinh \beta_j L\xi_2 + b_{2j} \cosh \beta_j L\xi_2 + \sin \beta_j L\xi_2 + b_{4j} \cos \beta_j L\xi_2\right)$$

(4.56d)

Equations (4.56a) – (4.56d) are used to evaluate the total displacements u_1 to u_4 at the nodal points of a segment of the vibrating fixed-pinned beam. The equations represent the summation of an infinite series but an evaluation of the first few terms of the series provides values of very good precision.

From Table 3.3

$$A_j = \frac{b\mu L^2}{M_j}\left[\frac{-(\cosh \beta_j L+\cos \beta_j L)}{\beta_j^2 L^2} + \frac{18(\cosh \beta_j L-\cos \beta_j L)}{\beta_j^4 L^4} - \frac{48(\sinh \beta_j L-\sin \beta_j L)}{\beta_j^5 L^5} + \right.$$

$$\left. b_{2j}\left(\frac{-2(\sinh \beta_j L+\sin \beta_j L)}{\beta_j^2 L^2} + \frac{18(\sinh \beta_j L-\sin \beta_j L)}{\beta_j^4 L^4} - \frac{48(\cosh \beta_j L+\cos \beta_j L-2)}{\beta_j^5 L^5}\right)\right] \qquad (4.57)$$

The values of $\beta_j L$, b_{2j}, and b_{4j} for j = 1, 2, 3, 4, 5, 6, 7 can be obtained from Table 3.3. The formula for calculating the generalized mass M_j can also be picked from Table 3.3.

For this beam there are two possible cases, and the method of obtaining the stiffness modification factors depends on the case being considered.

a) **When ξ_1 is greater or equal to zero and ξ_2 is less than 1**

In this case the segment of the fixed-pinned beam under consideration is not positioned to the far right of the beam (the end that is pinned). Hence the process of calculating the stiffness modification factors is similar to the one for fixed-fixed beam.

The values of the fixed end forces F_1, F_2, F_3 and F_4 are evaluated using the equations provided in Table 3.3 while the values of the nodal displacements u_1, u_2, u_3 and u_4 are calculated from equations (4.56a) – (4.56b). These are substituted into the equations for nodal forces (equations 4.35) – (4.38) from which the nodal forces P_1, P_2, P_3 and P_4 are obtained.

Equations (4.56a) – (4.56d) are evaluated for the first mode, j = 1 to obtain the nodal displacements u_{11}, u_{21}, u_{31} and u_{41} due to the first mode. These together with the calculated nodal forces P_1 to P_4 are substituted into equation (4.49) in order to obtain the stiffness modification factors ϕ_1, ϕ_2, ϕ_3 and ϕ_4.

b) **When ξ_2 is equal to 1**

In this case the segment under consideration is located at the far right of the fixed-pinned beam. This implies that the segment is fixed at the left end and pinned at the right end hence its stiffness matrix is different from that of a fixed-fixed beam. The stiffness matrix for a fixed pinned beam is given as (Okonkwo and Onyeyili, 2012)

$$[k] = \begin{bmatrix} \frac{3EI}{l^3} & \frac{3EI}{l^2} & -\frac{3EI}{l^3} & 0 \\ \frac{3EI}{l^2} & \frac{3EI}{l} & -\frac{3EI}{l^2} & 0 \\ -\frac{3EI}{l^3} & -\frac{3EI}{l^2} & \frac{3EI}{l^3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.58}$$

The proposed stiffness matrix for this beam segment is therefore

$$[k_d] = \begin{bmatrix} \frac{3EI}{l^3}\phi_1 & \frac{3EI}{l^2}\phi_2 & -\frac{3EI}{l^3}\phi_3 & 0 \\ \frac{3EI}{l^2}\phi_2 & \frac{3EI}{l}\phi_1 & -\frac{3EI}{l^2}\phi_4 & 0 \\ -\frac{3EI}{l^3}\phi_3 & -\frac{3EI}{l^2}\phi_4 & \frac{3EI}{l^3}\phi_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.59}$$

By substituting equations (4.40), (4.42) and (4.59) into equation (4.39) we obtain

$$\begin{bmatrix} \frac{\mu(\xi_2-\xi_1)}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu(\xi_2-\xi_1)}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} -\omega^2 u_{11} \\ -\omega^2 u_{21} \\ -\omega^2 u_{31} \\ -\omega^2 u_{41} \end{Bmatrix} +$$

$$\begin{bmatrix} \frac{3EI}{l^3}\phi_1 & \frac{3EI}{l^2}\phi_2 & -\frac{3EI}{l^3}\phi_3 & 0 \\ \frac{3EI}{l^2}\phi_2 & \frac{3EI}{l}\phi_1 & -\frac{3EI}{l^2}\phi_4 & 0 \\ -\frac{3EI}{l^3}\phi_3 & -\frac{3EI}{l^2}\phi_4 & \frac{3EI}{l^3}\phi_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} \tag{4.60}$$

By multiplying out the first row of equation (4.60) we obtain

$$-\frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{11} + \frac{3EI}{l^3}\phi_1 u_{11} + \frac{3EI}{l^2}\phi_2 u_{21} - \frac{3EI}{l^3}\phi_3 u_{31} + 0 = P_1$$

$$\frac{3EI}{l^3}\phi_1 u_{11} + \frac{3EI}{l^2}\phi_2 u_{21} - \frac{3EI}{l^3}\phi_3 u_{31} = P_1 + \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{11} \tag{4.61}$$

By multiplying out the second row of equation (4.60) we obtain

$$0 + \frac{3EI}{l^2}\phi_2 u_{11} + \frac{3EI}{l}\phi_1 u_{21} - \frac{3EI}{l^2}\phi_4 u_{31} + 0 = P_2$$

$$\frac{3EI}{l}\phi_1 u_{21} + \frac{3EI}{l^2}\phi_2 u_{11} - \frac{3EI}{l^2}\phi_4 u_{31} = P_2 \tag{4.62}$$

By multiplying out the third row of equation (4.60) we obtain

$$-\frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{31} - \frac{3EI}{l^3}\phi_3 u_{11} - \frac{3EI}{l^2}\phi_4 u_{21} + \frac{3EI}{l^3}\phi_1 u_{31} + 0 = P_3$$

$$\frac{3EI}{l^3}\phi_1 u_{31} - \frac{3EI}{l^3}\phi_3 u_{11} - \frac{3EI}{l^2}\phi_4 u_{21} = P_3 - \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{31} \tag{4.63}$$

By multiplying out the fourth row of equation (4.60) we obtain

$$P_4 = 0 \tag{4.64}$$

Hence P_4 for an element with the far end pinned must be zero. This will help in ascertaining the correctness of our equations.

By putting equations (4.61) – (4.63) in matrix form we obtain

$$\begin{bmatrix} \frac{3EI}{l^3}u_{11} & \frac{3EI}{l^2}u_{21} & -\frac{3EI}{l^3}u_{31} & 0 \\ \frac{3EI}{l}u_{21} & \frac{3EI}{l^2}u_{11} & 0 & \frac{3EI}{l^2}u_{31} \\ \frac{3EI}{l^3}u_{31} & 0 & -\frac{3EI}{l^3}u_{11} & -\frac{3EI}{l^2}u_{21} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} P_1 + \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{11} \\ P_2 \\ P_3 + \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{31} \end{Bmatrix} \quad (4.65)$$

Equation (4.65) is a set of three equations (equations 4.61 – 4.63) with four unknowns ($\phi_1 to\ \phi_4$). To solve it there is need to know the value of one of the unknowns.

Let $\phi_4 = 1$ (4.66)

By substituting equation (4.66) into (4.65) and rearranging the equation we obtain

$$\begin{bmatrix} \frac{3EI}{l^3}u_{11} & \frac{3EI}{l^2}u_{21} & -\frac{3EI}{l^3}u_{31} \\ \frac{3EI}{l}u_{21} & \frac{3EI}{l^2}u_{11} & 0 \\ \frac{3EI}{l^3}u_{31} & 0 & -\frac{3EI}{l^3}u_{11} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} P_1 + \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{11} \\ P_2 - \frac{3EI}{l^2}u_{31} \\ P_3 + \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{31} + \frac{3EI}{l^2}u_{21} \end{Bmatrix}$$

Therefore

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{bmatrix} \frac{3EI}{l^3}u_{11} & \frac{3EI}{l^2}u_{21} & -\frac{3EI}{l^3}u_{31} \\ \frac{3EI}{l}u_{21} & \frac{3EI}{l^2}u_{11} & 0 \\ \frac{3EI}{l^3}u_{31} & 0 & -\frac{3EI}{l^3}u_{11} \end{bmatrix}^{-1} \begin{Bmatrix} P_1 + \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{11} \\ P_2 - \frac{3EI}{l^2}u_{31} \\ P_3 + \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{31} + \frac{3EI}{l^2}u_{21} \end{Bmatrix}$$

(4.67)

Equation (4.67) is a mathematical expression for calculating the four stiffness modification factors for an element of a beam (having the far end pinned) under lateral vibration. Note that $\phi_4 = 1$.

Having presented the methods of calculating the stiffness modification factors for a fixed-pinned beam, some numerical demonstrations of these steps are presented below. For clarity the calculations will be presented in a tabular form.

Example 1: when $\xi_1 = 0, \xi_2 = 0.5$

Table 4.19: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0, \xi_2 = 0.5$ on a fixed-pinned beam under lateral vibration

$\xi_1 = 0, \ \xi_2 = 0.5$					
j	**Aj**	**F_{1j}**	**F_{2j}**	**F_{3j}**	**F_{4j}**
1	-0.17364907535931	-3.46795443001802	-0.4184932923117	-10.379501871211	0.71973306595434
2	-0.00158875089165	-0.67522094272502	-0.0770120802222	-1.1637605379658	0.10110850740348
3	-0.00147691803689	-3.13035585884167	-0.3210181183045	-0.8549012311576	0.22202170181458
4	-0.00006626491683	-0.34660132237135	-0.0291042316727	0.23742796910115	-0.0036895971616
5	-0.00013421224883	-1.21624196413605	-0.0736000402438	0.33795071940028	-0.0530628310870
6	-0.00000988628419	-0.14307523918566	-0.0064799781693	-0.1043775770341	0.00009496675733
7	-0.00000020890674	-0.00490120548876	-0.0002129996562	-0.0013714630787	0.00015565023283
	Total	**-8.98435096276653**	**-0.9259207405804**	**-11.928533991946**	**0.98636146391392**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	-0.17364907535931	0	0	-0.2508979869619	-0.2713326562263
2	-0.00158875089165	0	0	-0.0009061450552	0.01500096921473
3	-0.00147691803689	0	0	0.0019207280459	0.00825249030240
4	-0.00006626491683	0	0	0.00003577875740	-0.0011548709728
5	-0.00013421224883	0	0	-0.0001753919344	-0.0011974174623
6	-0.00000988628419	0	0	-0.0000053509572	0.00025363631599
7	-0.00000020890674	0	0	0.00000027294745	0.00000257515922
	Total	**0**	**0**	**-0.2500280951625**	**-0.2501752736690**
$u_{11} = 0$		u_{31} = -0.2500280951625			
$u_{21} = 0$		u_{41} = -0.2501752736690			

From Table 3. 3 F_1 = 8.98435096276653 F_3 = 11.92853399194589 F_2 = 0.92592074058035 F_4 = -0.98636146391392 From equations (4.35) – (4.38) P_1 = 26.98284153031153 P_3 = -6.06995657559912 P_2 = 5.92589392980464 P_4 = 3.01291063063430 From equation (4.49) taking EI = 1 ϕ_1 = 1.11502214241540 ϕ_3 = 1.45734448442994 ϕ_2 = 0.90229998347702 ϕ_4 = 1.24678725977081

Table 4.20: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.5, \xi_2 = 1.0$ on a fixed-pinned beam under lateral vibration

$\xi_1 = 0.5, \ \ \xi_2 = 1.0$					
j	**Aj**	**F_{1j}**	**F_{2j}**	**F_{3j}**	**F_{4j}**
1	-0.17364907535931	-13.9957762990768	-1.0948663443722	-7.6576426975165	0.82549618887594
2	-0.00158875089165	0.60907317204077	0.07545574747650	0.90216848365137	-0.0886461105590
3	-0.00147691803689	1.73552601656857	-0.0046743568665	-3.1173408929962	0.24972243460432
4	-0.00006626491683	-0.26058792146973	-0.0197932507008	0.27736871099274	-0.0137706183840
5	-0.00013421224883	-0.29649351035138	0.04642622845628	-0.8811548234496	0.01604507715516
6	-0.00000988628419	0.09676165271599	0.00286512421068	0.10685201827263	-0.0013100578916
7	-0.00000020890674	0.00158536678243	-0.0000875313883	-0.0037403522362	0.00007068397551
	Total	**-12.1099115227902**	**-0.9946743831844**	**-10.373489553282**	**0.98760759777631**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	-0.17364907535931	-0.25089798696619	-0.2713326562263	0	0.99154730530086

2	-0.00158875089165	-0.00090614505516	0.01500096921473	0	-0.0158628131389
3	-0.00147691803689	0.00192072804559	0.00825249030240	0	0.02132687502742
4	-0.00006626491683	0.00003577875740	-0.0011548709728	0	-0.0012512280745
5	-0.00013421224883	-0.00017539193443	-0.0011974174623	0.00000000000012	0.0031305191869
6	-0.00000988628419	-0.00000535095716	0.00025363631599	0	-0.0002745225210
7	-0.00000020890674	0.000000027294745	0.00000257515922	0	0.00000672907449
	Total	**-0.25002809516252**	**-0.2501752736690**	**0.00000000000012**	**0.99862286485519**

u_{11} = -0.25089798696619 u_{31} = 0

u_{21} = -0.2713326562263 u_{41} = 0.99154730530086

From Table 3. 3

F_1 = 12.10991152279019 F_3 = 10.37348955328170

F_2 = 0.99467438318438 F_4 = -0.98760759777631

From equations (4.35) – (4.38)

P_1 = 6.06995657564480 P_3 = 16.41344450042708

P_2 = -3.01291063065038 $P_4 = -5.751425291578016 \times 10^{-8}$

From equation (4.67) taking EI = 1

ϕ_1 = 1.31019183437732 ϕ_3 = 2.18506014881593

ϕ_2 = 0.29225810649324 ϕ_4 = 1

Table 4.21: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0, \xi_2 = 0.4$ on a fixed-pinned beam under lateral vibration

$\xi_1 = 0,\ \xi_2 = 0.4$					
j	**Aj**	$\mathbf{F_{1j}}$	$\mathbf{F_{2j}}$	$\mathbf{F_{3j}}$	$\mathbf{F_{4j}}$
1	-0.17364907535931	-1.95885130539335	-0.1908933873925	-6.3625124225160	0.34235311278584

2	-0.00158875089165	-0.43394284176040	-0.0407922390309	-1.0153455890456	0.06201407643782
3	-0.00147691803689	-2.40123420143900	-0.2126651325310	-2.9121117209021	0.23850572770471
4	-0.00006626491683	-0.32524583053818	-0.0260852315448	-0.0243416733237	0.01509778390815
5	-0.00013421224883	-1.32495004334296	-0.0901757378075	0.88020928018414	-0.0075968378043
6	-0.00000988628419	-0.15509835649696	-0.0082222737245	0.08028930755915	-0.0053795277931
7	-0.00000020890674	-0.00477286511963	-0.0001915999613	-00174848090669	-0.0000961213109
	Total	**-6.60409544409049**	**-0.5690256019924**	**-9.3555612989508**	**0.64489821392814**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	-0.17364907535931	0	0	-0.2095501987208	-0.5388622969652
2	-0.00158875089165	0	0	-0.0020959222175	0.00787482324347
3	-0.00147691803689	0	0	0.00030187176657	0.02131713302710
4	-0.00006626491683	0	0	0.00009224141154	0.00019997579844
5	-0.00013421224883	0	0	0.00008598650455	-0.0027862934131
6	-0.00000988628419	0	0	-0.0000098901221	-0.0001940413795
7	-0.00000020890674	0	0	-0.0000002632609	0.00000305546161
	Total	**0**	**0**	**-0.2111761746386**	**-0.5124476442272**

$u_{11} = 0$ $u_{31} = -0.2095501987208$

$u_{21} = 0$ $u_{41} = -0.5388622969652$

From Table 3. 3

$F_1 = 6.60409544409049$ $F_3 = 9.35556129895076$

$F_2 = 0.56902560199239$ $F_4 = -0.64489821392814$

From equations (4.35) – (4.38)

$P_1 = 26.98284153031158$ $P_3 = -11.02318478727032$

$P_2 = 5.92589392980464$ $P_4 = 2.14973189274804$

From equation (4.49) taking EI = 1

ϕ_1 = 1.04248077620655	ϕ_3 = 1.30465183700470
ϕ_2 = 0.98843685777153	ϕ_4 = 1.20143461764998

Table 4.22: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.4$, $\xi_2 = 0.7$ on a fixed-pinned beam under lateral vibration

$\xi_1 = 0.4,\ \ \xi_2 = 0.7$					
j	**Aj**	**F_{1j}**	**F_{2j}**	**F_{3j}**	**F_{4j}**
1	-0.17364907535931	-8.62846486345837	-0.4454268830278	-9.0486286219172	0.45616689269091
2	-0.00158875089165	-0.36737036896912	-0.0105135180580	0.28384180002719	-0.0062075646843
3	-0.00147691803689	2.38143485506075	0.13704682028785	2.00543405536515	-0.1270297533939
4	-0.00006626491683	0.17431842397132	0.00242763816785	-0.2850304744890	0.01036703463981
5	-0.00013421224883	-0.94044200180817	-0.0623405025903	-0.0179546022520	0.03480406905528
6	-0.00000988628419	-0.10672925569029	-0.0007707078843	0.12964287387314	-0.0070406794717
7	-0.00000020890674	0.00212233816722	0.00020827792590	-0.0035142364815	0.00000952906367
	Total	**-7.48513087272667**	**-0.3793688751788**	**-6.9392092058742**	**0.36106952789975**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	-0.17364907535931	-0.20955019872077	-0.5388622969652	-0.2370270716716	0.41706837652900
2	-0.00158875089165	-0.00209592221751	0.00787482324347	0.00190462781234	0.00837914022987
3	-0.00147691803689	0.00030187176657	0.02131713302710	-0.0001650357299	-0.0212481296734
4	-0.00006626491683	0.00009224141154	0.00019997579844	-0.0000712655121	0.00081268672021
5	-0.00013421224883	0.00008598650455	-0.0027862934131	0.00018455916205	0.00073082654981
6	-0.00000988628419	-0.00000989012208	-0.0001940413795	-0.0000053504291	-0.0002536255300
7	-0.00000020890674	-0.00000026326093	0.00000305546161	-0.0000001543664	0.00000573747975
	Total	**-0.21117617463862**	**-0.5124476442272**	**-0.2351796907346**	**0.40549501230523**
u_{11} = -0.20955019872077			u_{31} = -0.2370270716716		
u_{21} = -0.5388622969652			u_{41} = 0.41706837652900		

From Table 3. 3 F_1 = 7.48513087272667 F_3 = 6.93620920587415 F_2 = 0.37936887517875 F_4 = -0.36106952789975 From equations (4.35) – (4.38) P_1 = 11.02318478727057 P_3 = 3.39815529133027 P_2 = -2.14973189274783 P_4 = 3.22944741439000 From equation (4.49) taking EI = 1 ϕ_1 = 0.99486338610132 ϕ_3 = 0.97560437880874 ϕ_2 = 0.93372174004309 ϕ_4 = 0.97012166839347

Table 4.23: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.7, \xi_2 = 1.0$ on a fixed-pinned beam under lateral vibration

$\xi_1 = 0.7, \ \xi_2 = 1.0$					
j	**Aj**	**F_{1j}**	**F_{2j}**	**F_{3j}**	**F_{4j}**
1	-0.17364907535931	-6.48137328145983	-0.2864567110644	-3.0210448030777	0.19928186132517
2	-0.00158875089165	0.75817259323220	0.03666989034355	0.44690458151699	-0.0286680538133
3	-0.00147691803689	-2.28564824481658	-0.1351688664291	-2.1549467096943	0.13170175396620
4	-0.00006626491683	0.08394024483541	0.01044569016191	0.28396674580612	-0.0160161637421
5	-0.00013421224883	0.53824513976257	-0.0072880889098	-1.1910473518033	0.05891637688410
6	-0.00000988628419	-0.13536861027945	-0.0035057687902	0.14342489529287	-0.0057078323818
7	-0.00000020890674	0.00373893447016	0.00019114988964	-0.0042533466324	0.00011767982716
	Total	**-7.51829322425552**	**-0.3851127047983**	**-5.4969959885918**	**0.33962562206542**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	-0.17364907535931	-0.23702707167164	0.41706837652900	0	0.99154730530086

2	-0.00158875089165	0.00190462781234	0.00837914022987	0	-0.0158628131389
3	-0.00147691803689	-0.00016503572987	-0.0212481296734	0	0.02132687502742
4	-0.00006626491683	-0.00007126551212	0.00081268672021	0	-0.0012512280745
5	-0.00013421224883	0.00018455916205	0.00073082654981	0.00000000000012	0.00313051918690
6	-0.00000988628419	-0.00000535042906	-0.0002536255300	0	-0.0002745225210
7	-0.00000020890674	-0.00000015436635	0.00000573747975	0	0.00000672907449
	Total	**-0.23517969073464**	**0.40549501230523**	**0.00000000000012**	**-0.0002745225210**

u_{11} = -0.23702707167164 u_{31} = 0

u_{21} = 0.41706837652900 u_{41} = 0.99154730530086

From Table 3. 3

F_1 = 7.51829322425552 F_3 = 5.49699598859177

F_2 = 0.38511270479828 F_4 = -0.33962562206542

From equations (4.35) – (4.38)

P_1 = -3.39815529161311 P_3 = 16.41344450446040

P_2 = -3.22944741441518 $P_4 = -5.761254406877470 \times 10^{-8}$

From equation (4.67) taking EI = 1

ϕ_1 = 0.92287981019171 ϕ_3 = 1.15109852884999

ϕ_2 = 0.89590870859280 ϕ_4 = 1

Tables 4.19 to 4.23 provide illustrations on how the inherent nodal forces P_1 to P_4 and the stiffness modification factors ϕ_1 to ϕ_4 for a element of a fixed-pinned beam under lateral vibration are calculated. It would be observed that the values of the of fixed end forces calculated where negated, this is to take care of the sign convention used in the development of the equations of Table 3.2. Using the methods presented in Table

4.19 to 4.23 the values of stiffness modification factors at different values of ξ_1 and ξ_2 for the lateral vibration of a fixed-pinned beam are presented in Table 4.24. A sample matlab program for the calculation of the stiffness modification factors for a segment of a fixed-pinned beam can be found in Appendix E.

Table 4.24: Stiffness modification factors for the lateral vibration of a fixed-pinned beam

			ξ_2						
			0	0.05	0.10	0.15	0.20	0.25	0.30
ξ_1	0	$\emptyset_1$	-	1.033877	0.881306	-0.068863	1.339468	1.095985	1.032136
		$\emptyset_2$	-	1.025153	0.852781	-0.238489	1.394572	1.117229	1.038153
		$\emptyset_3$	-	1.122936	1.137995	1.153326	1.170708	1.192121	1.219770
		$\emptyset_4$	-	1.117139	1.126060	1.134404	1.143244	1.153613	1.166398
	0.05	$\emptyset_1$	1.033877	-	1.279392	1.241155	1.176808	1.116289	1.077191
		$\emptyset_2$	1.025153	-	1.290878	1.260557	1.197586	1.135201	1.090533
		$\emptyset_3$	1.122936	-	1.275561	1.231454	1.177547	1.144591	1.140538
		$\emptyset_4$	1.117139	-	1.264738	1.213846	1.156535	1.117719	1.103020
	0.10	$\emptyset_1$	0.881306	1.279392	-	1.188396	1.138297	1.091330	1.060427
		$\emptyset_2$	0.852781	1.290878	-	1.191860	1.145849	1.101072	1.069592
		$\emptyset_3$	1.137995	1.275561	-	1.188327	1.139927	1.100696	1.084064
		$\emptyset_4$	1.126060	1.264738	-	1.184873	1.131904	1.087551	1.063277
	0.15	$\emptyset_1$	-0.068863	1.241155	1.188396	-	1.092959	1.048665	1.020413
		$\emptyset_2$	-0.238489	1.260557	1.191860	-	1.094265	1.051930	1.024704
		$\emptyset_3$	1.153326	1.231454	1.188326	-	1.093086	1.050381	1.026759
		$\emptyset_4$	1.134404	1.213846	1.184873	-	1.091736	1.046325	1.018445
	0.20	$\emptyset_1$	1.339468	1.176808	1.138297	1.092959	-	0.998710	0.971047
		$\emptyset_2$	1.394572	1.197586	1.145847	1.094265	-	0.999314	0.972668
		$\emptyset_3$	1.170708	1.177547	1.139927	1.093086	-	0.998796	0.971973
		$\emptyset_4$	1.143244	1.156535	1.131904	1.091736	-	0.998140	0.969549
	0.25	$\emptyset_1$	1.095985	1.116289	1.091330	1.048665	0.998710	-	0.926831
		$\emptyset_2$	1.117229	1.135201	1.101072	1.051930	0.999314	-	0.927224
		$\emptyset_3$	1.192121	1.144591	1.100696	1.050382	0.998796	-	0.926868
		$\emptyset_4$	1.153613	1.117719	1.087551	1.046325	0.998140	-	0.926428
	0.30	$\emptyset_1$	1.032136	1.077191	1.060427	1.020413	0.971047	0.926831	-
		$\emptyset_2$	1.038153	1.090533	1.069592	1.024704	0.972668	0.927224	-
		$\emptyset_3$	1.219770	1.140538	1.084064	1.026759	0.971973	0.926868	-
		$\emptyset_4$	1.166398	1.103020	1.063277	1.018448	0.969549	0.926428	-
	0.35	$\emptyset_1$	1.023912	1.063765	1.049907	1.011332	0.962015	0.916877	0.886483
		$\emptyset_2$	1.007971	1.064407	1.053431	1.013851	0.963722	0.917931	0.886812
		$\emptyset_3$	1.256213	1.161672	1.091764	1.025199	0.965038	0.917211	0.886492
		$\emptyset_4$	1.182250	1.108094	1.059770	1.010443	0.959545	0.915470	0.886125
	0.40	$\emptyset_1$	1.042481	1.070584	1.056290	1.017851	0.967665	0.920644	0.887791
		$\emptyset_2$	0.988437	1.046949	1.045859	1.013173	0.966556	0.921389	0.888665
		$\emptyset_3$	1.304652	1.201630	1.118708	1.041331	0.973839	0.921581	0.887833
		$\emptyset_4$	1.201435	1.125756	1.071428	1.017765	0.963976	0.917746	0.886443
	0.45	$\emptyset_1$	1.073619	1.089900	1.073594	1.034306	0.982192	0.932022	0.895439
		$\emptyset_2$	0.957879	1.025527	1.037109	1.013841	0.972846	0.929700	0.895997
		$\emptyset_3$	1.369472	1.256989	1.159877	1.069524	0.992396	0.933727	0.895459
		$\emptyset_4$	1.223540	1.150378	1.092244	1.034610	0.977073	0.927272	0.892600
	0.50	$\emptyset_1$	1.115022	1.119190	1.099718	1.058605	1.003193	0.948205	0.906311
		$\emptyset_2$	0.902300	0.989655	1.018936	1.008764	0.976381	0.937422	0.904014
		$\emptyset_3$	1.457344	1.329203	1.214352	1.107639	1.017953	0.950568	0.906087
		$\emptyset_4$	1.246787	1.179413	1.119691	1.058690	0.996495	0.941408	0.901631

			ξ_2						
			0.35	0.40	0.45	0.50	0.55	0.60	0.65
ξ_1	0	$\emptyset_1$	1.023912	1.042481	1.073619	1.115022	1.178909	1.298098	1.565512
		$\emptyset_2$	1.007971	0.988437	0.957879	0.902300	0.806776	0.636098	0.274212
		$\emptyset_3$	1.256213	1.304652	1.369472	1.457344	1.579701	1.759370	2.053368
		$\emptyset_4$	1.182250	1.201435	1.223540	1.246787	1.266293	1.269036	1.215803
	0.05	$\emptyset_1$	1.063765	1.070584	1.089900	1.119190	1.166086	1.252514	1.439769
		$\emptyset_2$	1.064407	1.046949	1.025527	0.989655	0.928820	0.820505	0.590009
		$\emptyset_3$	1.161672	1.201630	1.256989	1.329203	1.423784	1.547124	1.689889
		$\emptyset_4$	1.108094	1.125756	1.150378	1.179412	1.212939	1.253575	1.319927
	0.10	$\emptyset_1$	1.049907	1.056290	1.073594	1.099718	1.141105	1.217102	1.390210
		$\emptyset_2$	1.053431	1.045859	1.037109	1.018936	0.983615	0.913988	0.742198
		$\emptyset_3$	1.091764	1.118708	1.159877	1.214352	1.284777	1.371088	1.437638
		$\emptyset_4$	1.059770	1.071428	1.092244	1.119691	1.156041	1.210566	1.327781
	0.15	$\emptyset_1$	1.011332	1.017851	1.034306	1.058605	1.096358	1.164507	1.315331
		$\emptyset_2$	1.013851	1.013173	1.013841	1.008764	0.992269	0.951975	0.836528
		$\emptyset_3$	1.025199	1.041331	1.069524	1.107639	1.157121	1.216605	1.252451
		$\emptyset_4$	1.010443	1.017765	1.034610	1.058690	1.093397	1.150198	1.273588
	0.20	$\emptyset_1$	0.962014	0.967665	0.982192	1.003193	1.035176	1.091335	1.203438
		$\emptyset_2$	0.963722	0.966556	0.972846	0.976381	0.973485	0.957113	0.898489
		$\emptyset_3$	0.965038	0.973839	0.992396	1.017953	1.051874	1.094707	1.129103
		$\emptyset_4$	0.959545	0.963976	0.977073	0.996495	1.025697	1.074854	1.171610
	0.25	$\emptyset_1$	0.916877	0.920644	0.932022	0.948205	0.972684	1.015385	1.093518
		$\emptyset_2$	0.917931	0.921389	0.929700	0.937422	0.942683	0.942915	0.924295
		$\emptyset_3$	0.917211	0.921581	0.933727	0.950568	0.973800	1.006330	1.042558
		$\emptyset_4$	0.915470	0.917746	0.927272	0.941408	0.963208	1.000885	1.068908
	0.30	$\emptyset_1$	0.886483	0.887791	0.895439	0.906311	0.923337	0.954701	1.010846
		$\emptyset_2$	0.886812	0.888665	0.895997	0.904014	0.912332	0.921622	0.925848
		$\emptyset_3$	0.886492	0.887833	0.895459	0.906087	0.921791	0.947272	0.982758
		$\emptyset_4$	0.886125	0.886443	0.892600	0.901631	0.916300	0.943779	0.992511
	0.35	$\emptyset_1$	-	0.871440	0.875172	0.880864	0.891391	0.914612	0.958926
		$\emptyset_2$	-	0.871734	0.875976	0.881452	0.889069	0.902055	0.918867
		$\emptyset_3$	-	0.871438	0.875130	0.880652	0.890512	0.910924	0.945471
		$\emptyset_4$	-	0.871118	0.873956	0.878282	0.886897	0.907022	0.945438
	0.40	$\emptyset_1$	0.871440	-	0.866052	0.866567	0.871307	0.888934	0.928794
		$\emptyset_2$	0.871734	-	0.866321	0.867339	0.871772	0.885372	0.910438
		$\emptyset_3$	0.871438	-	0.866049	0.866528	0.871061	0.887574	0.922994
		$\emptyset_4$	0.871118	-	0.865761	0.865435	0.868731	0.883858	0.918611
	0.45	$\emptyset_1$	0.875172	0.866052	-	0.855448	0.854261	0.867582	0.909029
		$\emptyset_2$	0.875976	0.866321	-	0.855716	0.855011	0.867338	0.900319
		$\emptyset_3$	0.875130	0.866049	-	0.855447	0.854221	0.867185	0.906732
		$\emptyset_4$	0.873956	0.865761	-	0.855158	0.853036	0.864429	0.900973
	0.50	$\emptyset_1$	0.880864	0.866567	0.855448	-	0.835749	0.845400	0.901008
		$\emptyset_2$	0.881452	0.867339	0.855716	-	0.836042	0.845908	0.889927
		$\emptyset_3$	0.880652	0.866528	0.855447	-	0.835746	0.845315	0.900184
		$\emptyset_4$	0.878282	0.865435	0.855158	-	0.835415	0.843832	0.888793

			ξ_2						
			0.70	0.75	0.80	0.85	0.90	0.95	1.00
ξ_1	0	$\emptyset_1$	2.595914	-2.172110	0.177611	0.476098	0.486790	0.325283	-
		$\emptyset_2$	-1.018911	4.281781	1.199283	0.261280	-0.615353	-1.601405	-
		$\emptyset_3$	2.672121	5.876722	-1.117326	0.915539	1.645079	2.263494	-
		$\emptyset_4$	0.944652	-1.376949	4.708916	3.533182	3.574035	3.927801	-
	0.05	$\emptyset_1$	2.406019	1.263019	8.676850	0.149628	0.517018	0.501695	-5.124731
		$\emptyset_2$	-0.529062	0.532814	-9.023118	1.315316	0.203700	-0.669598	-32.333667
		$\emptyset_3$	1.341264	3.674345	22.074381	-0.056971	1.210150	1.801691	713.258506
		$\emptyset_4$	1.897197	0.117870	-14.143094	3.756001	3.182342	3.257590	1
	0.10	$\emptyset_1$	4.857283	1.212148	1.881503	-1.255422	0.530840	0.631951	-1.410466
		$\emptyset_2$	-2.748434	0.786906	-0.196723	3.524698	0.720562	-0.136393	-14.413830
		$\emptyset_3$	-1.668272	2.594997	3.747179	-3.426459	0.858208	1.495322	154.766607
		$\emptyset_4$	4.671569	0.706843	0.068954	5.913190	2.855691	2.672692	1
	0.15	$\emptyset_1$	2.743700	1.030684	1.411778	7.284137	0.633354	0.733836	-0.233091
		$\emptyset_2$	-0.479304	1.056240	0.518456	-7.271644	0.841907	0.025170	-7.815071
		$\emptyset_3$	0.116914	2.166176	2.545986	15.805416	0.812959	1.318795	59.431587
		$\emptyset_4$	2.705275	0.741690	0.666071	-9.468059	2.319239	1.993284	1
	0.20	$\emptyset_1$	1.595085	0.685026	1.153122	3.049482	0.810269	0.710214	0.264989
		$\emptyset_2$	0.584734	1.374291	0.841189	-1.777032	0.606567	-0.151945	-4.283662
		$\emptyset_3$	0.949759	2.061940	2.017123	6.395831	1.029725	1.020466	28.794937
		$\emptyset_4$	1.546875	0.481327	0.749365	-2.606944	1.556480	1.210513	1
	0.25	$\emptyset_1$	1.260275	4.506745	0.918584	0.647380	0.809997	0.145598	0.481023
		$\emptyset_2$	0.825847	-2.041302	1.051814	1.171276	0.288174	-1.096990	-2.180505
		$\emptyset_3$	1.031972	-1.882180	1.749396	0.397315	0.932240	-0.347950	15.860689
		$\emptyset_4$	1.219270	4.649108	0.668866	1.731909	0.907141	0.018121	1
	0.30	$\emptyset_1$	1.105308	1.308447	0.175611	0.857180	0.341559	3.856357	0.563581
		$\emptyset_2$	0.898734	0.727975	1.797289	0.695899	-0.675944	5.429493	-0.894186
		$\emptyset_3$	1.010909	0.932727	2.156916	1.036061	-0.373641	7.815737	9.504482
		$\emptyset_4$	0.073318	1.250929	0.065448	0.822704	0.123245	4.652578	1
	0.35	$\emptyset_1$	1.027706	1.120185	1.027196	0.171517	1.568103	1.584875	0.564293
		$\emptyset_2$	0.923017	0.858695	0.789897	3.485079	2.670149	1.806860	-0.071200
		$\emptyset_3$	0.985988	0.999926	0.822494	4.372770	3.590924	2.815269	6.070767
		$\emptyset_4$	1.001972	1.057561	1.086947	-0.057100	1.640720	1.686265	1
	0.40	$\emptyset_1$	0.994863	0.881270	0.900268	-0.208520	0.804642	1.210537	0.399923
		$\emptyset_2$	0.933722	1.116344	0.799188	0.763577	1.502086	1.441033	0.614009
		$\emptyset_3$	0.975604	0.860653	0.863077	0.930544	1.952179	2.087233	4.096635
		$\emptyset_4$	0.970122	1.145231	0.873407	-0.445045	0.726622	1.222386	1
	0.45	$\emptyset_1$	1.042454	1.074389	1.719053	1.757123	-7.271986	0.903924	13.513977
		$\emptyset_2$	0.895998	1.031793	1.445889	1.224938	2.420337	1.357213	-12.149381
		$\emptyset_3$	1.031469	1.047945	1.552103	1.378081	2.866212	1.787913	2.913496
		$\emptyset_4$	0.921715	1.087383	1.788588	1.834015	-8.584750	0.865025	1
	0.50	$\emptyset_1$	1.061705	0.383753	0.970752	1.191346	1.493014	-6.403278	1.310192
		$\emptyset_2$	1.054167	0.422509	0.930595	1.052685	1.086303	3.187114	0.292258
		$\emptyset_3$	1.056671	0.396075	0.955443	1.129075	1.235296	3.756568	2.185060
		$\emptyset_4$	1.069211	0.343848	0.959581	1.195016	1.536377	-7.546021	1

			ξ_2						
			0	0.05	0.10	0.15	0.20	0.25	0.30
ξ_1	0.55	$\boldsymbol{\emptyset_1}$	1.178909	1.166086	1.141105	1.096358	1.035176	0.972684	0.923337
		$\boldsymbol{\emptyset_2}$	0.806776	0.928820	0.983615	0.992269	0.973485	0.942683	0.912332
		$\boldsymbol{\emptyset_3}$	1.579701	1.423784	1.284777	1.157121	1.051874	0.973800	0.921791
		$\boldsymbol{\emptyset_4}$	1.266293	1.212939	1.156041	1.093397	1.025697	0.963208	0.916300
	0.60	$\boldsymbol{\emptyset_1}$	1.298098	1.252514	1.217102	1.164507	1.091335	1.015385	0.954701
		$\boldsymbol{\emptyset_2}$	0.636098	0.820505	0.913988	0.951975	0.957113	0.942915	0.921622
		$\boldsymbol{\emptyset_3}$	1.759370	1.547124	1.371088	1.216605	1.094707	1.006330	0.947272
		$\boldsymbol{\emptyset_4}$	1.269036	1.253575	1.210566	1.150198	1.074854	1.000885	0.943779
	0.65	$\boldsymbol{\emptyset_1}$	1.565512	1.439769	1.390210	1.315331	1.203438	1.093518	1.010846
		$\boldsymbol{\emptyset_2}$	0.274212	0.590009	0.742198	0.836528	0.898489	0.924295	0.925848
		$\boldsymbol{\emptyset_3}$	2.053368	1.689889	1.437638	1.252451	1.129103	1.042558	0.982758
		$\boldsymbol{\emptyset_4}$	1.215803	1.319927	1.327781	1.273588	1.171610	1.068908	0.992511
	0.70	$\boldsymbol{\emptyset_1}$	2.595914	2.406019	4.857283	2.743700	1.595085	1.260275	1.105308
		$\boldsymbol{\emptyset_2}$	-1.018911	-0.529062	-2.748434	-0.479304	0.584734	0.825847	0.898734
		$\boldsymbol{\emptyset_3}$	2.672121	1.341264	-1.668272	0.116914	0.949759	1.031972	1.010909
		$\boldsymbol{\emptyset_4}$	0.944651	1.897197	4.671569	2.705275	1.546875	1.219270	0.073318
	0.75	$\boldsymbol{\emptyset_1}$	-2.172211	1.263302	1.212148	1.030684	0.685026	4.506745	1.308447
		$\boldsymbol{\emptyset_2}$	4.281781	0.532814	0.786906	1.056240	1.374291	-2.041302	0.727975
		$\boldsymbol{\emptyset_3}$	5.876722	3.674345	2.594997	2.166176	2.061940	-1.882180	0.932727
		$\boldsymbol{\emptyset_4}$	-1.376949	0.117870	0.706843	0.741690	0.481327	4.649108	1.250929
	0.80	$\boldsymbol{\emptyset_1}$	0.177611	8.676850	1.881503	1.411778	1.153122	0.918584	0.175611
		$\boldsymbol{\emptyset_2}$	1.199283	-9.023118	-0.196723	0.518456	0.841189	1.051814	1.797289
		$\boldsymbol{\emptyset_3}$	-1.117326	22.074381	3.747179	2.545986	2.017123	1.749396	2.156916
		$\boldsymbol{\emptyset_4}$	4.708916	-14.143094	0.068954	0.666071	0.749365	0.668866	0.065448
	0.85	$\boldsymbol{\emptyset_1}$	0.476098	0.149628	-1.255422	7.284137	3.049482	0.647380	0.857180
		$\boldsymbol{\emptyset_2}$	0.261280	1.315316	3.524698	-7.271644	-1.777032	1.171275	0.695899
		$\boldsymbol{\emptyset_3}$	0.915539	-0.056970	-3.426459	15.805416	6.395831	0.397315	1.036061
		$\boldsymbol{\emptyset_4}$	3.533182	3.756001	5.913190	-9.468059	-2.606944	1.731909	0.822704
	0.90	$\boldsymbol{\emptyset_1}$	0.486790	0.517018	0.530840	0.633354	0.810269	0.809997	0.341559
		$\boldsymbol{\emptyset_2}$	-0.615353	0.203700	0.720562	0.841907	0.606567	0.288174	-0.675944
		$\boldsymbol{\emptyset_3}$	1.645079	1.210150	0.858208	0.81259	1.029725	0.932240	-0.373641
		$\boldsymbol{\emptyset_4}$	3.574035	3.182342	2.855691	2.319239	1.556480	0.907141	0.123245
	0.95	$\boldsymbol{\emptyset_1}$	0.325283	0.501695	0.631951	0.733836	0.710214	0.145598	3.856357
		$\boldsymbol{\emptyset_2}$	-1.601405	-0.669598	-0.136393	0.025170	-0.151945	-1.096990	5.429493
		$\boldsymbol{\emptyset_3}$	2.263494	1.801691	1.495323	1.318795	1.020466	-0.347950	7.815737
		$\boldsymbol{\emptyset_4}$	3.927801	3.257590	2.672692	1.993284	1.210513	0.018121	4.652578
	1.00	$\boldsymbol{\emptyset_1}$	-	0.317635	0.509947	0.554246	0.090048	8.495648	2.063735
		$\boldsymbol{\emptyset_2}$	-	-1.589691	-0.937625	-0.767461	-1.449441	11.469416	1.913580
		$\boldsymbol{\emptyset_3}$	-	2.316843	1.851332	1.375339	0.064056	17.038901	3.755075
		$\boldsymbol{\emptyset_4}$	-	3.570686	2.766031	1.866645	0.460768	11.280457	2.435907

			ξ_2						
			0.35	0.40	0.45	0.50	0.55	0.60	0.65
ξ_1	0.55	$\emptyset_1$	0.891391	0.871307	0.854261	0.835749	-	0.831416	0.952436
		$\emptyset_2$	0.889069	0.871772	0.855011	0.836042	-	0.831600	0.952437
		$\emptyset_3$	0.890512	0.871061	0.854221	0.835746	-	0.831409	0.952195
		$\emptyset_4$	0.886897	0.868731	0.853036	0.835415	-	0.830963	0.952790
	0.60	$\emptyset_1$	0.914612	0.888934	0.867582	0.845400	0.831415	-	0.401331
		$\emptyset_2$	0.902055	0.885372	0.867338	0.845908	0.831600	-	0.402892
		$\emptyset_3$	0.910924	0.887574	0.867185	0.845315	0.831409	-	0.401341
		$\emptyset_4$	0.907022	0.883858	0.864429	0.843832	0.830963	-	0.399771
	0.65	$\emptyset_1$	0.958926	0.928794	0.909029	0.901008	0.952436	0.401331	-
		$\emptyset_2$	0.918867	0.910438	0.900319	0.889927	0.952437	0.402892	-
		$\emptyset_3$	0.945471	0.922994	0.906732	0.900184	0.952195	0.401341	-
		$\emptyset_4$	0.945438	0.918611	0.900973	0.888793	0.952790	0.399771	-
	0.70	$\emptyset_1$	1.027706	0.994863	1.042454	1.061705	0.327387	0.658202	0.674225
		$\emptyset_2$	0.923017	0.933722	0.895998	1.054167	0.343349	0.661321	0.675028
		$\emptyset_3$	0.985988	0.975604	1.031469	1.056671	0.328722	0.658168	0.674222
		$\emptyset_4$	1.001972	0.970121	0.921715	1.069211	0.311683	0.654682	0.673387
	0.75	$\emptyset_1$	1.120185	0.881270	1.074390	0.383753	0.746378	0.744364	0.725163
		$\emptyset_2$	0.858695	1.116344	1.031793	0.422509	0.750816	0.748097	0.727601
		$\emptyset_3$	0.999926	0.860653	1.047945	0.396075	0.745662	0.744097	0.725150
		$\emptyset_4$	1.057561	1.145231	1.087383	0.343848	0.734681	0.737770	0.722110
	0.80	$\emptyset_1$	1.027196	0.900268	1.719053	0.970752	0.896588	0.841969	0.800736
		$\emptyset_2$	0.789897	0.799188	1.445889	0.930595	0.883857	0.840931	0.803332
		$\emptyset_3$	0.822493	0.863077	1.552103	0.955443	0.892216	0.841203	0.800925
		$\emptyset_4$	1.086947	0.873407	1.788588	0.959581	0.884763	0.832848	0.795058
	0.85	$\emptyset_1$	0.171517	-0.208520	1.757123	1.191346	1.041276	0.953363	0.893274
		$\emptyset_2$	3.485079	0.763577	1.224938	1.052685	0.989120	0.936444	0.891075
		$\emptyset_3$	4.372770	0.930544	1.378081	1.129075	1.026627	0.951611	0.894685
		$\emptyset_4$	-0.057100	-0.445045	1.834015	1.195016	1.033992	0.944842	0.886552
	0.90	$\emptyset_1$	1.568103	0.804642	-7.271986	1.493014	1.196939	1.068525	0.988618
		$\emptyset_2$	2.670149	1.502086	2.420337	1.086303	1.060702	1.016933	0.972830
		$\emptyset_3$	3.590924	1.952179	2.866212	1.235296	1.147750	1.062662	0.993295
		$\emptyset_4$	1.640720	0.726622	-8.584750	1.536377	1.204493	1.067203	0.985307
	0.95	$\emptyset_1$	1.584875	1.210537	0.903924	-6.403278	1.435470	1.186332	1.074954
		$\emptyset_2$	1.806860	1.441033	1.357213	3.187114	1.063246	1.063473	1.031756
		$\emptyset_3$	2.815269	2.087233	1.787913	3.756568	1.218355	1.157721	1.083468
		$\emptyset_4$	1.686265	1.222386	0.865025	-7.546021	1.482626	1.203040	1.082527
	1.00	$\emptyset_1$	1.569110	1.331122	1.156544	0.934482	-2.701334	1.375316	1.155199
		$\emptyset_2$	1.420396	1.296707	1.264856	1.281661	2.348990	1.04484	1.055903
		$\emptyset_3$	2.673676	2.166603	1.866474	1.697629	2.792276	1.201529	1.153701
		$\emptyset_4$	1.721280	1.405144	1.188271	0.925040	-3.257790	1.432444	1.183802

			ξ_2						
			0.70	0.75	0.80	0.85	0.90	0.95	1.00
ξ_1	0.55	$\boldsymbol{\emptyset_1}$	0.327387	0.746378	0.896588	1.041276	1.196939	1.435470	1.109554
		$\boldsymbol{\emptyset_2}$	0.343349	0.750816	0.883857	0.989120	1.060702	1.063246	0.637042
		$\boldsymbol{\emptyset_3}$	0.328722	0.745662	0.892216	1.026627	1.147750	1.218355	1.729502
		$\boldsymbol{\emptyset_4}$	0.311683	0.734681	0.884763	1.033992	1.204493	1.482626	1
	0.60	$\boldsymbol{\emptyset_1}$	0.658202	0.744364	0.841969	0.953363	1.068525	1.186332	1.020474
		$\boldsymbol{\emptyset_2}$	0.661321	0.748097	0.840931	0.936444	1.016933	1.063473	0.790354
		$\boldsymbol{\emptyset_3}$	0.658168	0.744097	0.841203	0.951611	1.062662	1.157721	1.442831
		$\boldsymbol{\emptyset_4}$	0.654682	0.737770	0.832848	0.944842	1.067203	1.203040	1
	0.65	$\boldsymbol{\emptyset_1}$	0.674225	0.725163	0.800736	0.893274	0.988618	1.074954	0.958928
		$\boldsymbol{\emptyset_2}$	0.675028	0.727601	0.803332	0.891075	0.972830	1.031756	0.862876
		$\boldsymbol{\emptyset_3}$	0.674222	0.725150	0.800925	0.894685	0.993295	1.083468	1.262866
		$\boldsymbol{\emptyset_4}$	0.673388	0.722110	0.795058	0.886552	0.985307	1.082527	1
	0.70	$\boldsymbol{\emptyset_1}$	-	0.720292	0.781514	0.860637	0.942931	1.014072	0.922880
		$\boldsymbol{\emptyset_2}$	-	0.721028	0.783624	0.862688	0.941137	1.002651	0.895909
		$\boldsymbol{\emptyset_3}$	-	0.720296	0.781669	0.861873	0.947981	1.027675	1.151099
		$\boldsymbol{\emptyset_4}$	-	0.719518	0.778875	0.856457	0.939934	1.017773	1
	0.75	$\boldsymbol{\emptyset_1}$	0.720292	-	0.791130	0.858931	0.930799	0.991731	0.919768
		$\boldsymbol{\emptyset_2}$	0.721028	-	0.791758	0.860626	0.932474	0.991179	0.920490
		$\boldsymbol{\emptyset_3}$	0.720296	-	0.791149	0.859356	0.933475	1.001170	1.083017
		$\boldsymbol{\emptyset_4}$	0.719518	-	0.790477	0.857003	0.928763	0.993295	1
	0.80	$\boldsymbol{\emptyset_1}$	0.781514	0.791130	-	0.886534	0.949192	1.001907	0.950774
		$\boldsymbol{\emptyset_2}$	0.783624	0.791758	-	0.887000	0.950391	1.003291	0.956972
		$\boldsymbol{\emptyset_3}$	0.781669	0.791149	-	0.886573	0.949942	1.006121	1.042723
		$\boldsymbol{\emptyset_4}$	0.778875	0.790477	-	0.886068	0.948178	1.002244	1
	0.85	$\boldsymbol{\emptyset_1}$	0.860637	0.858931	0.886534	-	0.988889	1.034450	1.006453
		$\boldsymbol{\emptyset_2}$	0.862688	0.860626	0.887000	-	0.989170	1.035168	1.010174
		$\boldsymbol{\emptyset_3}$	0.861873	0.859356	0.886573	-	0.988949	1.035509	1.019814
		$\boldsymbol{\emptyset_4}$	0.856457	0.857003	0.886068	-	0.988635	1.034341	1
	0.90	$\boldsymbol{\emptyset_1}$	0.942931	0.930799	0.949192	0.988889	-	1.075036	1.068769
		$\boldsymbol{\emptyset_2}$	0.941137	0.932474	0.950391	0.989170	-	1.075157	1.069790
		$\boldsymbol{\emptyset_3}$	0.947981	0.933475	0.949942	0.988949	-	1.075114	1.007541
		$\boldsymbol{\emptyset_4}$	0.939934	0.928763	0.948178	0.988635	-	1.074967	1
	0.95	$\boldsymbol{\emptyset_1}$	1.014072	0.991731	1.001907	1.034450	1.075036	-	1.117068
		$\boldsymbol{\emptyset_2}$	1.002651	0.991179	1.003291	1.035168	1.075157	-	1.117144
		$\boldsymbol{\emptyset_3}$	1.027675	1.001170	1.006121	1.035509	1.075114	-	1.001703
		$\boldsymbol{\emptyset_4}$	1.017773	0.993295	1.002244	1.034341	1.074967		1
	1.00	$\boldsymbol{\emptyset_1}$	1.067201	1.032153	1.034078	1.060232	1.095390	1.124189	-
		$\boldsymbol{\emptyset_2}$	1.037194	1.026543	1.034993	1.061404	1.095758	1.124217	-
		$\boldsymbol{\emptyset_3}$	1.093062	1.055338	1.047698	1.065733	1.096669	1.124276	-
		$\boldsymbol{\emptyset_4}$	1.085210	1.043367	1.040051	1.062599	1.095942	1.124227	-

4.2.3 **Pinned-pinned beams**

These are beams hinged or pinned at both ends. From section 3.2.3 the first natural frequency w for such a beam is given as

$$\omega_1 = 9.869604693\sqrt{\frac{EI}{\mu L^4}} \tag{4.68}$$

In normalized coordinates, the length of the bar L = 1 hence

$$\omega_1 = 9.869604693\sqrt{\frac{EI}{\mu}} \tag{4.69}$$

Likewise

$$u_1 = u(x_1, 0) = \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j x_1 + c_{2j} \sinh \beta_j x_1 - \cos \beta_j x_1 + c_{4j} \sin \beta_j x_1\right)$$

$$= \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j L\xi_1 + c_{2j} \sinh \beta_j L\xi_1 - \cos \beta_j L\xi_1 + c_{4j} \sin \beta_j L\xi_1\right)$$

(4.70a)

$$u_2 = u'(x_1, 0) = \sum_{j=1}^{\infty} A_j \beta_j L\left(\sinh \beta_j x_1 + c_{2j} \cosh \beta_j x_1 + \sin \beta_j x_1 + c_{4j} \cos \beta_j x_1\right)$$

$$= \sum_{j=1}^{\infty} A_j \beta L\left(\sinh \beta_j L\xi_1 + c_{2j} \cosh \beta_j L\xi_1 + \sin \beta_j L\xi_1 + c_{4j} \cos \beta_j L\xi_1\right)$$

(4.70b)

$$u_3 = u(x_2, 0) = \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j x_2 + c_{2j} \sinh \beta_j x_2 - \cos \beta_j x_2 + c_{4j} \sin \beta_j x_2\right)$$

$$= \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j L\xi_2 + c_{2j} \sinh \beta_j L\xi_2 - \cos \beta_j L\xi_2 + c_{4j} \sin \beta_j L\xi_2\right)$$

(4.70c)

$$u_4 = u'(x_2, 0) = \sum_{j=1}^{\infty} A_j \beta L\left(\sinh \beta_j x_2 + c_{2j} \cosh \beta_j x_2 + \sin \beta_j x_2 + c_{4j} \cos \beta_j x_2\right)$$

$$= \sum_{j=1}^{\infty} A_j \beta_j L\left(\sinh \beta_j L\xi_2 + c_{2j} \cosh \beta_j L\xi_2 + \sin \beta_j L\xi_2 + c_{4j} \cos \beta_j L\xi_2\right)$$

(4.70d)

Equations (4.70a – 4.70d) are used to evaluate the total displacements $u_1 - u_4$ at the nodal points of a segment of the vibrating pinned-pinned beam.

From Table 3.4

$$A_j = \frac{c\mu L^2}{M_j}\left[\frac{-\sinh \beta_j L}{\beta_j^2 L^2} + \frac{12 \sinh \beta_j L}{\beta_j^4 L^4} - \frac{24(\cosh \beta_j L - 1)}{\beta_j^5 L^5} + c_{4j}\left(\frac{-\sin \beta_j L}{\beta_j^2 L^2} - \frac{12 \sin \beta_j L}{\beta_j^4 L^4} - \frac{24(\cos \beta_j L - 1)}{\beta_j^5 L^5}\right)\right] \tag{4.71}$$

The values of $\beta_j L$, c_{2j}, and c_{4j} for j = 1, 2, 3, 4, 5, 6, 7 can be obtained from Table 3.4. The formula for calculating the generalized mass M_j can also be picked from Table 3.4.

For this beam there are three possible cases, and the method of obtaining the stiffness modification factors depends on the case being considered.

a) **When ξ_1 is greater than zero and ξ_2 is less than 1**

In this case the segment of the pinned-pinned beam under consideration is not positioned to the far left or far right of the beam (the ends that are pinned). Hence the process of calculating the stiffness modification factors is similar to the one for fixed-fixed beam.

The values of the fixed end forces F_1, F_2, F_3 and F_4 are evaluated using the equations provided in Table 3.4 while the values of the nodal displacements u_1, u_2, u_3 and u_4 are calculated from equations (4.70a – 4.70b). These are substituted into the equations for nodal forces (equations 4.35 – 4.38) from which the nodal forces P_1, P_2, P_3 and P_4 are obtained.

Equations (4.70a) – (4.70d) are evaluated for the first mode, j = 1 to obtain the nodal displacements u_{11}, u_{21}, u_{31} and u_{41} due to the first mode. These together with the calculated nodal forces P_1 to P_4 are substituted into equation (4.49) in order to obtain the stiffness modification factors ϕ_1, ϕ_2, ϕ_3 and ϕ_4.

b) **When ξ_2 is equal to 1**

In this case the segment under consideration is located at the far right of the pinned-pinned beam. This implies that the segment is fixed at the left end and pinned at the right. Hence the process of calculating the stiffness modification factors is similar to the one for a segment at the far right of a fixed-pinned beam.

The values of the fixed end forces F_1, F_2, F_3 and F_4 are evaluated using the equations provided in Table 3.4 while the values of the nodal displacements u_1, u_2, u_3 and u_4 are calculated from equations (4.70a) – (4.70b). These are substituted into the equations for nodal forces (equations (4.35) – (4.38)) from which the nodal forces P_1, P_2, P_3 and P_4 are obtained.

Equations (4.70a) – (4.70d) are evaluated for the first mode, j = 1 to obtain the nodal displacements u_{11}, u_{21}, u_{31} and u_{41} due to the first mode. These together with the calculated nodal forces P_1 to P_4 are substituted into equation (4.67) in order to obtain the stiffness modification factors ϕ_1, ϕ_2 and ϕ_3. The value of ϕ_4 is taken to be equal to unity.

c) **When ξ_1 is equal to 0**

In this case the segment under consideration is located at the far left of the pinned-pinned beam. This implies that the segment is pinned at the left end and fixed at the right end hence its stiffness matrix is different from that of a fixed-

fixed or fixed-pinned beam. The stiffness matrix for a pinned-fixed beam is given as (Okonkwo , 2012)

$$[k] = \begin{bmatrix} \frac{3EI}{l^3} & 0 & -\frac{3EI}{l^3} & \frac{3EI}{l^2} \\ 0 & 0 & 0 & 0 \\ -\frac{3EI}{l^3} & 0 & \frac{3EI}{l^3} & -\frac{3EI}{l^2} \\ \frac{3EI}{l^2} & 0 & -\frac{3EI}{l^2} & \frac{3EI}{l} \end{bmatrix} \tag{4.72}$$

The proposed stiffness matrix for this beam segment is therefore

$$[k_d] = \begin{bmatrix} \frac{3EI}{l^3}\phi_1 & 0 & -\frac{3EI}{l^3}\phi_3 & \frac{3EI}{l^2}\phi_4 \\ 0 & 0 & 0 & 0 \\ -\frac{3EI}{l^3}\phi_3 & 0 & \frac{3EI}{l^3}\phi_1 & -\frac{3EI}{l^2}\phi_2 \\ \frac{3EI}{l^2}\phi_4 & 0 & -\frac{3EI}{l^2}\phi_2 & \frac{3EI}{l}\phi_1 \end{bmatrix} \tag{4.73}$$

By substituting equations (4.40), (4.42) and (4.73) into equation (4.39) we obtain

$$\begin{bmatrix} \frac{\mu(\xi_2-\xi_1)}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu(\xi_2-\xi_1)}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} -\omega^2 u_{11} \\ -\omega^2 u_{21} \\ -\omega^2 u_{31} \\ -\omega^2 u_{41} \end{Bmatrix} +$$

$$\begin{bmatrix} \frac{3EI}{l^3}\phi_1 & 0 & -\frac{3EI}{l^3}\phi_3 & \frac{3EI}{l^2}\phi_4 \\ 0 & 0 & 0 & 0 \\ -\frac{3EI}{l^3}\phi_3 & 0 & \frac{3EI}{l^3}\phi_1 & -\frac{3EI}{l^2}\phi_2 \\ \frac{3EI}{l^2}\phi_4 & 0 & -\frac{3EI}{l^2}\phi_2 & \frac{3EI}{l}\phi_1 \end{bmatrix} \begin{Bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} \tag{4.74}$$

By multiplying out the first row of equation (4.74) we obtain

$$-\frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{11} + \frac{3EI}{l^3}\phi_1 u_{11} + 0 - \frac{3EI}{l^3}\phi_3 u_{31} + \frac{3EI}{l^2}\phi_4 u_{41} = P_1$$

$$\frac{3EI}{l^3}\phi_1 u_{11} - \frac{3EI}{l^3}\phi_3 u_{31} + \frac{3EI}{l^2}\phi_4 u_{41} = P_1 + \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{11} \tag{4.75}$$

By multiplying out the second row of equation (4.60) we obtain

$$P_2 = 0 \quad (4.76)$$

Hence P_2 for an element with the near (left) end pinned must be zero. This will help in ascertaining the correctness of our equations.

By multiplying out the third row of equation (4.74) we obtain

$$-\frac{3EI}{l^3}\phi_3 u_{11} + 0 + \frac{3EI}{l^3}\phi_1 u_{31} - \frac{3EI}{l^2}\phi_2 u_{41} = P_2$$

$$\frac{3EI}{l^3}\phi_1 u_{31} - \frac{3EI}{l^2}\phi_2 u_{41} - \frac{3EI}{l^3}\phi_3 u_{11} = P_2 + \frac{3EI}{l^3}\phi_3 u_{11} \quad (4.77)$$

By multiplying out the fourth row of equation (4.74) we obtain

$$\frac{3EI}{l^2}\phi_4 u_{11} + 0 - \frac{3EI}{l^2}\phi_2 u_{31} + \frac{3EI}{l}\phi_1 u_{41} + 0 = P_4$$

$$\frac{3EI}{l}\phi_1 u_{41} - \frac{3EI}{l^2}\phi_2 u_{31} + \frac{3EI}{l^2}\phi_4 u_{11} = P_4 \quad (4.78)$$

By putting equations (4.75),(4.77) and (4.78) in matrix form we obtain

$$\begin{bmatrix} \frac{3EI}{l^3}u_{11} & 0 & -\frac{3EI}{l^3}u_{31} & \frac{3EI}{l^2}u_{41} \\ \frac{3EI}{l^3}u_{31} & -\frac{3EI}{l^2}u_{41} & -\frac{3EI}{l^3}u_{11} & 0 \\ \frac{3EI}{l}u_{41} & -\frac{3EI}{l^2}u_{31} & 0 & \frac{3EI}{l^2}u_{11} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} P_1 + \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{11} \\ P_3 + \frac{\mu(\xi_2-\xi_1)}{2}\omega^2 u_{31} \\ P_4 \end{Bmatrix} \quad (4.79)$$

Equation (4.79) is a set of three equations (equations 4.75, 4.77 and 4.78) with four unknowns $(\phi_1 - \phi_4)$. To solve it there is need to know the value of one of the unknowns.

Let $\phi_4 = 1$. (4.80)

By substituting equation (4.80) into (4.79) and rearranging the equation we obtain

$$\begin{bmatrix} \frac{3EI}{l^3}u_{11} & 0 & -\frac{3EI}{l^3}u_{31} \\ \frac{3EI}{l^3}u_{31} & -\frac{3EI}{l^2}u_{41} & -\frac{3EI}{l^3}u_{11} \\ \frac{3EI}{l}u_{41} & -\frac{3EI}{l^2}u_{31} & 0 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} P_1 + \frac{\mu(\xi_2 - \xi_1)}{2}\omega^2 u_{11} - \frac{3EI}{l^2}u_{41} \\ P_3 + \frac{\mu(\xi_2 - \xi_1)}{2}\omega^2 u_{31} \\ P_4 - \frac{3EI}{l^2}u_{11} \end{Bmatrix}$$

Therefore

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{bmatrix} \frac{3EI}{l^3}u_{11} & 0 & -\frac{3EI}{l^3}u_{31} \\ \frac{3EI}{l^3}u_{31} & -\frac{3EI}{l^2}u_{41} & -\frac{3EI}{l^3}u_{11} \\ \frac{3EI}{l}u_{41} & -\frac{3EI}{l^2}u_{31} & 0 \end{bmatrix}^{-1} \begin{Bmatrix} P_1 + \frac{\mu(\xi_2 - \xi_1)}{2}\omega^2 u_{11} - \frac{3EI}{l^2}u_{41} \\ P_3 + \frac{\mu(\xi_2 - \xi_1)}{2}\omega^2 u_{31} \\ P_4 - \frac{3EI}{l^2}u_{11} \end{Bmatrix}$$

. (4.81)

Equation (4.81) is a mathematical expression for calculating the four stiffness modification factors for an element of a beam (having the near end pinned i.e. pinned-fixed) under lateral vibration. Note that $\phi_4 = 1$. Below are some examples on how to calculate the stiffness modification factors for a pinned-pinned beam.

Table 4.25: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0, \xi_2 = 0.5$ on a pinned-pinned beam under lateral vibration

$\xi_1 = 0,\ \xi_2 = 0.5$					
j	**Aj**	$\mathbf{F_{1j}}$	$\mathbf{F_{2j}}$	$\mathbf{F_{3j}}$	$\mathbf{F_{4j}}$
1	0.00000000001114	3.26394701128700	0.35534693813438	6.46288662152144	-0.4906429423346
2	0.00000000000000	-0.00000059666034	-0.0000000604543	-0.0000005966603	0.00000006045433
3	0.00000000000000	1.49670241732092	0.12831996121837	-0.4159431242878	-0.0350205216652
4	0.00000000000000	-0.00000000018717	-0.0000000000109	0.00000000018717	-0.0000000000109
5	0.00000000000000	0.41728083505831	0.01020558221588	-0.0282074898687	0.02866734116343
6	0.00000000000000	-0.00000000000010	0	-0.0000000000001	0
7	0.00000000000000	0.21014990221729	0.00373170988491	-0.0116430934458	-0.0069368316916

	Total	5.38807956903590	0.49760413098831	6.00709231744594	-0.5039328940846
j	Aj	u_{1j}	u_{2j}	u_{3j}	u_{4j}
1	0.00000000001114	0	0.98553429629092	0.31370530965612	-0.0000000001143
2	0.00000000000000	0	-0.0000000151136	0	0.00000001511358
3	0.00000000000000	0	0.01216709008996	-0.0012909683539	0.00000000000143
4	0.00000000000000	0	-0.0000000000009	0	-0.0000000000009
5	0.00000000000000	0	0.00157685487433	0.00010038569912	0
6	0.00000000000000	0	0	0	0
7	0.00000000000000	0	0.00041046826175	-0.0000186651580	0
	Total	**0**	**0.99968869440247**	**0.3124906184341**	**0.00000001499980**

$u_{11} = 0$ $u_{31} = 0.31370530965612$

$u_{21} = 0.98553429629092$ $u_{41} = -0.0000000001143$

From Table 3. 4

$F_1 = -5.38807956903590$ $F_3 = -6.00709231744594$

$F_2 = -0.49760413098831$ $F_4 = 0.50393289408460$

From equations (4.35) – (4.38)

$P_1 = -11.39517248034896$ $P_3 = 5.938671222500384 \times 10^{-7}$

$P_2 = -1.123273595847988 \times 10^{-11}$ $P_4 = -2.99721769254901$

From equation (4.81) taking EI = 1

$\phi_1 = 1.01467811086715$ $\phi_3 = 1.51351872675740$

$\phi_2 = 0.79618716451293$ $\phi_4 = 1$

Table 4.26: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.5, \xi_2 = 1.0$ on a pinned-pinned beam under lateral vibration

$\xi_1 = 0.5,\ \ \xi_2 = 1.0$					
j	**Aj**	**F_{1j}**	**F_{2j}**	**F_{3j}**	**F_{4j}**
1	0.00000000001114	6.46288662222064	0.49064294242366	3.26394700921441	-0.3553469375521
2	0.00000000000000	0.00000059666039	0.00000006045434	0.00000059666033	-0.0000000604543
3	0.00000000000000	-0.41594312386947	0.03502052170401	1.49670241739663	-0.1283199612233
4	0.00000000000000	-0.00000000018717	-0.0000000000109	0.00000000018717	-0.0000000000109
5	0.00000000000000	-0.02820748986902	-0.0286673411634	0.41728083505868	-0.0102055822159
6	0.00000000000000	0.00000000000010	0	0.00000000000010	0
7	0.00000000000000	-0.01164309344546	0.00693683169164	0.21014990221753	-0.0037317098849
	Total	**6.00709351151001**	**0.50393301509931**	**5.38808076073485**	**-0.4976042513414**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	0.00000000001114	0.31370530965612	-0.0000000001143	0	-0.9855342958502
2	0.00000000000000	0	0.00000001511358	0	-0.0000000151136
3	0.00000000000000	-0.00129096835387	0.00000000000143	0	-0.0121670900872
4	0.00000000000000	0	-0.0000000000009	0	-0.0000000000009
5	0.00000000000000	0.00010038569912	0	0	-0.0015768548743
6	0.00000000000000	0	0	0	0
7	0.00000000000000	-0.00001866515795	0.00000001499980	0	-0.0004104682618
	Total	**0.31249606184341**	**0.00000001499980**	**0**	**-0.9996887241879**

u_{11} = 0.31370530965612 u_{31} = -0.0000000001143

u_{21} = 0 u_{41} = -0.9855342958502

From Table 3. 4

F_1 = -6.00709351151001 F_3 = -5.38808076073485

F_2 = -0.50393301509931 F_4 = 0.49760425134140

From equations (4.35) – (4.38)

$P_1 = -5.950556705158761 \times 10{-7}$	$P_3 = -11.39517367718919$
$P_2 = 2.99721769238958$	$P_4 = 2.079678651512040 \times 10^{-9}$
From equation (4.67) taking EI = 1	
$\phi_1 = 1.01467795324328$	$\phi_3 = 1.51351886199768$
$\phi_2 = 0.79618716484029$	$\phi_4 = 1$

Table 4.27: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0, \xi_2 = 0.3$ on a pinned-pinned beam under lateral vibration

$\xi_1 = 0, \quad \xi_2 = 0.3$					
j	**Aj**	**F_{1j}**	**F_{2j}**	**F_{3j}**	**F_{4j}**
1	0.00000000001114	1.25109871009849	0.08281233325346	2.75844556193896	-0.1206970219608
2	0.00000000000000	-0.00000027613218	-0.0000000178672	-0.0000005049063	0.00000002371198
3	0.00000000000000	0.94315654019168	0.05848852173748	1.16546592128938	-0.0643337129916
4	0.00000000000000	-0.00000000017401	-0.0000000000101	-0.0000000000857	0.00000000000762
5	0.00000000000000	0.53881286975170	0.02771711159877	-0.1497395245624	-0.0075644326750
6	0.00000000000000	-0.00000000000014	-0.0000000000000	0.00000000000012	-0.0000000000000
7	0.00000000000000	0.24933310455506	0.00769091018978	-0.2396174897946	0.01034637252347
	Total	**-6.60409544409049**	**0.17670885890221**	**3.53455396387949**	**-0.1822487713844**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	0.00000000001114	0	0.98553429629092	0.25379292675153	0.57928252492129
2	0.00000000000000	0	-0.0000000151136	-0.0000000022877	0.00000000467035
3	0.00000000000000	0	0.01216709008996	0.00039893116046	-0.0115715903147
4	0.00000000000000	0	-0.0000000000009	0.00000000000004	0.00000000000074
5	0.00000000000000	0	0.00157685487433	-0.0001003856991	0.00000000000000
6	0.00000000000000	0	-0.0000000000000	0.00000000000000	-0.0000000000000

7	0.00000000000000	0	0.00041046826175	0.00000576785101	0.00039037851507
	Total	**0**	**0.99968869440247**	**0.25409723777626**	**0.56810131779279**

u_{11} = 0 u_{31} = 0.25379292675153

u_{21} = 0.98553429629092 u_{41} = 0.57928252492129

From Table 3. 4

F_1 = -2.98240094829059 F_3 = -3.53455396387949

F_2 = -0.17670885890221 F_4 = 0.18224877138435

From equations (4.35) – (4.38)

P_1 = -11.39517248027852 P_3 = 4.87821756810844

P_2 = -1.454836251468805 x 10-12 P_4 = -2.51829154711278

From equation (4.81) taking EI = 1

ϕ_1 = 0.95709405015697 ϕ_3 = 1.07562866764841

ϕ_2 = 0.95304903449419 ϕ_4 = 1

Table 4.28: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.3, \xi_2 = 0.8$ on a fixed-pinned beam under lateral vibration

$\xi_1 = 0.3,\ \ \xi_2 = 0.8$					
j	**Aj**	$\mathbf{F_{1j}}$	$\mathbf{F_{2j}}$	$\mathbf{F_{3j}}$	$\mathbf{F_{4j}}$
1	0.00000000001114	7.14708502739753	0.60580616824732	6.43937804289837	-0.5758743972175
2	0.00000000000000	-0.00000025485494	-0.0000000008132	0.00000062361138	-0.0000000381760
3	0.00000000000000	-1.29491509294964	-0.1328614716375	-0.0669210635622	0.07295954406183
4	0.00000000000000	0.00000000015142	0.00000000000211	-0.0000000001514	0.00000000001554
5	0.00000000000000	0.02820748986900	0.02866734116344	-0.4172808350583	0.01020558221590
6	0.00000000000000	-0.00000000000008	-0.0000000000000	-0.0000000000001	-0.0000000000000

7	0.00000000000000	0.20346234764431	0.00140546812313	-0.0760131310429	-0.0054441572102
	Total	**6.08383951725761**	**0.50301750508537**	**5.87916363669486**	**-0.4981534663105**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	0.00000000001114	0.25379292675153	0.57928252492129	0.18439135455168	-0.7973139941830
2	0.00000000000000	-0.00000000228767	0.00000000467035	0.00000000228767	-0.0000000046703
3	0.00000000000000	0.00039893116046	-0.0115715903147	0.00122778386540	0.00375983760818
4	0.00000000000000	0.00000000000004	0.00000000000074	0.00000000000004	0.00000000000074
5	0.00000000000000	-0.00010038569912	0.00000000000000	0.00000000000000	0.00157685487433
6	0.00000000000000	0.00000000000000	-0.0000000000000	-0.0000000000000	0.00000000000000
7	0.00000000000000	0.00000576785101	0.00039037851507	-0.0000177516201	0.00012684166853
	Total	**0.25409723777626**	**0.56810131779279**	**0.18560138908470**	**-0.7918504647016**

u_{11} = 0.25379292675153 u_{31} = 0.18439135455168

u_{21} = -0.57928252492129 u_{41} = -0.7973139941830

From Table 3. 4

F_1 = -6.08383951725761 F_3 = -5.87916363669486

F_2 = -0.50301750508537 F_4 = 0.49815346631049

From equations (4.35) – (4.38)

P_1 = -4.87821756867864 P_3 = -7.08478558527382

P_2 = 2.51829154704809 P_4 = -1.92034461153348

From equation (4.49) taking EI = 1

ϕ_1 = 1.42114242852867 ϕ_3 = 1.04372357801254

ϕ_2 = 0.93702083568665 ϕ_4 = 1.45667820922367

Table 4.29: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.8, \xi_2 = 1.0$ on a pinned-pinned beam under lateral vibration

$\xi_1 = 0.8, \quad \xi_2 = 1.0$					
j	**Aj**	**F_{1j}**	**F_{2j}**	**F_{3j}**	**F_{4j}**
1	0.00000000001114	1.29061461108650	0.03721061709105	0.56704531082364	-0.0251226784691
2	0.00000000000000	0.00000027954035	0.00000000829430	0.00000013274180	-0.0000000058243
3	0.00000000000000	0.91456091451343	0.02863376173383	0.50017136707759	-0.0215758458927
4	0.00000000000000	0.00000000015386	0.00000000000528	0.00000000010580	-0.0000000000045
5	0.00000000000000	0.38907334518992	0.01576854874333	0.38907334518938	-0.0157685487433
6	0.00000000000000	0.00000000000006	0.00000000000000	0.00000000000012	-0.0000000000000
7	0.00000000000000	-0.00782944172292	0.00435427966360	0.26767822790464	-0.0096036915647
	Total	**2.58641970876120**	**0.08596721553139**	**1.72396838384296**	**-0.0720707704986**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	0.00000000001114	0.18439135455168	-0.7973139941830	0	-0.9855342958501
2	0.00000000000000	0.00000000228767	-0.0000000046704	0	-0.0000000151136
3	0.00000000000000	0.00122778386540	0.00375983760818	0	-0.0121670900872
4	0.00000000000000	0.00000000000004	0.00000000000074	0	-0.0000000000009
5	0.00000000000000	0.00000000000000	0.00157685487433	0	-0.0015768548743
6	0.00000000000000	-0.0000000000000	0.00000000000000	0.0000000000000	-0.0000000000000
7	0.00000000000000	-0.00001775162010	0.00012684166853	-0.0000000000000	-0.0004104682618
	Total	**0.18560138908470**	**-0.7918504647016**	**0.0000000000000**	**-0.9996887241879**

u_{11} = 0.18439135455168 u_{31} = -0.7973139941830

u_{21} = 0 u_{41} = -0.9855342958501

From Table 3. 4

F_1 = -2.58641970876120 F_3 = -1.72396838384296

F_2 = -0.08596721553139 F_4 = 0.07207077049861

From equations (4.35) – (4.38)

$P_1 = 7.08478558487801$	$P_3 = -11.39517367748218$
$P_2 = 1.92034461126394$	$P_4 = 2.43112552311686 \times 10^{-9}$
From equation (4.67) taking EI = 1	
$\phi_1 = 0.98577135090287$	$\phi_3 = 1.02960319969885$
$\phi_2 = 0.99136130264460$	$\phi_4 = 1$

Tables 4.25 to 4.29 provide illustrations on how the inherent nodal forces P_1 to P_4 and the stiffness modification factors ϕ_1 to ϕ_4 for a element of a pinned-pinned beam under lateral vibration are calculated. It would be observed that the values of the of fixed end forces calculated where negated, this is to take care of the sign convention used in the development of the equations of Table 3.4. Using the methods presented in Table 4.25 to 4.29 the values of stiffness modification factors at different values of ξ_1 and ξ_2 for the lateral vibration of a pinned-pinned beam are presented in Table 4.30. A sample Matlab program for the calculation of the stiffness modification factors for a segment of a pinned-pinned beam can be found in Appendix F.

Table 4.30: Stiffness modification factors for the lateral vibration of a pinned-pinned beam

			ξ_2						
			0	0.05	0.10	0.15	0.20	0.25	0.30
ξ_1	0	$\varnothing_1$	-	1.148799	1.092695	1.030531	0.985771	0.964402	0.957094
		$\varnothing_2$	-	1.148877	1.093700	1.033929	0.991361	0.968428	0.953049
		$\varnothing_3$	-	1.001437	1.006066	1.014871	1.029603	1.052953	1.088846
		$\varnothing_4$	-	1	1	1	1	1	1
	0.05	$\varnothing_1$	-	-	1.098047	1.054748	1.025753	1.016801	1.022804
		$\varnothing_2$	-	-	1.098152	1.055372	1.027036	1.017636	1.020499
		$\varnothing_3$	-	-	1.098106	1.055480	1.028409	1.022536	1.032350
		$\varnothing_4$	-	-	1.097987	1.054603	1.025747	1.017325	1.024437
	0.10	$\varnothing_1$	-	-	-	1.011144	0.980355	0.969141	0.972091
		$\varnothing_2$	-	-	-	1.011347	0.981177	0.970430	0.972655
		$\varnothing_3$	-	-	-	1.011182	0.980782	0.970581	0.975147
		$\varnothing_4$	-	-	-	1.010959	0.979646	0.967771	0.970173
	0.15	$\varnothing_1$	-	-	-	-	0.933250	0.920399	0.920631
		$\varnothing_2$	-	-	-	-	0.933530	0.921367	0.922125
		$\varnothing_3$	-	-	-	-	0.933269	0.920590	0.921259
		$\varnothing_4$	-	-	-	-	0.932970	0.919327	0.918457
	0.20	$\varnothing_1$	-	-	-	-	-	0.885874	0.883311
		$\varnothing_2$	-	-	-	-	-	0.886181	0.884353
		$\varnothing_3$	-	-	-	-	-	0.885881	0.883380
		$\varnothing_4$	-	-	-	-	-	0.885557	0.882120
	0.25	$\varnothing_1$	-	-	-	-	-	-	0.863445
		$\varnothing_2$	-	-	-	-	-	-	0.863749
		$\varnothing_3$	-	-	-	-	-	-	0.863447
		$\varnothing_4$	-	-	-	-	-	-	0.863133
	0.30	$\varnothing_1$	-	-	-	-	-	-	-
		$\varnothing_2$	-	-	-	-	-	-	-
		$\varnothing_3$	-	-	-	-	-	-	-
		$\varnothing_4$	-	-	-	-	-	-	-
	0.35	$\varnothing_1$	-	-	-	-	-	-	-
		$\varnothing_2$	-	-	-	-	-	-	-
		$\varnothing_3$	-	-	-	-	-	-	-
		$\varnothing_4$	-	-	-	-	-	-	-
	0.40	$\varnothing_1$	-	-	-	-	-	-	-
		$\varnothing_2$	-	-	-	-	-	-	-
		$\varnothing_3$	-	-	-	-	-	-	-
		$\varnothing_4$	-	-	-	-	-	-	-
	0.45	$\varnothing_1$	-	-	-	-	-	-	-
		$\varnothing_2$	-	-	-	-	-	-	-
		$\varnothing_3$	-	-	-	-	-	-	-
		$\varnothing_4$	-	-	-	-	-	-	-
	0.50	$\varnothing_1$	-	-	-	-	-	-	-
		$\varnothing_2$	-	-	-	-	-	-	-
		$\varnothing_3$	-	-	-	-	-	-	-
		$\varnothing_4$	-	-	-	-	-	-	-
			-	-	-	-	-	-	-
			-	-	-	-	-	-	-
			-	-	-	-	-	-	-
			-	-	-	-	-	-	-

			ξ_2						
			0.35	0.40	0.45	0.50	0.55	0.60	0.65
ξ_1	0	$\boldsymbol{Ø_1}$	0.953337	0.953748	0.969896	1.014678	1.101528	1.319367	-3.255413
		$\boldsymbol{Ø_2}$	0.933241	0.904699	0.864033	0.796187	0.664847	0.351318	4.763754
		$\boldsymbol{Ø_3}$	1.142893	1.223107	1.341019	1.513519	1.765936	2.137493	2.691491
		$\boldsymbol{Ø_4}$	1	1	1	1	1	1	1
	0.05	$\boldsymbol{Ø_1}$	1.037334	1.061978	1.108982	1.204719	1.490000	0.301853	1.004578
		$\boldsymbol{Ø_2}$	1.027520	1.036467	1.048415	1.056337	1.004663	1.447389	1.235187
		$\boldsymbol{Ø_3}$	1.051078	1.078334	1.117704	1.164618	1.159988	1.745506	1.607624
		$\boldsymbol{Ø_4}$	1.040993	1.068900	1.120899	1.226231	1.547679	0.177559	0.980931
	0.10	$\boldsymbol{Ø_1}$	0.982262	1.000082	1.034277	1.096759	1.207130	1.507745	0.240974
		$\boldsymbol{Ø_2}$	0.979770	0.989984	1.006248	1.027523	1.040150	0.983761	1.451766
		$\boldsymbol{Ø_3}$	0.987489	1.007340	1.039922	1.086912	1.136609	1.124060	1.736221
		$\boldsymbol{Ø_4}$	0.980192	0.998482	1.033967	1.099427	1.218280	1.556263	0.095175
	0.15	$\boldsymbol{Ø_1}$	0.926503	0.937914	0.962461	1.008214	1.078763	1.186181	1.473788
		$\boldsymbol{Ø_2}$	0.927614	0.936427	0.952862	0.978400	1.005259	1.016567	0.957924
		$\boldsymbol{Ø_3}$	0.927907	0.940277	0.964539	1.004612	1.054863	1.09942	1.079651
		$\boldsymbol{Ø_4}$	0.923146	0.933481	0.957202	1.002791	1.075252	1.190647	1.514579
	0.20	$\boldsymbol{Ø_1}$	0.884727	0.889907	0.906433	0.941479	0.994443	1.060554	1.152721
		$\boldsymbol{Ø_2}$	0.886510	0.891686	0.905510	0.931279	0.962960	0.987832	0.993468
		$\boldsymbol{Ø_3}$	0.884993	0.890531	0.907054	0.939919	0.985005	1.030608	1.065032
		$\boldsymbol{Ø_4}$	0.882255	0.885875	0.900682	0.934143	0.986446	1.054398	1.154225
	0.25	$\boldsymbol{Ø_1}$	0.860023	0.859017	0.868738	0.896862	0.941657	0.993294	1.047320
		$\boldsymbol{Ø_2}$	0.861109	0.860990	0.870563	0.894998	0.929221	0.960208	0.977701
		$\boldsymbol{Ø_3}$	0.860060	0.859178	0.868968	0.896306	0.937858	0.981428	1.017089
		$\boldsymbol{Ø_4}$	0.858822	0.856403	0.864264	0.890318	0.933409	0.984658	1.040815
	0.30	$\boldsymbol{Ø_1}$	0.844036	0.836200	0.839296	0.862449	0.904246	0.952614	0.996664
		$\boldsymbol{Ø_2}$	0.844345	0.837351	0.841254	0.863534	0.900326	0.937532	0.962528
		$\boldsymbol{Ø_3}$	0.844038	0.836235	0.839387	0.862308	0.902791	0.947660	0.984290
		$\boldsymbol{Ø_4}$	0.843721	0.834917	0.836391	0.857441	0.897125	0.944090	0.988158
	0.35	$\boldsymbol{Ø_1}$	-	0.811395	0.806468	0.824086	0.864246	0.914051	-
		$\boldsymbol{Ø_2}$	-	0.811742	0.807719	0.825874	0.864345	0.908453	-
		$\boldsymbol{Ø_3}$	-	0.811398	0.806487	0.824055	0.863739	0.912053	-
		$\boldsymbol{Ø_4}$	-	0.811038	0.804972	0.820735	0.858699	0.906579	-
	0.40	$\boldsymbol{Ø_1}$	-	-	0.769745	0.779025	0.815273	-	0.914049
		$\boldsymbol{Ø_2}$	-	-	0.770153	0.780375	0.816904	-	0.908451
		$\boldsymbol{Ø_3}$	-	-	0.769746	0.779017	0.815133	-	0.912052
		$\boldsymbol{Ø_4}$	-	-	0.769316	0.777285	0.811574	-	0.906578
	0.45	$\boldsymbol{Ø_1}$	-	-	-	0.739677	-	0.815271	0.864245
		$\boldsymbol{Ø_2}$	-	-	-	0.740132	-	0.816903	0.864344
		$\boldsymbol{Ø_3}$	-	-	-	0.739676	-	0.815131	0.863739
		$\boldsymbol{Ø_4}$	-	-	-	0.739193	-	0.811573	0.858698
	0.50	$\boldsymbol{Ø_1}$	-	-	-	-	0.739675	0.779024	0.824086
		$\boldsymbol{Ø_2}$	-	-	-	-	0.740130	0.780374	0.825874
		$\boldsymbol{Ø_3}$	-	-	-	-	0.739675	0.779016	0.824054
		$\boldsymbol{Ø_4}$	-	-	-	-	0.739192	0.777284	0.820735

			ξ_2						
			0.70	0.75	0.80	0.85	0.90	0.95	1.00
ξ_1	0	$\emptyset_1$	0.572582	0.687158	0.692418	0.648055	0.535998	0.325845	-
		$\emptyset_2$	0.705852	0.279811	-0.173470	-0.800686	-1.651712	-2.695135	-
		$\emptyset_3$	3.535758	4.867786	7.087778	11.138151	19.862313	47.518020	-
		$\emptyset_4$	1	1	1	1	1	1	-
	0.05	$\emptyset_1$	1.156859	1.272939	1.392716	1.507009	1.581464	-	0.325845
		$\emptyset_2$	1.204548	1.185032	1.149029	1.063117	0.898520	-	-2.695135
		$\emptyset_3$	1.718009	1.897632	2.131726	2.384472	2.597429	-	47.518027
		$\emptyset_4$	1.150083	1.270986	1.391745	1.518262	1.639732	-	1
	0.10	$\emptyset_1$	0.980072	1.149480	1.289854	1.427157	-	1.581464	0.535998
		$\emptyset_2$	1.236124	1.222475	1.218899	1.183406	-	0.898520	-1.651712
		$\emptyset_3$	1.594773	1.732839	1.948454	2.201690	-	2.597431	19.862316
		$\emptyset_4$	0.936263	1.116106	1.255638	1.396018	-	1.639732	1
	0.15	$\emptyset_1$	0.210826	0.950444	1.138096	-	1.427157	1.507010	0.648055
		$\emptyset_2$	1.427016	1.228913	1.232121	-	1.183406	1.063118	-0.800686
		$\emptyset_3$	1.689198	1.569223	1.735788	-	2.201691	2.384473	11.138153
		$\emptyset_4$	0.056467	0.892833	1.087876	-	1.396018	1.518262	1
	0.20	$\emptyset_1$	1.421143	0.202983	-	1.138096	1.289854	1.392716	0.692418
		$\emptyset_2$	0.937021	1.401913	-	1.232121	1.218899	1.149029	-0.173470
		$\emptyset_3$	1.043724	1.647501	-	1.735788	1.948455	2.131726	7.087779
		$\emptyset_4$	1.456679	0.047510	-	1.087876	1.255638	1.391745	1
	0.25	$\emptyset_1$	1.127751	-	0.202984	0.950444	1.149480	1.272939	0.687158
		$\emptyset_2$	0.979235	-	1.401912	1.228912	1.222475	1.185032	0.279811
		$\emptyset_3$	1.044551	-	1.647501	1.569223	1.732839	1.897632	4.867787
		$\emptyset_4$	1.128501	-	0.047510	0.892833	1.116106	1.270986	1
	0.30	$\emptyset_1$	-	1.127750	1.421142	0.210827	0.980072	1.156859	0.572582
		$\emptyset_2$	-	0.979234	0.937021	1.427016	1.236124	1.204548	0.705852
		$\emptyset_3$	-	1.044550	1.043724	1.689197	1.594773	1.718009	3.535758
		$\emptyset_4$	-	1.128500	1.456678	0.056468	0.936263	1.150083	1
	0.35	$\emptyset_1$	0.996663	1.047319	1.152720	1.473787	0.240974	1.004578	-3.255411
		$\emptyset_2$	0.962527	0.977700	0.993468	0.957923	1.451765	1.235187	4.763752
		$\emptyset_3$	0.984289	1.017088	1.065032	1.079651	1.736221	1.607624	2.691491
		$\emptyset_4$	0.988157	1.040815	1.154224	1.514578	0.095175	0.980931	1
	0.40	$\emptyset_1$	0.952614	0.993294	1.060554	1.186181	1.507744	0.301854	1.319367
		$\emptyset_2$	0.937532	0.960207	0.987832	1.016567	0.983761	1.447389	0.351318
		$\emptyset_3$	0.947659	0.981427	1.030608	1.099419	1.124060	1.745505	2.137494
		$\emptyset_4$	0.944089	0.984658	1.054398	1.190647	1.556263	0.177560	1
	0.45	$\emptyset_1$	0.904246	0.941657	0.994443	1.078762	1.207129	1.490000	1.101528
		$\emptyset_2$	0.900326	0.929220	0.962960	1.005259	1.040150	1.044663	0.664847
		$\emptyset_3$	0.902790	0.937858	0.985005	1.054863	1.136609	1.159988	1.765936
		$\emptyset_4$	0.897124	0.933408	0.986446	1.075252	1.218280	1.547679	1
	0.50	$\emptyset_1$	0.862449	0.896862	0.941479	1.008214	1.096759	1.204719	1.014678
		$\emptyset_2$	0.863534	0.894997	0.931279	0.978400	1.027523	1.056337	0.796187
		$\emptyset_3$	0.862308	0.896306	0.939919	1.004612	1.086912	1.164618	1.513519
		$\emptyset_4$	0.857441	0.890318	0.934143	1.002791	1.099428	1.226231	1

			ξ_2						
			0.35	0.40	0.45	0.50	0.55	0.60	0.65
ξ_1	0.55	$\boldsymbol{\emptyset_1}$	-	-	-	-	-	0.769745	0.806467
		$\boldsymbol{\emptyset_2}$	-	-	-	-	-	0.770153	0.807719
		$\boldsymbol{\emptyset_3}$	-	-	-	-	-	0.769745	0.806487
		$\boldsymbol{\emptyset_4}$	-	-	-	-	-	0.769316	0.804971
	0.60	$\boldsymbol{\emptyset_1}$	-	-	-	-	-	-	0.811395
		$\boldsymbol{\emptyset_2}$	-	-	-	-	-	-	0.811742
		$\boldsymbol{\emptyset_3}$	-	-	-	-	-	-	0.811397
		$\boldsymbol{\emptyset_4}$	-	-	-	-	-	-	0.811037
	0.65	$\boldsymbol{\emptyset_1}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_2}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_3}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_4}$	-	-	-	-	-	-	-
	0.70	$\boldsymbol{\emptyset_1}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_2}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_3}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_4}$	-	-	-	-	-	-	-
	0.75	$\boldsymbol{\emptyset_1}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_2}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_3}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_4}$	-	-	-	-	-	-	-
	0.80	$\boldsymbol{\emptyset_1}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_2}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_3}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_4}$	-	-	-	-	-	-	-
	0.85	$\boldsymbol{\emptyset_1}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_2}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_3}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_4}$	-	-	-	-	-	-	-
	0.90	$\boldsymbol{\emptyset_1}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_2}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_3}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_4}$	-	-	-	-	-	-	-
	0.95	$\boldsymbol{\emptyset_1}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_2}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_3}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_4}$	-	-	-	-	-	-	-
	1.00	$\boldsymbol{\emptyset_1}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_2}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_3}$	-	-	-	-	-	-	-
		$\boldsymbol{\emptyset_4}$	-	-	-	-	-	-	-

			ξ_2						
			0.70	0.75	0.80	0.85	0.90	0.95	1.00
ξ_1	0.55	$Ø_1$	0.839296	0.868738	0.906433	0.962461	1.034277	1.108983	0.969896
		$Ø_2$	0.841254	0.870563	0.905510	0.952862	1.006248	1.048415	0.864032
		$Ø_3$	0.839387	0.868968	0.907054	0.964539	1.039922	1.117704	1.341019
		$Ø_4$	0.836391	0.864264	0.900682	0.957202	1.033967	1.120899	1
	0.60	$Ø_1$	0.836200	0.859017	0.889907	0.937914	1.000082	1.061978	0.953748
		$Ø_2$	0.837351	0.860990	0.891686	0.936427	0.989984	1.036467	0.904699
		$Ø_3$	0.836234	0.859178	0.890531	0.940277	1.007340	1.078334	1.223107
		$Ø_4$	0.834917	0.856403	0.885875	0.933481	0.998482	1.068900	1
	0.65	$Ø_1$	0.844036	0.860023	0.884727	0.926503	0.982262	1.037334	0.953337
		$Ø_2$	0.844345	0.861109	0.886510	0.927614	0.979770	1.027520	0.933241
		$Ø_3$	0.844038	0.860060	0.884992	0.927907	0.987489	1.051078	1.142893
		$Ø_4$	0.843721	0.858822	0.882255	0.923146	0.980192	1.040993	1
	0.70	$Ø_1$	-	0.863445	0.883311	0.920631	0.972091	1.022804	0.957094
		$Ø_2$	-	0.863749	0.884353	0.922125	0.972655	1.020499	0.953049
		$Ø_3$	-	0.863447	0.883380	0.921259	0.975147	1.032350	1.088846
		$Ø_4$	-	0.863133	0.882120	0.918457	0.970173	1.024437	1
	0.75	$Ø_1$	-	-	0.885874	0.920400	0.969141	1.016801	0.964402
		$Ø_2$	-	-	0.886181	0.921367	0.970430	1.017636	0.968428
		$Ø_3$	-	-	0.885881	0.920590	0.970581	1.022536	1.052953
		$Ø_4$	-	-	0.885557	0.919327	0.967771	1.017325	1
	0.80	$Ø_1$	-	-	-	0.933250	0.980354	1.025753	0.985771
		$Ø_2$	-	-	-	0.933530	0.981178	1.027036	0.991361
		$Ø_3$	-	-	-	0.933269	0.980782	1.028409	1.029603
		$Ø_4$	-	-	-	0.932970	0.979646	1.025747	1
	0.85	$Ø_1$	-	-	-	-	1.011144	1.054748	1.030531
		$Ø_2$	-	-	-	-	1.011348	1.055372	1.033929
		$Ø_3$	-	-	-	-	1.011182	1.055480	1.014871
		$Ø_4$	-	-	-	-	1.010959	1.054603	1
	0.90	$Ø_1$	-	-	-	-	-	1.098047	1.092695
		$Ø_2$	-	-	-	-	-	1.098152	1.093700
		$Ø_3$	-	-	-	-	-	1.098106	1.006066
		$Ø_4$	-	-	-	-	-	1.097987	1
	0.95	$Ø_1$	-	-	-	-	-	-	1.148799
		$Ø_2$	-	-	-	-	-	-	1.148877
		$Ø_3$	-	-	-	-	-	-	1.001437
		$Ø_4$	-	-	-	-	-	-	1
	1.00	$Ø_1$	-	-	-	-	-	-	-
		$Ø_2$	-	-	-	-	-	-	-
		$Ø_3$	-	-	-	-	-	-	-
		$Ø_4$	-	-	-	-	-	-	-

In locating an element using ξ_1 and ξ_2 for pinned pinned beam, ξ_2 must be greater than ξ_1, hence the reason for many of the blank spaces.

4.2.4 **Fixed-free beams**

These are beams rigidly restrained at one end and completely free at the other end. From section 3.2.4 the first natural frequency w for such a beam is given as

$$\omega_1 = 3.516015273\sqrt{\frac{EI}{\mu L^4}} \tag{4.82}$$

In normalized coordinates, the length of the bar L = 1 hence

$$\omega_1 = 3.516015273\sqrt{\frac{EI}{\mu}} \tag{4.83}$$

Likewise

$$u_1 = u(x_1, 0) = \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j x_1 + d_{2j} \sinh \beta_j x_1 - \cos \beta_j x_1 + d_{4j} \sin \beta_j x_1\right)$$

$$= \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j L\xi_1 + d_{2j} \sinh \beta_j L\xi_1 - \cos \beta_j L\xi_1 + d_{4j} \sin \beta_j L\xi_1\right)$$

(4.84a)

$$u_2 = u'(x_1, 0) = \sum_{j=1}^{\infty} A_j \beta_j L\left(\sinh \beta_j x_1 + d_{2j} \cosh \beta_j x_1 + \sin \beta_j x_1 + d_{4j} \cos \beta_j x_1\right)$$

$$= \sum_{j=1}^{\infty} A_j \beta L\left(\sinh \beta_j L\xi_1 + d_{2j} \cosh \beta_j L\xi_1 + \sin \beta_j L\xi_1 + d_{4j} \cos \beta_j L\xi_1\right)$$

(4.84b)

$$u_3 = u(x_2, 0) = \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j x_2 + d_{2j} \sinh \beta_j x_2 - \cos \beta_j x_2 + d_{4j} \sin \beta_j x_2\right)$$

$$= \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j L\xi_2 + d_{2j} \sinh \beta_j L\xi_2 - \cos \beta_j L\xi_2 + d_{4j} \sin \beta_j L\xi_2\right)$$

(4.84c)

$$u_4 = u'(x_2, 0) = \sum_{j=1}^{\infty} A_j \beta L\left(\sinh \beta_j x_2 + d_{2j} \cosh \beta_j x_2 + \sin \beta_j x_2 + d_{4j} \cos \beta_j x_2\right)$$

$$= \sum_{j=1}^{\infty} A_j \beta_j L\left(\sinh \beta_j L\xi_2 + d_{2j} \cosh \beta_j L\xi_2 + \sin \beta_j L\xi_2 + d_{4j} \cos \beta_j L\xi_2\right)$$

(4.84d)

Equations (4.84a) – (4.84d) are used to evaluate the total displacements u_1 to u_4 at the nodal points of a segment of the vibrating fixed-pinned beam. The equations represent the summation of an infinite series but an evaluation of the first few terms of the series provides values of very good precision.

From Table 3.5

$$A_j = \frac{a\mu L^2}{M_j}\left[\frac{-2\left(\sinh \beta_j L - \sin \beta L\right)}{\beta_j^3 L^3} + \frac{12\left(\cosh \beta_j L - \cos \beta_j L\right)}{\beta_j^4 L^4} - \frac{24\left(\sinh \beta_j L - \sin \beta_j L\right)}{\beta_j^5 L^5} + \right.$$

$$\left. d_{2j}\left(\frac{-2\left(\cosh \beta_j L - \cos \beta_j L\right)}{\beta_j^3 L^3} + \frac{12\left(\sinh \beta_j L - \sin \beta_j L\right)}{\beta_j^4 L^4} - \frac{24\left(\cosh \beta_j L + \cos \beta_j L - 2\right)}{\beta_j^5 L^5}\right)\right] \quad (4.85)$$

The values of $\beta_j L$, d_{2j}, and d_{4j} for j = 1, 2, 3, 4, 5, 6, 7 can be obtained from Table 3.5. The formula for calculating the generalized mass M_j can also be picked from Table 3.5.

For this beam the process of calculating the stiffness modification factors is similar to the one for fixed-fixed beam. The values of the fixed end forces F_1, F_2, F_3 and F_4 are evaluated using the equations provided in Table 3.5 while the values of the nodal displacements u_1, u_2, u_3 and u_4 are calculated from equations (4.84a) – (4.84b). These are substituted into the equations for nodal forces (equations (4.35) – (4.38)) from which the nodal forces P_1, P_2, P_3 and P_4 are obtained.

Equations (4.84a) – (4.84d) are evaluated for the first mode, j = 1 to obtain the nodal displacements u_{11}, u_{21}, u_{31} and u_{41} due to the first mode. These together with the calculated nodal forces P_1 to P_4 are substituted into equation (4.49) in order to obtain the stiffness modification factors ϕ_1, ϕ_2, ϕ_3 and ϕ_4.

Having presented the methods of calculating the stiffness modification factors for a fixed-free beam, some numerical demonstrations of these steps are presented below. For clarity the calculations will be presented in a tabular form.

Table 4.31: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0, \xi_2 = 1$ on a fixed-free beam under lateral vibration

$\xi_1 = 0,\ \xi_2 = 1$					
j	**A_j**	**F_{1j}**	**F_{2j}**	**F_{3j}**	**F_{4j}**
1	-1.52008168387560	-3.34067504559817	-0.8178837164478	-11.373151156019	1.50177401221623
2	-0.02145008105734	-3.80341494562209	-0.7924919339710	-0.7157747526374	0.56298369678467
3	-0.00160413375147	-1.66615863638835	-0.2290528034721	0.11258451007437	-0.0814728758508
4	-0.00029865709609	-0.76184503745894	-0.0626642896686	-0.0322403070909	0.02268816645668
5	-0.00008500158509	-0.49271451399008	-0.0377634546137	0.01238012128935	-0.0085934208140
6	-0.00003126635674	-0.31685307070100	-0.0168837081853	-0.0057325350555	0.00394675488356
7	-0.00001421912502	-0.24529756186162	-0.0128492968622	0.00314305482716	-0.0021522468612
	Total	**-10.6269588116202**	**-1.9695892032207**	**-11.998791064611**	**1.99917408681513**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	-1.52008168387560	0	0	-3.0401633706258	-4.1848015528847
2	-0.02145008105734	0	0	0.04290016504320	0.20509617176097
3	-0.00160413375147	0	0	-0.0032082675054	-0.0251806202208
4	-0.00029865709609	0	0	0.00059731419220	0.00656801290244
5	-0.00008500158509	0	0	-0.0001700031704	-0.0024033599594
6	-0.00003126635674	0	0	0.00006253271341	0.00108048778402
7	-0.00001421912502	0	0	-0.0000284382501	-0.0005807190825
	Total	**0**	**0**	**-3.0000100676028**	**-4.0002215797000**

$u_{11} = 0$ $\quad$ $u_{31} = -3.0401633706258$

$u_{21} = 0$ $\quad$ $u_{41} = -4.1848015528847$

From Table 3. 5

F_1 = 10.62695881162024 F_3 = 11.99879106461140

F_2 = 1.96958920322065 F_4 = -1.99917408681513

From equations (4.35) – (4.38)

P_1 = 22.62575014465383 $P_3 = -2.6842219114087 \times 10^{-7}$

P_2 = 11.96920644943740 $P_4 = 1.715960706860642 \times 10^{-12}$

From equation (4.49) taking EI = 1

ϕ_1 = 1.39816304291937 ϕ_3 = 1.56649109790865

ϕ_2 = 1.28305273103156 ϕ_4 = 1.37493242687651

Table 4.32: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0, \xi_2 = 0.5$ on a fixed-free beam under lateral vibration

$\xi_1 = 0, \quad \xi_2 = 0.5$					
j	**Aj**	**F_{1j}**	**F_{2j}**	**F_{3j}**	**F_{4j}**
1	-1.52008168387560	-0.48308887562856	-0.0598254327852	-1.7985449116688	0.11518435690387
2	-0.02145008105734	-1.11346066394695	-0.1327306939295	-2.9773818843250	0.21569568578587
3	-0.00160413375147	-1.12001531648915	-0.1251760620645	-1.5325279827593	0.14791128388993
4	-0.00029865709609	-0.83398742239521	-0.0822457526386	0.04315451235387	0.04114386769791
5	-0.00008500158509	-0.52110689712795	-0.0407710132156	0.38042045709329	-0.0135630638691
6	-0.00003126635674	-0.31833615354616	-0.0176075071317	-0.0041923547408	-0.0121385762947
7	-0.00001421912502	-0.23229927629899	-0.0102159415230	-0.1810843238668	0.00328550464345
	Total	**-12.1099115227902**	**-0.4685724032880**	**-6.0701564879136**	**0.49751905875728**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	-1.52008168387560	0	0	-1.0322057314524	-3.5358755383026
2	-0.02145008105734	0	0	-0.0306163797522	0.01943986981316
3	-0.00160413375147	0	0	-0.0000631630698	0.01781229808071

4	-0.00029865709609	0	0	0.00042237200171	0.00002690143067
5	-0.00008500158509	0	0	-0.0000001447471	-0.0016994333570
6	-0.00003126635674	0	0	-0.0000442173071	0.00000019124271
7	-0.00001421912502	0	0	-0.0000000010464	0.00041063040123
	Total	**-0.25002809516252**	**-0.2501752736690**	**-1.0625072653733**	**-3.4998850806912**

$u_{11} = 0$ $u_{31} = -1.0322057314524$

$u_{21} = -0$ $u_{41} = -3.5358755383026$

From Table 3. 5

$F_1 = 4.62229460543298$ $F_3 = 6.07015648791363$

$F_2 = 0.46857240328801$ $F_4 = -0.49751905875728$

From equations (4.35) – (4.38)

$P_1 = 22.62575014467687$ $P_3 = -11.93329905133027$

$P_2 = 11.96920644948130$ $P_4 = -2.99642533532860$

From equation (4.49) taking EI = 1

$\phi_1 = 2.21557484234068$ $\phi_3 = 1.25639506293829$

$\phi_2 = 2.40890082732463$ $\phi_4 = 1.20046452551605$

Table 4.33: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.5, \xi_2 = 1.0$ on a fixed-free beam under lateral vibration

$\xi_1 = 0.5, \quad \xi_2 = 1.0$					
j	**Aj**	$\mathbf{F_{1j}}$	$\mathbf{F_{2j}}$	$\mathbf{F_{3j}}$	$\mathbf{F_{4j}}$
1	-1.52008168387560	-4.96450921635365	-0.4644782862649	-7.4676831979656	0.56906876387313
2	-0.02145008105734	-2.09707074997230	-0.1138770427682	1.66872359998485	-0.0458682217481

3	-0.00160413375147	1.22405946857281	0.11336142464737	-0.1250902956381	-0.0547132629898
4	-0.00029865709609	0.13823105918218	-0.0287769338101	-0.1414834936900	0.04845309637066
5	-0.00008500158509	-0.4218085831993	-0.0191612335917	0.08216063055100	-0.0219480109423
6	-0.00003126635674	0.01563119814659	0.01496280219283	-0.0156882951774	0.00608266678954
7	-0.00001421912502	0.16987693708827	0.00164422355375	0.00135214757295	-0.0023207382242
	Total	**-6.60409544409049**	**-0.5690256019924**	**-5.9977089043624**	**0.49875429312898**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	-1.52008168387560	-1.03220573145244	-3.5358755383026	-3.0401633706258	-4.1848015528847
2	-0.02145008105734	-0.03061637975215	0.01943986981316	0.04290016504320	0.20509617176097
3	-0.00160413375147	-0.00006316306981	0.01781229808071	-0.0032082675054	-0.0251806202208
4	-0.00029865709609	0.00042237200171	0.00002690143067	0.00059731419220	0.00656801290244
5	-0.00008500158509	-0.00000014474706	-0.0016994333570	-0.0001700031704	-0.0024033599594
6	-0.00003126635674	-0.00004421730713	0.00000019124271	0.00006253271341	0.00108048778402
7	-0.00001421912502	-0.00000000104636	0.00041063040123	-0.0000284382501	-0.0005807190825
	Total	**-1.06250726537325**	**-3.4998850806912**	**-3.0000100676028**	**-4.0002215797000**

u_{11} = -1.03220573145244 u_{31} = -3.0401633706258

u_{21} = -3.5358755383026 u_{41} = -4.1848015528847

From Table 3. 5

F_1 = 5.93558988653541 F_3 = 5.99770890436235

F_2 = 0.49632504604100 F_4 = -0.49875429312898

From equations (4.35) – (4.38)

P_1 = 11.93329905118384 $P_3 = -2.602861002287682 \times 10^{-7}$

P_2 = 2.99642533522072 $P_4 = 1.552535877635819 \times 10^{-11}$

From equation (4.49) taking EI = 1

ϕ_1 = 1.01642802444861 ϕ_3 = 1.01490275693350

ϕ_2 = 1.00584030019720	ϕ_4 = 1.00946349670344

Table 4.34: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.5, \xi_2 = 0.8$ on a fixed-free beam under lateral vibration

$\xi_1 = 0.5,\ \ \xi_2 = 0.8$					
j	**Aj**	$\mathbf{F_{1j}}$	$\mathbf{F_{2j}}$	$\mathbf{F_{3j}}$	$\mathbf{F_{4j}}$
1	-1.52008168387560	-2.53754305197243	-0.1375125778971	-3.4114270846113	0.15938931191541
2	-0.02145008105734	-1.81230484003075	-0.0813120361971	-0.8129250959937	0.05615897916132
3	-0.00160413375147	0.68250088526875	0.04364876265484	1.03367218732848	-0.0528296174527
4	-0.00029865709609	0.42971511615191	0.01033201399443	-0.4767067798360	0.01405662526632
5	-0.00008500158509	-0.45167962124413	-0.0248332700103	-0.0563715809027	0.01360674930991
6	-0.00003126635674	-0.05545949556406	0.00710391405670	0.26373431775862	-0.0168112150316
7	-0.00001421912502	0.22682087454497	0.00837148019507	-0.2267515443707	0.00709372245167
	Total	**-3.51795013284574**	**-0.1742017132035**	**-3.6867755806273**	**0.18066455562036**
j	**Aj**	$\mathbf{u_{1j}}$	$\mathbf{u_{2j}}$	$\mathbf{u_{3j}}$	$\mathbf{u_{4j}}$
1	-1.52008168387560	-1.03220573145244	-3.5358755383026	-2.2055707048045	-4.1381378773295
2	-0.02145008105734	-0.03061637975215	0.01943986981316	0.00300455195851	0.18394031403512
3	-0.00160413375147	-0.00006316306981	0.01781229808071	0.00126686038451	-0.0151924993847
4	-0.00029865709609	0.00042237200171	0.00002690143067	-0.0003840955140	0.00109116172477
5	-0.00008500158509	-0.00000014474706	-0.0016994333570	0.00010207807821	0.00070044382380
6	-0.00003126635674	-0.00004421730713	0.00000019124271	-0.0000190873900	-0.0006636944469
7	-0.00001421912502	-0.00000000104636	0.00041063040123	-0.0000033851557	0.00040068558120
	Total	**-1.06250726537325**	**-3.4998850806912**	**-2.2016037824429**	**-3.9678614659962**

u_{11} = -1.03220573145244 u_{31} = -3.5358755383026

u_{21} = -2.2055707048045 u_{41} = -4.1381378773295

From Table 3. 5

F_1 = 3.51795013284574 F_3 = 3.68677558062730
F_2 = 0.17420171320345 F_4 = -0.18066455562036
From equations (4.35) – (4.38)
P_1 = 11.93329905132885 P_3 = -4.72857333785578
P_2 = 2.99642533532619 P_4 = -0.47828350219818
From equation (4.49) taking EI = 1
ϕ_1 = 0.89670920379525 ϕ_3 = 0.89752780657083
ϕ_2 = 0.89634218110213 ϕ_4 = 0.89577528454164

Table 4.35: Calculation of the Stiffness modification factor for an element positioned at $\xi_1 = 0.8$, $\xi_2 = 1.0$ on a fixed-free beam under lateral vibration

$\xi_1 = 0.8, \quad \xi_2 = 1.0$					
j	**Aj**	**F_{1j}**	**F_{2j}**	**F_{3j}**	**F_{4j}**
1	-1.52008168387560	-3.03513805200607	-0.1046062437585	-3.4480842257295	0.11149004221555
2	-0.02145008105734	0.70893573100767	0.03000826162572	1.48794705502932	-0.0430153871327
3	-0.00160413375147	0.03951999552946	-0.0040436238453	-0.6567238951921	0.01573178522022
4	-0.00029865709609	-0.27949913605781	-0.0052720507610	0.32323836523339	-0.0049395806331
5	-0.00008500158509	0.32108770241990	0.00837555511122	-0.1526844529580	-0.0002505736636
6	-0.00003126635674	-0.26824307318398	-0.0084241237301	0.05991115391168	0.00272979574560
7	-0.00001421912502	0.18173438131982	0.00717638027610	-0.0105746162333	-0.0038495614701
	Total	**-2.33160245097101**	**-0.0767858450818**	**-2.3969706159385**	**0.07789652028189**
j	**Aj**	**u_{1j}**	**u_{2j}**	**u_{3j}**	**u_{4j}**
1	-1.52008168387560	-2.20557070480445	-4.1381378773295	-3.0401633706258	-4.1848015528847
2	-0.02145008105734	0.00300455195851	0.18394031403512	0.04290016504320	0.20509617176097
3	-0.00160413375147	0.00126686038451	-0.0151924993847	-0.0032082675054	-0.0251806202208

4	-0.00029865709609	-0.00038409551400	0.00109116172477	0.00059731419220	0.00656801290244
5	-0.00008500158509	0.00010207807821	0.00070044382380	-0.0001700031704	-0.0024033599594
6	-0.00003126635674	-0.00001908739002	-0.0006636944469	0.00006253271341	0.00108048778402
7	-0.00001421912502	-0.00000338515570	0.00040068558120	-0.0000284382501	-0.0005807190825
	Total	**-2.20160378244294**	**-3.9678614659962**	**-3.0000100676028**	**-4.0002215797000**

u_{11} = -2.20557070480445 u_{31} = -3.0401633706258

u_{21} = -4.1381378773295 u_{41} = -4.1848015528847

From Table 3. 5

F_1 = 2.33160245097101 F_3 = 2.39697061593850

F_2 = 0.07678584508183 F_4 = -0.07789652028189

From equations (4.35) – (4.38)

P_1 = 4.72857333629486 $P_3 = -2.693852820812026 \times 10^{-7}$

P_2 = 0.47828350213275 $P_4 = -2.681161959117162 \times 10^{-10}$

From equation (4.49) taking EI = 1

ϕ_1 = 0.77135265793859 ϕ_3 = 0.77112957607026

ϕ_2 = 0.77098907611122 ϕ_4 = 0.77113934157876

Tables 4.31 to 4.35 provide illustrations on how the inherent nodal forces P_1 to P_4 and the stiffness modification factors ϕ_1 to ϕ_4 for an element of a fixed-free beam under lateral vibration are calculated. Using the methods presented in the Tables above, the values of stiffness modification factors at different values of ξ_1 and ξ_2 for the lateral vibration of a fixed-free beam are presented in Table 4.36. A sample matlab program for the calculation of the stiffness modification factors for a segment of a fixed-fixed beam can be found in Appendix G.

Table 4.36: Stiffness modification factors for the lateral vibration of a fixed-free beam

			ξ_2						
			0	0.05	0.10	0.15	0.20	0.25	0.30
ξ_1	0	$\boldsymbol{\emptyset_1}$	-	1.071905	1.017044	0.961068	0.906933	0.849510	0.774592
		$\boldsymbol{\emptyset_2}$	-	1.066470	1.005492	0.943900	0.884905	0.821986	0.738272
		$\boldsymbol{\emptyset_3}$	-	1.133808	1.147308	1.160371	1.173158	1.185861	1.198696
		$\boldsymbol{\emptyset_4}$	-	1.129009	1.137682	1.145848	1.153621	1.161137	1.168542
	0.05	$\boldsymbol{\emptyset_1}$	1.071905	-	1.944790	2.453196	2.427962	2.139531	1.874784
		$\boldsymbol{\emptyset_2}$	1.066470	-	1.991825	2.584192	2.586096	2.279186	1.989061
		$\boldsymbol{\emptyset_3}$	1.133808	-	1.924067	2.315523	2.181436	1.872797	1.636011
		$\boldsymbol{\emptyset_4}$	1.129009	-	1.880491	2.207457	2.064233	1.777101	1.560254
	0.10	$\boldsymbol{\emptyset_1}$	1.017044	1.944790	-	1.492052	1.491886	1.464626	1.425453
		$\boldsymbol{\emptyset_2}$	1.005492	1.991825	-	1.501784	1.515706	1.497953	1.463630
		$\boldsymbol{\emptyset_3}$	1.147308	1.924067	-	1.490731	1.483252	1.445901	1.398987
		$\boldsymbol{\emptyset_4}$	1.137682	1.880491	-	1.481223	1.460882	1.415587	1.364655
	0.15	$\boldsymbol{\emptyset_1}$	0.961068	2.453196	1.492052	-	1.372580	1.343215	1.315239
		$\boldsymbol{\emptyset_2}$	0.943900	2.584192	1.501784	-	1.376539	1.353658	1.331193
		$\boldsymbol{\emptyset_3}$	1.160371	2.315523	1.490731	-	1.372357	1.341613	1.311489
		$\boldsymbol{\emptyset_4}$	1.145848	2.207457	1.481223	-	1.368434	1.331395	1.295918
	0.20	$\boldsymbol{\emptyset_1}$	0.906933	2.427962	1.491886	1.372580	-	1.280274	1.252039
		$\boldsymbol{\emptyset_2}$	0.884905	2.586096	1.515706	1.376539	-	1.282170	1.257418
		$\boldsymbol{\emptyset_3}$	1.173158	2.181436	1.483252	1.372357	-	1.280226	1.251683
		$\boldsymbol{\emptyset_4}$	1.153621	2.064233	1.460882	1.368434	-	1.278335	1.246313
	0.25	$\boldsymbol{\emptyset_1}$	0.849510	2.139531	1.464626	1.343215	1.280274	-	1.209020
		$\boldsymbol{\emptyset_2}$	0.821986	2.279186	1.497954	1.353658	1.282170	-	1.210041
		$\boldsymbol{\emptyset_3}$	1.185860	1.872797	1.445901	1.341613	1.280226	-	1.209009
		$\boldsymbol{\emptyset_4}$	1.161137	1.777101	1.415587	1.331395	1.278335	-	1.207987
	0.30	$\boldsymbol{\emptyset_1}$	0.774592	1.874784	1.425453	1.315239	1.252039	1.209020	-
		$\boldsymbol{\emptyset_2}$	0.738272	1.989061	1.463630	1.331193	1.257418	1.210041	-
		$\boldsymbol{\emptyset_3}$	1.198696	1.636011	1.398987	1.311489	1.251683	1.209009	-
		$\boldsymbol{\emptyset_4}$	1.168542	1.560254	1.364655	1.295918	1.246313	1.207987	-
	0.35	$\boldsymbol{\emptyset_1}$	0.655531	1.678222	1.382755	1.287632	1.227582	1.184881	1.153670
		$\boldsymbol{\emptyset_2}$	0.603023	1.771273	1.422635	1.307481	1.236354	1.187940	1.154273
		$\boldsymbol{\emptyset_3}$	1.211901	1.483629	1.353450	1.282245	1.226787	1.184812	1.153669
		$\boldsymbol{\emptyset_4}$	1.176000	1.420427	1.316989	1.262593	1.217934	1.181730	1.153064
	0.40	$\boldsymbol{\emptyset_1}$	0.4098260	1.528694	1.336957	1.257592	1.202228	1.160759	1.129258
		$\boldsymbol{\emptyset_2}$	0.321299	1.604642	1.376351	1.279744	1.213682	1.165940	1.131114
		$\boldsymbol{\emptyset_3}$	1.225732	1.385308	1.311146	1.252476	1.201464	1.160727	1.129277
		$\boldsymbol{\emptyset_4}$	1.183683	1.328951	1.273290	1.229739	1.189607	1.155424	1.127401
	0.45	$\boldsymbol{\emptyset_1}$	-0.822768	1.409628	1.289048	1.224627	1.174496	1.134705	1.103135
		$\boldsymbol{\emptyset_2}$	-1.100241	1.471454	1.326398	1.247668	1.187715	1.141627	1.106314
		$\boldsymbol{\emptyset_3}$	1.240466	1.321164	1.273494	1.222796	1.175001	1.135190	1.103347
		$\boldsymbol{\emptyset_4}$	1.191774	1.267564	1.234215	1.197542	1.160638	1.127856	1.100054
	0.50	$\boldsymbol{\emptyset_1}$	2.215575	1.317195	1.244206	1.192534	1.147329	1.109243	1.077717
		$\boldsymbol{\emptyset_2}$	2.408901	1.367368	1.278533	1.215337	1.161419	1.117344	1.081986
		$\boldsymbol{\emptyset_3}$	1.256395	1.282367	1.244171	1.197123	1.150772	1.111094	1.078538
		$\boldsymbol{\emptyset_4}$	1.200465	1.228205	1.202713	1.169322	1.134087	1.101884	1.073848

			ξ_2						
			0.35	0.40	0.45	0.50	0.55	0.60	0.65
ξ_1	0	$\boldsymbol{\emptyset_1}$	0.655531	0.409826	-0.822768	2.215575	1.384370	1.219207	1.161188
		$\boldsymbol{\emptyset_2}$	0.603023	0.321299	-1.100241	2.408901	1.447830	1.253100	1.179377
		$\boldsymbol{\emptyset_3}$	1.211901	1.225732	1.240466	1.256395	1.273833	1.293117	1.314608
		$\boldsymbol{\emptyset_4}$	1.176000	1.183683	1.191774	1.200465	1.209956	1.220462	1.232211
	0.05	$\boldsymbol{\emptyset_1}$	1.678222	1.528694	1.409628	1.317195	1.251917	1.210850	1.186751
		$\boldsymbol{\emptyset_2}$	1.771273	1.604642	1.471454	1.367368	1.292434	1.242857	1.210391
		$\boldsymbol{\emptyset_3}$	1.483629	1.385308	1.321164	1.282367	1.264070	1.261351	1.269160
		$\boldsymbol{\emptyset_4}$	1.420427	1.328951	1.267564	1.228205	1.206504	1.198063	1.198343
	0.10	$\boldsymbol{\emptyset_1}$	1.382755	1.336957	1.289048	1.244206	1.208640	1.184841	1.170626
		$\boldsymbol{\emptyset_2}$	1.422635	1.376351	1.326398	1.278533	1.239341	1.211333	1.192110
		$\boldsymbol{\emptyset_3}$	1.353450	1.311146	1.273494	1.244171	1.226466	1.220550	1.223862
		$\boldsymbol{\emptyset_4}$	1.316989	1.273290	1.234215	1.202713	1.181530	1.170671	1.167689
	0.15	$\boldsymbol{\emptyset_1}$	1.287632	1.257592	1.224627	1.192534	1.166599	1.149396	1.139602
		$\boldsymbol{\emptyset_2}$	1.307481	1.279744	1.247668	1.215337	1.188268	1.169091	1.156393
		$\boldsymbol{\emptyset_3}$	1.282245	1.252476	1.222796	1.197123	1.179773	1.172182	1.172683
		$\boldsymbol{\emptyset_4}$	1.262593	1.229739	1.197542	1.169322	1.148820	1.137165	1.132658
	0.20	$\boldsymbol{\emptyset_1}$	1.227582	1.202228	1.174496	1.147329	1.125359	1.110985	1.103139
		$\boldsymbol{\emptyset_2}$	1.236354	1.213682	1.187715	1.161419	1.139516	1.124423	1.115008
		$\boldsymbol{\emptyset_3}$	1.226787	1.201464	1.175001	1.150772	1.133195	1.124167	1.122425
		$\boldsymbol{\emptyset_4}$	1.217934	1.189607	1.160638	1.134087	1.113937	1.101783	1.096277
	0.25	$\boldsymbol{\emptyset_1}$	1.184881	1.160759	1.134705	1.109243	1.088651	1075195	1.067884
		$\boldsymbol{\emptyset_2}$	1.187940	1.165940	1.141627	1.117344	1.097335	1.083832	1.075775
		$\boldsymbol{\emptyset_3}$	1.184812	1.160727	1.135190	1.111094	1.092779	1.082321	1.078640
		$\boldsymbol{\emptyset_4}$	1.181730	1.155424	1.127856	1.101884	1.081589	1.068812	1.062381
	0.30	$\boldsymbol{\emptyset_1}$	1.153670	1.129258	1.103135	1.077717	1.057135	1.043558	1.035989
		$\boldsymbol{\emptyset_2}$	1.154273	1.131114	1.106314	1.081986	1.062125	1.048824	1.041001
		$\boldsymbol{\emptyset_3}$	1.153669	1.129277	1.103347	1.078538	1.059111	1.047204	1.041792
		$\boldsymbol{\emptyset_4}$	1.153064	1.127401	1.100054	1.073848	1.052953	1.039352	1.031942
	0.35	$\boldsymbol{\emptyset_1}$	-	1.102514	1.075399	1.049174	1.027925	1.013735	1.00525
		$\boldsymbol{\emptyset_2}$	-	1.102886	1.076542	1.051122	1.030521	1.016710	1.008513
		$\boldsymbol{\emptyset_3}$	-	1.102516	1.075451	1.049452	1.028736	1.015423	1.008445
		$\boldsymbol{\emptyset_4}$	-	1.102143	1.074286	1.047363	1.025630	1.011130	1.002741
	0.40	$\boldsymbol{\emptyset_1}$	1.102514	-	1.047165	1.019831	0.997710	0.982767	0.973775
		$\boldsymbol{\emptyset_2}$	1.102886	-	1.047390	1.020514	0.998860	0.984264	0.975400
		$\boldsymbol{\emptyset_3}$	1.102516	-	1.047168	1.019887	0.997958	0.983408	0.975041
		$\boldsymbol{\emptyset_4}$	1.102143	-	1.046941	1.019172	0.996627	0.981324	0.972042
	0.45	$\boldsymbol{\emptyset_1}$	1.075399	1.047165	-	0.989049	0.966131	0.950469	0.940633
		$\boldsymbol{\emptyset_2}$	1.076542	1.047390	-	0.989179	0.966521	0.951113	0.941426
		$\boldsymbol{\emptyset_3}$	1.075451	1.047168	-	0.989052	0.966175	0.950642	0.941061
		$\boldsymbol{\emptyset_4}$	1.074286	1.046941	-	0.988919	0.965745	0.949803	0.939700
	0.50	$\boldsymbol{\emptyset_1}$	1.049174	1.019831	0.989049	-	0.936304	0.919949	0..909140
		$\boldsymbol{\emptyset_2}$	1.051122	1.020514	0.989179	-	0.936378	0.920168	0.909486
		$\boldsymbol{\emptyset_3}$	1.049452	1.019887	0.989052	-	0.936306	0.919975	0.909241
		$\boldsymbol{\emptyset_4}$	1.047363	1.019172	0.988919	-	0.936230	0.919719	0.908726

			ξ_2						
			0.70	0.75	0.80	0.85	0.90	0.95	1.00
	0	$\emptyset_1$	1.137236	1.131436	1.144870	1.181049	1.238777	1.312599	1.398163
		$\emptyset_2$	1.142500	1.123612	1.122484	1.141446	1.178267	1.226960	1.283053
		$\emptyset_3$	1.338702	1.365833	1.396486	1.431205	1.470612	1.515426	1.566491
		$\emptyset_4$	1.245454	1.260466	1.277557	1.297086	1.319472	1.345218	1.374932
	0.05	$\emptyset_1$	1.172950	1.168129	1.176270	1.202046	1.246295	1.305911	1.377876
		$\emptyset_2$	1.187566	1.172465	1.168393	1.179206	1.205073	1.242625	1.288816
		$\emptyset_3$	1.284022	1.304988	1.332874	1.368631	1.412307	1.463330	1.521561
		$\emptyset_4$	1.204174	1.214649	1.230470	1.252471	1.280633	1.314308	1.353212
	0.10	$\emptyset_1$	1.162728	1.161074	1.169547	1.192714	1.232050	1.285574	1.351120
		$\emptyset_2$	1.178130	1.168976	1.167983	1.179033	1.203078	1.237976	1.281578
		$\emptyset_3$	1.233805	1.249929	1.273774	1.306991	1.349822	1.401349	1.461031
		$\emptyset_4$	1.170119	1.177487	1.191188	1.212750	1.242369	1.279090	1.322261
	0.15	$\emptyset_1$	1.134776	1.135064	1.144052	1.166070	1.202825	1.252924	1.314722
		$\emptyset_2$	1.147582	1.142537	1.144368	1.156800	1.181084	1.215703	1.259030
		$\emptyset_3$	1.179096	1.191257	1.211162	1.240991	1.281280	1.331060	1.389639
		$\emptyset_4$	1.133157	1.138441	1.150351	1.170941	1.200702	1.238643	1.283978
	0.20	$\emptyset_1$	1.099680	1.100763	1.109730	1.130715	1.165536	1.213163	1.272294
		$\emptyset_2$	1.109010	1.106361	1.109981	1.123465	1.148218	1.183084	1.226762
		$\emptyset_3$	1.125974	1.134785	1.151128	1.177573	1.214935	1.262295	1.318917
		$\emptyset_4$	1.095420	1.099113	1.109482	1.128973	1.158361	1.196711	1.243213
ξ_1	0.25	$\emptyset_1$	1.064684	1.065700	1.074088	1.093857	1.126907	1.172462	1.229479
		$\emptyset_2$	1.071019	1.069468	1.073910	1.087871	1.112873	1.147998	1.192179
		$\emptyset_3$	1.079802	1.085811	1.099112	1.122573	1.157266	1.202364	1.257146
		$\emptyset_4$	1.060341	1.062632	1.071568	1.089905	1.118679	1.157056	1.204261
	0.30	$\emptyset_1$	1.032389	1.032785	1.040206	1.058596	1.089949	1.133692	1.188999
		$\emptyset_2$	1.036508	1.035191	1.039754	1.053719	1.078726	1.114049	1.158806
		$\emptyset_3$	1.040911	1.044545	1.055250	1.076148	1.108557	1.151767	1.205117
		$\emptyset_4$	1.028741	1.029675	1.037191	1.054309	1.082315	1.120506	1.168182
	0.35	$\emptyset_1$	1.001177	1.000625	1.006815	1.023695	1.053371	1.-095457	1.149325
		$\emptyset_2$	1..003706	1.002054	1.006228	1.019803	1.044562	1.0799500	1.125256
		$\emptyset_3$	1.005744	1.007242	1.015585	1.034130	1.064442	1.105955	1.158096
		$\emptyset_4$	0.998392	0.997965	1.004011	1.019799	1.046868	1.084666	1.132602
	0.40	$\emptyset_1$	0.968483	0.966727	0.971426	0.986593	1.014483	1.054884	1.107357
		$\emptyset_2$	0.969910	0.967482	0.970799	0.983556	1.007708	1.042870	1.088493
		$\emptyset_3$	0.970656	0.970098	0.976101	0.992265	1.020415	1.060143	1.110995
		$\emptyset_4$	0.970656	0.964727	0.969175	0.983435	1.009310	1.046416	1.094297
	0.45	$\emptyset_1$	0.934205	0.930943	0.933804	0.946934	0.972772	1.011282	1.062206
		$\emptyset_2$	0.934947	0.931305	0.933258	0.944685	0.967756	1.002253	1.047796
		$\emptyset_3$	0.935055	0.932402	0.935951	0.949566	0.975352	1.013077	1.062423
		$\emptyset_4$	0.933059	0.929642	0.932243	0.944662	0.968985	1.005006	1.052422
	0.50	$\emptyset_1$	0.901291	0.896126	0.896709	0.907402	0.930862	0.967227	1.016428
		$\emptyset_2$	0.901662	0.896313	0.896342	0.905888	0.927376	0.960762	1.005840
		$\emptyset_3$	0.901548	0.896640	0.897528	0.908384	0.931615	0.967175	1.014903
		$\emptyset_4$	0.900707	0.895399	0.895775	0.905955	0.928327	0.962875	1.009464

			ξ_2						
			0	0.05	0.10	0.15	0.20	0.25	0.30
ξ_1	0.55	$\boldsymbol{\emptyset_1}$	1.384370	1.251917	1.208640	1.166599	1.125359	1.088651	1.057135
		$\boldsymbol{\emptyset_2}$	1.447830	1.292434	1.239341	1.188268	1.139516	1.097335	1.062125
		$\boldsymbol{\emptyset_3}$	1.273833	1.264070	1.226466	1.179773	1.133195	1.092779	1.059111
		$\boldsymbol{\emptyset_4}$	1.209956	1.206504	1.181530	1.148820	1.113937	1.081589	1.052953
	0.60	$\boldsymbol{\emptyset_1}$	1.219207	1.210850	1.184841	1.149396	1.110985	1.075195	1.043558
		$\boldsymbol{\emptyset_2}$	1.253100	1.242857	1.211333	1.169091	1.124423	1.083832	1.048824
		$\boldsymbol{\emptyset_3}$	1.293117	1.261351	1.220550	1.172182	1.124167	1.082321	1.047204
		$\boldsymbol{\emptyset_4}$	1.220462	1.198063	1.170671	1.137165	1.101783	1.068812	1.039352
	0.65	$\boldsymbol{\emptyset_1}$	1.161188	1.186751	1.170626	1.139602	1.103139	1.067884	1.035989
		$\boldsymbol{\emptyset_2}$	1.179377	1.210391	1.192110	1.156393	1.115008	1.075775	1.041001
		$\boldsymbol{\emptyset_3}$	1.314608	1.269160	1.223862	1.172683	1.122425	1.078640	1.041792
		$\boldsymbol{\emptyset_4}$	1.232211	1.198343	1.167689	1.132658	1.096277	1.062381	1.031942
	0.70	$\boldsymbol{\emptyset_1}$	1.137236	1.172950	1.162728	1.134776	1.099680	1.064684	1.032389
		$\boldsymbol{\emptyset_2}$	1.142500	1.187566	1.178130	1.147582	1.109010	1.071019	1.036508
		$\boldsymbol{\emptyset_3}$	1.338702	1.284022	1.233805	1.179096	1.125974	1.079802	1.040911
		$\boldsymbol{\emptyset_4}$	1.245454	1.204174	1.170119	1.133157	1.095420	1.060341	1.028741
	0.75	$\boldsymbol{\emptyset_1}$	1.131436	1.168129	1.161074	1.135064	1.100763	1.065700	1.032785
		$\boldsymbol{\emptyset_2}$	1.123612	1.172465	1.168976	1.142537	1.106361	1.069468	1.035191
		$\boldsymbol{\emptyset_3}$	1.365833	1.304988	1.249929	1.191257	1.134785	1.085811	1.044545
		$\boldsymbol{\emptyset_4}$	1.260466	1.214649	1.177487	1.138441	1.099113	1.065632	1.029675
	0.80	$\boldsymbol{\emptyset_1}$	1.144870	1.176270	1.169547	1.144052	1.109730	1.074088	1.040206
		$\boldsymbol{\emptyset_2}$	1.122484	1.168393	1.167983	1.144368	1.109981	1.073910	1.039754
		$\boldsymbol{\emptyset_3}$	1.396486	1.332874	1.273775	1.211162	1.151128	1.099112	1.055250
		$\boldsymbol{\emptyset_4}$	1.277557	1.230470	1.191188	1.150351	1.109482	1.071568	1.037191
	0.85	$\boldsymbol{\emptyset_1}$	1.181049	1.202046	1.192714	1.166070	1.130715	1.093857	1.058596
		$\boldsymbol{\emptyset_2}$	1.141446	1.179206	1.179033	1.156800	1.123465	1.087871	1.053719
		$\boldsymbol{\emptyset_3}$	1.431205	1.368631	1.306991	1.240991	1.177573	1.122573	1.076148
		$\boldsymbol{\emptyset_4}$	1.297086	1.252471	1.212750	1.170941	1.128973	1.089905	1.054309
	0.90	$\boldsymbol{\emptyset_1}$	1.238777	1.246295	1.232050	1.202825	1.165536	1.126907	1.089949
		$\boldsymbol{\emptyset_2}$	1.178267	1.205073	1.203078	1.181084	1.148218	1.112873	1.078726
		$\boldsymbol{\emptyset_3}$	1.470612	1.412307	1.349822	1.281280	1.214935	1.157266	1.108557
		$\boldsymbol{\emptyset_4}$	1.319472	1.280633	1.242369	1.200702	1.158361	1.118679	1.082315
	0.95	$\boldsymbol{\emptyset_1}$	1.312599	1.305911	1.285574	1.252924	1.213163	1.172462	1.133692
		$\boldsymbol{\emptyset_2}$	1.226960	1.242625	1.237976	1.215703	1.183084	1.147998	1.114049
		$\boldsymbol{\emptyset_3}$	1.515426	1.463330	1.401349	1.331060	1.262295	1.202364	1.151767
		$\boldsymbol{\emptyset_4}$	1.345218	1.314308	1.279090	1.238643	1.196711	1.157056	1.120506
	1.00	$\boldsymbol{\emptyset_1}$	1.398163	1.277876	1.351120	1.314722	1.272294	1.229479	1.188999
		$\boldsymbol{\emptyset_2}$	1.283053	1.288816	1.281578	1.259030	1.226762	1.192179	1.158806
		$\boldsymbol{\emptyset_3}$	1.566491	1.521561	1.461031	1.389639	1.318917	1.257146	1.205117
		$\boldsymbol{\emptyset_4}$	1.374932	1.353212	1.322261	1.283978	1.243212	1.204261	1.168182

			ξ_2						
			0.35	0.40	0.45	0.50	0.55	0.60	0.65
ξ_1	0.55	$\emptyset_1$	1.027925	0.997710	0.966131	0.936304	-	0.894940	0.882868
		$\emptyset_2$	1.030521	0.998860	0.966521	0.936378	-	0.894983	0.882993
		$\emptyset_3$	1.028736	0.997958	0.966175	0.936306	-	0.894941	0.882881
		$\emptyset_4$	1.025630	0.996627	0.965745	0.936230	-	0.894896	0.882727
	0.60	$\emptyset_1$	1.013735	0.982767	0.950469	0.919949	0.894940	-	0.862758
		$\emptyset_2$	1.016710	0.984264	0.951113	0.920168	0.894983	-	0.862784
		$\emptyset_3$	1.015423	0.983408	0.950642	0.919975	0.894941	-	0.862758
		$\emptyset_4$	1.011130	0.981324	0.949803	0.919719	0.894896	-	0.862731
	0.65	$\emptyset_1$	1.005525	0.973775	0.940633	0.909140	0.882868	0.862758	-
		$\emptyset_2$	1.008513	0.975400	0.941426	0.909486	0.882993	0.862784	-
		$\emptyset_3$	1.008445	0.975041	0.941061	0.909241	0.882881	0.862758	-
		$\emptyset_4$	1.002741	0.972042	0.939700	0.908726	0.882727	0.862731	-
	0.70	$\emptyset_1$	1.001177	0.968483	0.934205	0.901291	0.873175	0.850655	0.831363
		$\emptyset_2$	1.003706	0.969910	0.934947	0.901662	0.873353	0.850727	0.831379
		$\emptyset_3$	1.005744	0.970656	0.935055	0.901548	0.873231	0.850663	0.831364
		$\emptyset_4$	0.998392	0.966570	0.933059	0.900707	0.872924	0.850570	0.831347
	0.75	$\emptyset_1$	1.000625	0.966727	0.930943	0.896126	0.865609	0.840115	0.817224
		$\emptyset_2$	1.002054	0.967482	0.931305	0.896313	0.865728	0.840189	0.817258
		$\emptyset_3$	1.007242	0.970098	0.932402	0.896640	0.865756	0.840149	0.817229
		$\emptyset_4$	0.997965	0.964727	0.929642	0.895399	0.865258	0.839967	0.817174
	0.80	$\emptyset_1$	1.006815	0.971426	0.933804	0.896709	0.863431	0.834669	0.807976
		$\emptyset_2$	1.006228	0.970799	0.933258	0.896342	0.863250	0.834610	0.807970
		$\emptyset_3$	1.015585	0.976101	0.935951	0.897528	0.863691	0.834740	0.807992
		$\emptyset_4$	1.004011	0.969175	0.932243	0.895775	0.862939	0.834435	0.807879
	0.85	$\emptyset_1$	1.023695	0.986593	0.946934	0.907402	0.871285	0.839309	0.808998
		$\emptyset_2$	1.019803	0.983556	0.944685	0.905888	0.870386	0.838845	0.808793
		$\emptyset_3$	1.034130	0.992265	0.949566	0.908384	0.871565	0.839361	0.808994
		$\emptyset_4$	1.019799	0.983435	0.944662	0.905955	0.870455	0.838869	0.808786
	0.90	$\emptyset_1$	1.053371	1.014483	0.972772	0.930862	0.892090	0.857241	0.823782
		$\emptyset_2$	1.044562	1.007708	0.967756	0.927376	0.889859	0.855936	0.823095
		$\emptyset_3$	1.064442	1.020415	0.975352	0.931615	0.892109	0.857086	0.823643
		$\emptyset_4$	1.046861	1.009310	0.968985	0.928327	0.890523	0.856329	0.823289
	0.95	$\emptyset_1$	1.095457	1.054884	1.011282	0.967227	0.926159	0.888948	0.852965
		$\emptyset_2$	1.079950	1.042870	1.002253	0.960762	0.921805	0.886209	0.851373
		$\emptyset_3$	1.105955	1.060143	1.013077	0.967175	0.925473	0.888260	0.852463
		$\emptyset_4$	1.084666	1.046416	1.005006	0.962875	0.923317	0.887182	0.851930
	1.00	$\emptyset_1$	1.149325	1.107357	1.062206	1.016428	0.973621	0.934792	0.897180
		$\emptyset_2$	1.125256	1.088493	1.047796	1.005840	0.966215	0.929888	0.894130
		$\emptyset_3$	1.158096	1.110995	1.062423	1.014903	0.971682	0.933138	0.895986
		$\emptyset_4$	1.132602	1.094297	1.052422	1.009463	0.968896	0.931717	0.895273

			ξ_2						
			0.70	0.75	0.80	0.85	0.90	0.95	1.00
ξ_1	0.55	**$Ø_1$**	0.873175	0.865609	0.863431	0.871285	0.892090	0.926159	0.973621
		$Ø_2$	0.873353	0.865727	0.863250	0.870388	0.889859	0.921805	0.966215
		$Ø_3$	0.873231	0.865756	0.863691	0.871565	0.892109	0.925473	0.971682
		$Ø_4$	0.872924	0.865258	0.862939	0.870455	0.890523	0.923317	0.968896
	0.60	**$Ø_1$**	0.850655	0.840115	0.834669	0.839309	0.857241	0.888948	0.934792
		$Ø_2$	0.850727	0.840189	0.834610	0.838845	0.855936	0.886209	0.929888
		$Ø_3$	0.850663	0.840149	0.834740	0.839361	0.857086	0.888260	0.933138
		$Ø_4$	0.850570	0.839967	0.834435	0.838869	0.856329	0.887182	0.931717
	0.65	**$Ø_1$**	0.831363	0.817224	0.807976	0.808998	0.823782	0.852965	0.897180
		$Ø_2$	0.831379	0.817258	0.807970	0.808793	0.823095	0.851373	0.894130
		$Ø_3$	0.831364	0.817229	0.807992	0.808994	0.823643	0.852463	0.895986
		$Ø_4$	0.831347	0.817174	0.807879	0.808786	0.823289	0.851930	0.895273
	0.70	**$Ø_1$**	-	0.794117	0.780528	0.777450	0.788636	0.814823	0.856946
		$Ø_2$	-	0.794125	0.780533	0.777380	0.788325	0.813993	0.855204
		$Ø_3$	-	0.794117	0.780529	0.777440	0.788547	0.814506	0.856162
		$Ø_4$	-	0.794108	0.780498	0.777368	0.788406	0.814276	0.855859
	0.75	**$Ø_1$**	0.794117	-	0.752746	0.745074	0.752053	0.774437	0.813486
		$Ø_2$	0.794125	-	0.752748	0.745059	0.751945	0.774075	0.812611
		$Ø_3$	0.794117	-	0.752746	0.745069	0.752011	0.774268	0.813025
		$Ø_4$	0.794108	-	0.752742	0.745053	0.751972	0.774201	0.812949
	0.80	**$Ø_1$**	0.780528	0.752746	-	0.716725	0.718913	0.736584	0.771353
		$Ø_2$	0.780533	0.752748	-	0.716724	0.718890	0.736465	0.770989
		$Ø_3$	0.780529	0.752746	-	0.716724	0.718902	0.736517	0.771130
		$Ø_4$	0.780498	0.752741	-	0.716723	0.718896	0.736509	0.771139
	0.85	**$Ø_1$**	0.777450	0.745074	0.716725	-	0.695970	0.708163	0.737671
		$Ø_2$	0.777380	0.745059	0.716724	-	0.695968	0.708139	0.737558
		$Ø_3$	0.777440	0.745069	0.716724	-	0.695969	0.708147	0.737593
		$Ø_4$	0.777368	0.745053	0.716723	-	0.695969	0.708149	0.737609
	0.90	**$Ø_1$**	0.788636	0.752053	0.718913	0.695970	-	0.693376	0.716869
		$Ø_2$	0.788325	0.751945	0.718890	0.695968	-	0.693375	0.716847
		$Ø_3$	0.788547	0.752011	0.718902	0.695969	-	0.693375	0.716852
		$Ø_4$	0.788406	0.751972	0.718896	0.695969	-	0.693375	0.716858
	0.95	**$Ø_1$**	0.814823	0.774437	0.736584	0.708163	0.693376	-	0.707168
		$Ø_2$	0.813993	0.774075	0.736465	0.708139	0.693375	-	0.707167
		$Ø_3$	0.814506	0.774268	0.736517	0.708147	0.693375	-	0.707167
		$Ø_4$	0.814278	0.774201	0.736509	0.708149	0.693375	-	0.707167
	1.00	**$Ø_1$**	0.856946	0.813486	0.771353	0.737671	0.716869	0.707168	-
		$Ø_2$	0.855205	0.812611	0.770989	0.737558	0.716847	0.707167	-
		$Ø_3$	0.856162	0.813025	0.771130	0.737593	0.716852	0.707167	-
		$Ø_4$	0.855859	0.812949	0.771139	0.737609	0.716858	0.707168	-

4.2.5 **Free-free beams**

These are beams not restrained from lateral vibration at both ends. From section 3.2.5 the fundamental or lowest natural frequency ω for such a bar is given as

$$\omega_1 = 0 \tag{4.86}$$

Likewise

$$u_1 = u(x_1, 0) = \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j x_1 + e_{2j} \sinh \beta_j x_1 + \cos \beta_j x_1 + e_{4j} \sin \beta_j x_1\right)$$

$$= \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j L\xi_1 + e_{2j} \sinh \beta_j L\xi_1 + \cos \beta_j L\xi_1 + e_{4j} \sin \beta_j L\xi_1\right)$$

(4.87a)

$$u_2 = u'(x_1, 0) = \sum_{j=1}^{\infty} A_j \beta_j L\left(\sinh \beta_j x_1 + e_{2j} \cosh \beta_j x_1 - \sin \beta_j x_1 + e_{4j} \cos \beta_j x_1\right)$$

$$= \sum_{j=1}^{\infty} A_j \beta L\left(\sinh \beta_j L\xi_1 + e_{2j} \cosh \beta_j L\xi_1 - \sin \beta_j L\xi_1 + e_{4j} \cos \beta_j L\xi_1\right)$$

(4.87b)

$$u_3 = u(x_2, 0) = \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j x_2 + e_{2j} \sinh \beta_j x_2 + \cos \beta_j x_2 + e_{4j} \sin \beta_j x_2\right)$$

$$= \sum_{j=1}^{\infty} A_j \left(\cosh \beta_j L\xi_2 + e_{2j} \sinh \beta_j L\xi_2 + \cos \beta_j L\xi_2 + e_{4j} \sin \beta_j L\xi_2\right)$$

(4.87c)

$$u_4 = u'(x_2, 0) = \sum_{j=1}^{\infty} A_j \beta L\left(\sinh \beta_j x_2 + e_{2j} \cosh \beta_j x_2 - \sin \beta_j x_2 + e_{4j} \cos \beta_j x_2\right)$$

$$= \sum_{j=1}^{\infty} A_j \beta_j L\left(\sinh \beta_j L\xi_2 + e_{2j} \cosh \beta_j L\xi_2 - \sin \beta_j L\xi_2 + e_{4j} \cos \beta_j L\xi_2\right)$$

(4.87d)

Equations (4.87a) – (4.87d) are used to evaluate the total displacements u_1 to u_4 at the nodal points of a segment of the vibrating free-free beam. Though the equations represent the summation of an infinite series, an evaluation of the first few terms provides values of very good precision.

From Table 3.6

$$A_j =$$

$$\frac{2\mu d L_1^2}{M_j}\left[\frac{-12(\sinh\beta_j L_1-\sin\beta_j L_1)}{\beta_j^3 L_1^3}+\frac{24(\cosh\beta_j L_1+\cos\beta_j L_1)}{\beta_j^4 L_1^4}-\frac{24(\sinh\beta_j L_1+\sin\beta_j L_1)}{\beta_j^5 L_1^3}-\right.$$

$$\left.e_{2j}\left(\frac{12(\cosh\beta_j L_1+\cos\beta_j L_1)}{\beta_j^3 L_1^3}-\frac{24(\sinh\beta_j L_1+\sin\beta_j L_1)}{\beta_j^4 L_1^4}+\frac{24(\cosh\beta_j L_1-\cos\beta_j L_1)}{\beta_j^5 L^5}\right)\right] \quad (4.88)$$

$$L_1 = {}^{L}/_{2} \quad (4.89)$$

The values of $\beta_j L$, e_{2j}, and e_{4j} for j = 0,1, 2, 3, 4, 5, 6, 7 can be obtained from Table 3.6. The values of the fixed end forces F_1, F_2, F_3 and F_4 are evaluated using the equations provided in Table 3.6. These are substituted into the equations for nodal forces (equations 4.35) – (4.38) from which the nodal forces P_1, P_2, P_3 and P_4 are obtained.

Equations (4.87a) – (4.87d) are evaluated for the first mode, j = 1 to obtain the nodal displacements u_{11}, u_{21}, u_{31} and u_{41} due to the first mode. These together with the calculated nodal forces P_1 to P_4 are substituted into equation (4.49) in order to obtain the stiffness modification factors ϕ_1, ϕ_2, ϕ_3 and ϕ_4.

It is important to note that since the lowest frequency of a free-free beam ($\omega = 0$) correspond to a rigid body motion, it is not possible to evaluate equation (4.49) for a free-free beam when $\omega = 0$. From Table 3.6 we would observe that the lowest frequency, $B_jL = 0$, $\omega = 0$ and A_j is undefined. Hence equation (4.49) can only be evaluated for values of natural frequency w close to zero.

Because this frequency correspond to a rigid body motion, what is of essence is the stiffness modification factor for the full length of the beam (ie when $\xi_1 = 0$ and $\xi_2 = 1$). The calculated values of the stiffness modification factors at different values of natural frequency w are presented in the Table 4.37.

Table 4.37: Calculated values of stiffness modification factors for a Free-free beam at different values of natural frequency

Natural frequency ω (Hz)	$\boldsymbol{\emptyset_1}$	$\boldsymbol{\emptyset_2}$	$\boldsymbol{\emptyset_3}$	$\boldsymbol{\emptyset_4}$
0.5	5.000681	4.998074	5.000678	5.002418
0.4	4.993327	4.992262	4.993328	4.994038
0.3	4.974148	4.973813	4.974150	4.974373
0.2	4.866563	4.866499	4.866565	4.866607
0.1	2.863414	2.863412	2.863416	2.863416
0.05	-29.18560	-29.185598	-29.185599	-25.185601
0.01	-18298.68	-18298.68	-18298.68	-18298.68
0.005	37611.44	37611.44	37611.44	37611.44
0.001	-1668.115	-1668.115	-1668.115	-1668.115

From Table 4.37 it would be observed that the calculated values of stiffness modification factors vary widely with the natural frequency. As the value of ω tend to zero the values of the calculated stiffness modification factors for each value of natural frequency becomes equal. Therefore the calculated value of the stiffness modification factor for a free-free beam will be taken to be a constant.

$$\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_{frfr} \tag{4.90}$$

4.2.6 Pinned-free beams

These are beams pinned at one end and completely unrestrained at the other. From section 3.2.6 the fundamental or lowest natural frequency ω for such a bar is given as

$$\omega_1 = 0 \tag{4.91}$$

Likewise

$$u_1 = u(x_1, 0) = \sum_{j=1}^{\infty} A_j \left(\sinh \beta_j x_1 + f_{4j} \sin \beta_j x_1\right)$$

$$= \sum_{j=1}^{\infty} A_j \left(\sinh \beta_j L\xi_1 + f_{4j} \sin \beta_j L\xi_1\right) \tag{4.92a}$$

$$u_2 = u'(x_1, 0) = \sum_{j=1}^{\infty} A_j \beta_j L\left(\cosh \beta_j x_1 + f_{4j} \cos \beta_j x_1\right)$$

$$= \sum_{j=1}^{\infty} A_j \beta L\left(\cosh \beta_j L\xi_1 + f_{4j} \cos \beta_j L\xi_1\right) \tag{4.92b}$$

$$u_3 = u(x_2, 0) = \sum_{j=1}^{\infty} A_j \left(\sinh \beta_j x_2 + f_{4j} \sin \beta_j x_2\right)$$

$$= \sum_{j=1}^{\infty} A_j \left(\sinh \beta_j L\xi_2 + f_{4j} \sin \beta_j L\xi_2\right) \tag{4.92c}$$

$$u_4 = u'(x_2, 0) = \sum_{j=1}^{\infty} A_j \beta L\left(\cosh \beta_j x_2 + f_{4j} \cos \beta_j x_2\right)$$

$$= \sum_{j=1}^{\infty} A_j \beta_j L\left(\cosh \beta_j L\xi_2 + f_{4j} \cos \beta_j L\xi_2\right) \tag{4.92d}$$

Equations (4.92a) – (4.92d) are used to evaluate the total displacements u_1 to u_4 at the nodal points of a segment of the vibrating pinned-free beam.

From Table 3.8

$$A_j = \frac{d\mu L_1^2}{M_j}\left[\frac{-10\cosh\ 2\beta_j L_1}{\beta_j L_1} + \frac{19\sinh\ 2\beta_j L_1}{\beta_j^2 L_1^2} - \frac{24\cosh\ 2\beta_j L_1}{\beta_j^3 L_1^3} + \frac{24\sinh\ 2\beta_j L_1}{\beta_j^4 L_1^4} - \right.$$

$$\frac{24\cosh\ 2\beta_j L_1}{\beta_j^5 L_1^5} + f_{4j}\left(\frac{10\cos\ 2\beta_j L_1}{\beta_j L_1} - \frac{19\sin\ 2\beta_j L_1}{\beta_j^2 L_1^2} + \frac{24\cos\ 2\beta_j L_1}{\beta_j^3 L_1^3} + \frac{24\sin\ 2\beta_j L_1}{\beta_j^4 L_1^4} + \right.$$

$$\left.\left.\frac{24\cosh\ 2\beta_j L_1}{\beta_j^5 L_1^5}\right)\right] - \frac{d\mu L_1^2}{M_j}\left[\frac{24}{\beta_j^3 L_1^3} + \frac{24\sinh\ \beta_j L_1}{\beta_j^4 L_1^4} + f_{4j}\left(\frac{-6\sin\ \beta_j L_1}{\beta_j^2 L_1^2} + \frac{24}{\beta_j^3 L_1^3} - \frac{24\sin\ \beta_j L_1}{\beta_j^4 L_1^4}\right)\right] \quad (4.93)$$

Note $L_1 = {}^{L}/_{2}$

The values of $\beta_j L$, f_{2j}, and f_{4j} for j = 0,1, 2, 3, 4, 5, 6, 7 can be obtained from Table 3.8.

The values of the fixed end forces F_1, F_2, F_3 and F_4 are evaluated using the equations provided in Table 3.7. These are substituted into the equations for nodal forces (equations (4.35) – (4.38)) from which the nodal forces P_1, P_2, P_3 and P_4 are obtained. Just as in the fixed-fixed beam, equations (4.92a) – (4.92d) are evaluated for the first mode, j = 0 to obtain the nodal displacements u_{11}, u_{21}, u_{31} and u_{41} due to the first mode. These together with the calculated nodal forces P_1 to P_4 are substituted into equation (4.49) in order to obtain the stiffness modification factors ϕ_1, ϕ_2, ϕ_3 and ϕ_4.

The lowest frequency of a pinned-free beam (ω = 0) corresponds to a rigid body motion. Just as in the case of a free-free beam it is not possible to evaluate equation (4.49) for a pinned-free beam when ω = 0. We would observe that for the first mode of vibration, $B_j L$ = 0, ω = 0 and A_j is undefined. Hence equation (4.49) can only be evaluated for values of natural frequency w close to zero.

Because this frequency correspond to a rigid body motion, what is of essence is the stiffness modification factor for the full length of the beam (i.e. when $\xi_1 = 0$ and

$\xi_2 = 1$). The calculated values of the stiffness modification factors at different values of natural frequency w are presented in the Table 4.38.

Table 4.38: Calculated values of stiffness modification factors for a Pinned-free beam at different values of natural frequency

Natural frequency (Hz)	**$Ø_1$**	**$Ø_2$**	**$Ø_3$**	**$Ø_4$**
0.5	41.027407	41.012268	41.046107	41.047888
0.4	41.006295	41.000103	41.013965	41.014684
0.3	40.996305	40.994347	40.998733	40.998959
0.2	40.992598	40.992211	40.993078	40.993122
0.1	40.991743	40.991719	40.991773	40.991776
0.05	40.991688	40.991688	40.991688	40.991688
0.01	40.991730	40.991730	40.991730	40.991730
0.005	40.990614	40.990614	40.990614	40.990614

From Table 4.38 it would be observed that as the value of ω tends to zero the values of the calculated stiffness modification factors for each value of natural frequency becomes equal. Therefore the calculated value of the stiffness modification factor for a free-free beam will be taken to be equal to 40.991.

$$\phi_1 = \phi_2 = \phi_3 = \phi_4 = 40.991 \qquad (4.94)$$

Chapter 5

ANALYSIS AND DISCUSSION OF RESULTS

In the previous chapter the fixed end forces were used to calculate the stiffness modification factors for beams of different end conditions and for both longitudinal and lateral vibrations. The modification factors will now be applied in the analysis of the free vibration of lumped-massed beams using Lagrange equation. This will enable us see the effect of the stiffness modification factors on the calculated results. In order to carry out this comparative study some set of beams were selected and are presented below.

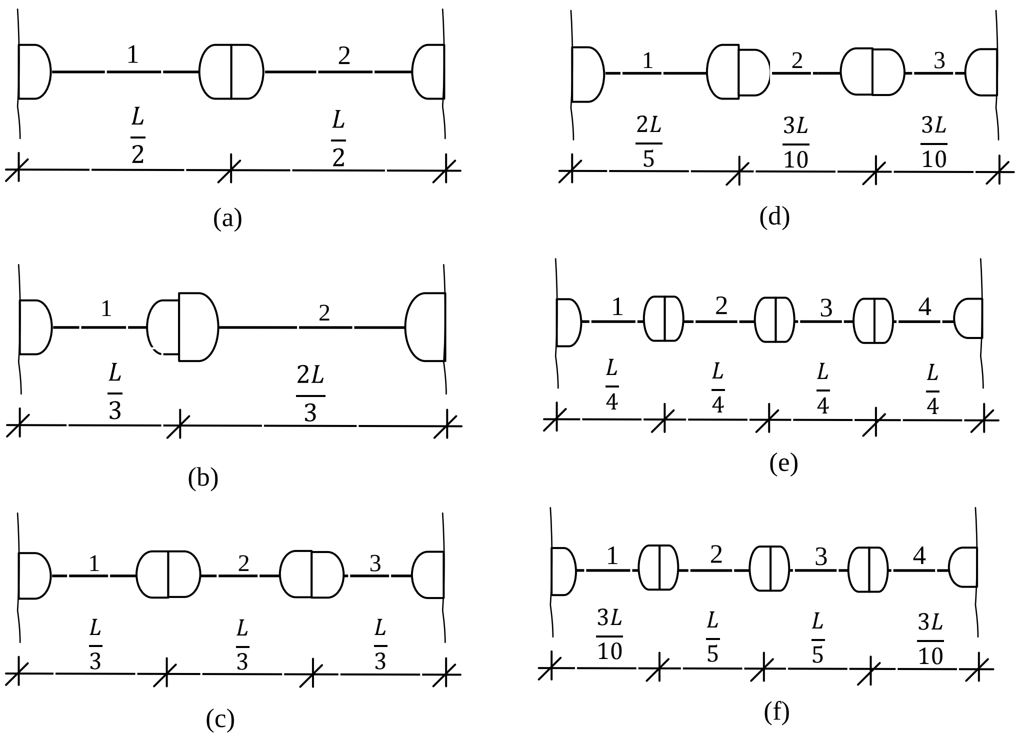


Figure 5.1: Some lumped massed beams constrained at both ends used for illustration of Lagrange equation

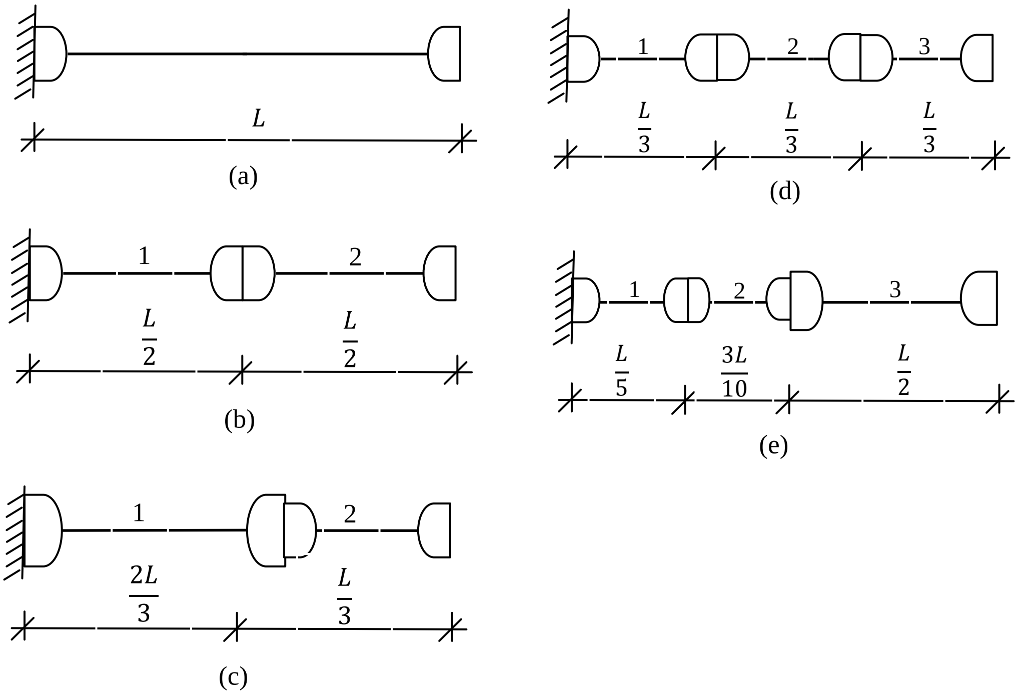


Figure 5.2: Some lumped massed beams fixed at one end and free at the other used for illustration of Lagrange equation

Some of the beams of Figure 5.1 and Figure 5.2 have an even distribution of lumped masses while the others have uneven distribution of lumped masses. The natural and mode shapes of these beams will be analyzed twice using Lagrange equations. In the first analysis the system will be analyzed using the ordinary stiffness of the structure while in the second analysis the stiffness modification factors will be applied. The results of the two analyses will then be compared.

5.1 Longitudinal Vibrations

The analysis of systems under longitudinal vibrations using the Lagrange equations was presented in section 2.7.1. From equation (2.37) the dynamical matrix is the product of the flexibility matrix and the inertia matrix which is the same as the product of the stiffness inverse and the inertia matrix.

$$[D] = [k]^{-1}[m] \tag{5.1}$$

where [D] is the dynamical matrix, [k] is the stiffness matrix and [m] is the inertia matrix. By substituting equation (5.1) into the eigenvalue problem of equation (2.39) we obtain

$$([k]^{-1}[m] - \lambda[I])\{\phi\} = 0 \tag{5.2}$$

Equation (5.2) is evaluated to obtain the natural frequencies $\omega = \sqrt{\frac{1}{\lambda}}$ and mode shapes $\{\phi\}$.

The two cases of fixed-fixed bars and fixed-free bars will be considered.

5.1.1 Fixed-fixed/Fixed-pinned/pinned-pinned bars

For the beam of Figure 5.1a (if restrained at both ends) the stiffness matrix and inertia matrix of the bar with respect to the coordinate of the lumped mass are

$$k = \frac{4EA}{L} \tag{5.3a}$$

$$m = \frac{1}{2}\mu L \tag{5.3b}$$

By substituting equations (5.3a) and (5.3b) into equation (5.2) and solving we obtain

$$\lambda = \frac{0.125\mu L^2}{EA} \tag{5.4a}$$

$$\{\phi\} = 1 \tag{5.4b}$$

$$\omega = 2.8284\sqrt{\frac{EA}{\mu L^2}} \tag{5.4c}$$

From Table 4.7 the stiffness modification factors of the two segments/elements of the bar are

For element 1: $\xi_1 = 0$, $\xi_2 = 0.5$, $\alpha_1 = 1.233701$, $\alpha_2 = 1.859611$

For element2: $\xi_1 = 0.5$, $\xi_2 = 1$, $\alpha_1 = 1.233701$, $\alpha_2 = 1.859611$

By applying these stiffness modification factors, the modified stiffness matrix of the bar with respect to the coordinate of the lumped mass becomes

$$k = \frac{4.934804\,EA}{L} \tag{5.5}$$

By using this modified stiffness on equation (5.2) the new values of λ, natural frequency and mode shape obtained are

$$\lambda = \frac{0.1013211467\ \mu L^2}{EA} \tag{5.6a}$$

$$\{\phi\} = 1 \tag{5.6b}$$

$$\omega = 3.14159\sqrt{\frac{EA}{\mu L^2}} \tag{5.6c}$$

For the beam of Figure 5.1b (if restrained at both ends) the stiffness matrix and inertia matrix of the bar with respect to the coordinate of the lumped mass are

$$k = \frac{4.5EA}{L} \tag{5.7a}$$

$$m = \frac{1}{2}\mu L \tag{5.7b}$$

By substituting equations (5.3a) and (5.3b) into equation (5.2) and solving we obtain

$$\lambda = \frac{0.1111111111\ \mu L^2}{EA} \tag{5.8a}$$

$$\{\phi\} = 1 \tag{5.8b}$$

$$\omega = 3.0000\sqrt{\frac{EA}{\mu L^2}} \tag{5.8c}$$

The stiffness modification factors of the two segments/elements of the bar cannot be obtained with precision from Table 4.7 because there was no provision for $\xi_1 = 1/3$. The modification factors can be obtained for any of ξ_1 and ξ_2 using the attached matlab program in Appendix A. Using this matlab program program the stiffness modification factors of the two segments/elements of the bar were obtained as

For element 1: $\xi_1 = 0$, $\xi_2 = 1/3$, $\alpha_1 = 1.040150$, $\alpha_2 = 1.143529$

For element2: $\xi_1 = 1/3$, $\xi_2 = 1$, $\alpha_1 = 1.209568$, $\alpha_2 = 2.863058$

By applying these stiffness modification factors, the modified stiffness matrix of the bar with respect to the coordinate of the lumped mass becomes

$$k = \frac{4.934802\,EA}{L} \tag{5.5}$$

By using this modified stiffness on equation (5.2) the new values of λ, natural frequency and mode shape obtained are

$$\lambda = \frac{0.10132118776\,\mu L^2}{EA} \tag{5.6a}$$

$$\{\phi\} = 1 \tag{5.6b}$$

$$\omega = 3.1416\sqrt{\frac{EA}{\mu L^2}} \tag{5.6c}$$

These were repeated for the bars of Figures 5.1(c), 5.1(d), 5.1(e) and 5.1(f) and a summary of the obtained natural frequencies presented in Table 5.1

Table 5.1: Comparism of the obtained natural frequencies of different lump-massed fixed-fixed bar under longitudinal vibration with the exact results.

	Mode No	Hamilton (Exact) (Hz)	Lagrange (Hz)	Percentage Error (%)	Lagrange with modified stiffness (Hz)	Percentage Error (%)
Figure 5.1(a)	1	3.1416	2.8284	9.97	3.1416	0
Figure 5.1(b)	1	3.1416	3.0000	4.51	3.1416	0
Figure 5.1(c)	1	3.1416	3.0000	4.51	3.1416	0
	2	6.2832	5.1962	17.30	5.4486	13.28
Figure 5.1(d)	1	3.1416	2.9646	5.63	3.1416	0
	2	6.2832	5.4863	12.68	5.6096	10.72
Figure 5.1(e)	1	3.1416	3.0615	2.55	3.1416	0
	2	6.2832	5.6569	9.97	5.6593	9.93
	3	9.4248	7.3910	21.58	7.3611	21.90
Figure 5.1(f)	1	3.1416	2.9940	4.70	3.1416	0
	2	6.2832	5.7735	8.11	5.6649	9.84
	3	9.4248	8.6237	8.50	8.2736	12.21

From Table 5.1, it would be observed that the natural frequencies obtained from the use of Lagrange equation on the continuous system had some measure of errors as seen from its comparison with exact results (results from the use of Hamilton's principle). However when the stiffness of the system was modified using the stiffness modification factors, the use of Lagrange equation was able to predict accurately the fundamental frequencies hence their percentage errors were zero. However the values of the higher frequencies remained approximate.

5.1.2 **Fixed-free bars**

For the beam of Figure 5.2a the stiffness matrix and inertia matrix of the bar with respect to the coordinate of the lumped mass are

$$k = \frac{EA}{L} \tag{5.7a}$$

$$m = \frac{1}{2}\mu L \tag{5.7b}$$

By substituting equations (5.7a) and (5.7b) into equation (5.2) and solving we obtain

$$\lambda = \frac{0.5\mu L^2}{EA} \tag{5.8a}$$

$$\{\phi\} = 1 \tag{5.8b}$$

$$\omega = 1.4142\sqrt{\frac{EA}{\mu L^2}} \tag{5.8c}$$

From Table 4.13 the stiffness modification factors of the element of the bar are

$\xi_1 = 0$, $\xi_2 = 1$, $\alpha_1 = 1.233701$, $\alpha_2 = 1.894303$

By applying these stiffness modification factors, the modified stiffness matrix of the bar with respect to the coordinate of the lumped mass becomes

$$k = \frac{1.233701\,EA}{L} \tag{5.5}$$

By using this modified stiffness on equation (5.2) the new values of λ, natural frequency and mode shape obtained are

$$\lambda = \frac{0.405284586784\ \mu L^2}{EA} \tag{5.9a}$$

$$\{\phi\} = 1 \tag{5.9b}$$

$$\omega = 1.5708\sqrt{\frac{EA}{\mu L^2}} \tag{5.9c}$$

For the beam of Figure 5.2b the stiffness matrix and inertia matrix of the bar with respect to the coordinate of the lumped mass are

$$k = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}\frac{EA}{L} \tag{5.10a}$$

$$m = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\mu L \tag{5.10b}$$

By substituting equations (5.10a) and (5.10b) into equation (5.2) and solving we obtain

$$\lambda_1 = \frac{0.42677669529664\ \mu L^2}{EA}, \lambda_2 = \frac{0.07322330470336\ \mu L^2}{EA} \tag{5.11a}$$

$$\{\phi_1\} = \begin{Bmatrix} 0.5773503 \\ 0.8164966 \end{Bmatrix}, \{\phi_2\} = \begin{Bmatrix} -0.5773503 \\ 0.8164966 \end{Bmatrix} \tag{5.11b}$$

$$\omega_1 = 1.5307\sqrt{\frac{EA}{\mu L^2}}\ ,\ \omega_2 = 3.6955\sqrt{\frac{EA}{\mu L^2}} \tag{5.11c}$$

The stiffness modification factors of the two segments/elements of the bar can be obtained from Table 4.13 as

For element 1: $\xi_1 = 0$, $\xi_2 = 0.5$, $\alpha_1 = 0.995936$, $\alpha_2 = 1.339475$

For element2: $\xi_1 = 0.5$, $\xi_2 = 1$, $\alpha_1 = 0.995936$, $\alpha_2 = 0.972287$

By applying these stiffness modification factors, the modified stiffness matrix of the bar with respect to the coordinate of the lumped mass becomes

$$k = \begin{bmatrix} 3.983744 & -1.944574 \\ -1.944574 & 1.991872 \end{bmatrix}\frac{EA}{L} \tag{5.12}$$

By using this modified stiffness on equation (5.2) the new values of λ, natural frequency and mode shape obtained are

$$\lambda_1 = \frac{0.40528456176396\ \mu L^2}{EA}, \lambda_2 = \frac{0.07425242373351\ \mu L^2}{EA} \qquad (5.13a)$$

$$\{\phi_1\} = \begin{Bmatrix} 0.5773503 \\ 0.8164966 \end{Bmatrix}, \{\phi_2\} = \begin{Bmatrix} -0.5773503 \\ 0.8164966 \end{Bmatrix}. \qquad (5.13b)$$

$$\omega_1 = 1.5708\sqrt{\frac{EA}{\mu L^2}}\ ,\ \omega_2 = 3.6698\sqrt{\frac{EA}{\mu L^2}} \qquad (5.13c)$$

These were repeated for the bars of Figures 5.2(c), 5.2(d), 5.2(e) and 5.2(f) and a summary of the obtained natural frequencies presented in Table 5.2.

Table 5.2: Comparison of the obtained natural frequencies of different lump-massed fixed-free bar under longitudinal vibration with the exact results.

	Mode No	Hamilton (Exact) (Hz)	Lagrange (Hz)	Percentage Error (%)	Lagrange with modified stiffness (Hz)	Percentage Error (%)
Figure 5.2 (a)	1	1.5708	1.4142	9.95	1.5708	0
Figure 5.2 (b)	1	1.5708	1.5307	2.55	1.5708	0
	2	4.7124	3.6955	21.43	3.6698	22.12
Figure 5.2 (c)	1	1.5708	1.4749	6.11	1.5708	0
	2	4.7124	4.9824	-5.73	4.6750	0.79
Figure 5.2 (d)	1	1.5708	1.5529	1.14	1.5708	0
	2	4.7124	4.2426	9.97	4.1054	12.88
	3	7.8540	5.7956	26.21	5.5171	29.75
Figure 5.2 (e)	1	1.5708	1.5755	-0.30	1.5708	0
	2	4.7124	3.7558	20.30	3.7100	21.27
	3	7.8540	6.1708	21.43	6.2305	20.67

From Table 5.2, it would be observed that the natural frequencies obtained from the use of Lagrange equation on the continuous system had some measure of errors as seen in the percentage error column. But when the stiffness of the system was modified using the stiffness modification factors, the use of Lagrange equation was able to predict accurately the fundamental frequencies hence their percentage errors were zero. However the values of the higher frequencies remained approximate. Some of its predictions for higher frequencies where less accurate than that obtained without the application of the stiffness modification factors.

5.2 Lateral Vibrations

The analysis of systems under lateral vibrations using the lagrange equations is the same as that for longitudinal vibration and was presented in section 2.7.1. The only different is that the stiffness matrix [k] will now be for a beam element. For a beam element the stiffness matrix is a 6 x 6 matrix but when axial deformation/vibration is ignored it reduces to a 4 x 4 matrix. Just like in the analysis for longitudinal vibration, the natural frequency and mode shapes for the lateral vibration of a beam is obtained by evaluating equation (5.2).

5.2.1 Fixed-fixed beams

The stiffness matrix for beams with respect to any set of arbitrary coordinate is obtained by the by implementing the matrix calculations presented in section 2.6.2.

For the beam of Figure 5.1a (if restrained at both ends) the stiffness matrix and inertia matrix of the bar with respect to the coordinate of the lumped mass are

$$k = \frac{192EI}{L^3} \tag{5.14a}$$

$$m = \frac{1}{2}\mu L \tag{5.14b}$$

By substituting equations (5.14a) and (5.14b) into equation (5.2) and solving we obtain

$$\lambda = \frac{0.00182898948331\ \mu L^4}{EI} \tag{5.15a}$$

$$\{\phi\} = 1 \tag{5.15b}$$

$$\omega = 19.5959\sqrt{\frac{EI}{\mu L^4}} \tag{5.15c}$$

From Table 4.18 the stiffness modification factors of the two segments/elements of the beam are

For element 1: $\xi_1 = 0$, $\xi_2 = 0.5$, $\phi_1 = 1.303552$, $\phi_2 = 0.663259$, $\phi_3 = 1.731522$, $\phi_4 = 1.293649$

For element2: $\xi_1 = 0.5$, $\xi_2 = 1$, $\phi_1 = 1.303552$, $\phi_2 = 0.663259$, $\phi_3 = 1.731522$, $\phi_4 = 1.293649$

By applying these stiffness modification factors, the modified stiffness matrix of the bar with respect to the coordinate of the lumped mass becomes

$$k = \frac{250.281984\,EI}{L^3} \tag{5.16}$$

By using this modified stiffness on equation (5.2) the new values of λ, natural frequency and mode shape obtained are

$$\lambda = \frac{0.00199774666961\ \mu L^4}{EI} \tag{5.17a}$$

$$\{\phi\} = 1 \tag{5.17b}$$

$$\omega = 22.3733\sqrt{\frac{EI}{\mu L^4}} \tag{5.17c}$$

For the beam of Figure 5.1b (if restrained at both ends) the stiffness matrix (using the method of section 2.6.2) and inertia matrix of the bar with respect to the coordinate of the lumped mass are

$$k = \frac{2187EI}{8L^3} \tag{5.18a}$$

$$m = \frac{1}{2}\mu L \tag{5.18b}$$

By substituting equations (5.3a) and (5.3b) into equation (5.2) and solving we obtain

$$\lambda = \frac{0.00182898948331\ \mu L^4}{EI} \tag{5.19a}$$

$$\{\phi\} = 1 \tag{5.19b}$$

$$\omega = 23.3827\sqrt{\frac{EI}{\mu L^4}} \tag{5.19c}$$

The stiffness modification factors of the two segments/elements of the beam cannot be obtained with precision from Table 4.18 because there was no provision for ξ_1 = 1/3. The modification factors can however be obtained for any value of ξ_1 and ξ_2 using the attached matlab program in Appendix C. Using this matlab program program the stiffness modification factors of the two segments/elements of the bar were obtained as

For element 1: $\xi_1 = 0$, $\xi_2 = 1/3$, $\phi_1 = 1.017826$, $\phi_2 = 0.968328$, $\phi_3 = 1.294663$, $\phi_4 = 1.197040$

For element2: $\xi_1 = 1/3$, $\xi_2 = 1$, $\phi_1 = -2.180818$, $\phi_2 = 4.078733$, $\phi_3 = 8.584341$, $\phi_4 = -3.117611$

By applying these stiffness modification factors, the modified stiffness matrix of the bar with respect to the coordinate of the lumped mass becomes

$$k = \frac{250.282095508\, EI}{L^3} \tag{5.20}$$

By using this modified stiffness on equation (5.2) the new values of λ, natural frequency and mode shape obtained are

$$\lambda = \frac{0.00199774577955\, \mu L^4}{EI} \tag{5.21a}$$

$$\{\phi\} = 1 \tag{5.21b}$$

$$\omega = 22.3733\sqrt{\frac{EI}{\mu L^4}} \tag{5.21c}$$

These were repeated for the bars of Figures 5.1 (c), 5.1(d), 5.1(e) and 5.1(f) and a summary of the obtained natural frequencies presented in Table 5.3.

Table 5.3: Comparison of the obtained natural frequencies of different lump-massed fixed-fixed beam under lateral vibration with the exact results.

	Mode No	Hamilton (Exact) (Hz)	Lagrange (Hz)	Percentage Error (%)	Lagrange with modified stiffness (Hz)	Percentage Error (%)
Figure 5.1(a)	1	22.3733	19.5959	12.41	22.3733	0
Figure 5.1 (b)	1	22.3733	23.3827	-4.51	22.3733	0
Figure 5.1 (c)	1	22.3733	22.0454	1.47	22.3733	0
	2	61.6728	51.2289	16.93	51.8139	15.99
Figure 5.1 (d)	1	22.3733	21.2230	5.14	22.3733	0
	2	61.6728	56.0138	9.18	56.1097	9.02
Figure 5.1 (e)	1	22.3733	22.3024	0.32	22.3733	0

	2	61.6728	59.2525	3.92	59.0436	4.26
	3	120.9034	97.3992	19.44	96.5191	20.17
Figure 5.1 (f)	1	22.3733	21.2654	4.95	22.3733	0
	2	61.6728	57.1662	7.31	56.2869	8.73
	3	120.9034	121.4176	-0.42	116.9119	3.30

Just like in the longitudinal vibration of beams from Table 5.3, it would be observed that the natural frequencies obtained from the use of Lagrange equation on the continuous system had some measure of errors as seen from its comparison with exact results (results from the use of Hamilton's principle). However when the stiffness of the system was modified using the stiffness modification factors, the use of Lagrange equation was able to predict accurately the fundamental frequencies hence their percentage errors were zero. However the values of the higher frequencies remained approximate.

5.2.2 Fixed-pinned beams

For the beam of Figure 5.1a (if rigidly fixed at the left side and pinned at the other) the stiffness matrix and inertia matrix of the bar with respect to the coordinate of the lumped mass are

$$k = \frac{768EI}{7L^3} \tag{5.22a}$$

$$m = \frac{1}{2}\mu L \tag{5.22b}$$

By substituting equations (5.22a) and (5.22b) into equation (5.2) and solving we obtain

$$\lambda = \frac{0.00455729\,\mu L^4}{EI} \tag{5.23a}$$

$$\{\phi\} = 1 \tag{5.23b}$$

$$\omega = 14.8131\sqrt{\frac{EI}{\mu L^4}} \tag{5.23c}$$

From Table 4.24 the stiffness modification factors of the two segments/elements of the beam are

For element 1: $\xi_1 = 0$, $\xi_2 = 0.5$, $\phi_1 = 1.115022$, $\phi_2 = 0.902300$, $\phi_3 = 1.437344$, $\phi_4 = 1.246787$

For element2: $\xi_1 = 0.5$, $\xi_2 = 1$, $\phi_1 = 1.310192$, $\phi_2 = 0.292258$, $\phi_3 = 2.185060$, $\phi_4 = 1$

By applying these stiffness modification factors, the modified stiffness matrix of the bar with respect to the coordinate of the lumped mass becomes

$$k = \frac{5111EI}{43L^3} \tag{5.24}$$

By using this modified stiffness on equation (5.2) the new values of λ, natural frequency and mode shape obtained are

$$\lambda = \frac{0.00420661120098\ \mu L^4}{EI} \tag{5.25a}$$

$$\{\phi\} = 1 \tag{5.25b}$$

$$\omega = 15.4182\sqrt{\frac{EI}{\mu L^4}} \tag{5.25c}$$

For the beam of Figure 5.1b (if rigidly fixed at the left side and pinned at the other) the stiffness matrix and inertia matrix of the bar with respect to the coordinate of the lumped mass are

$$k = \frac{2187EI}{11L^3} \tag{5.26a}$$

$$m = \frac{1}{2}\mu L \tag{5.26b}$$

By substituting equations (5.3a) and (5.3b) into equation (5.2) and solving we obtain

$$\lambda = \frac{0.00251486053955\ \mu L^3}{EI} \tag{5.27a}$$

$$\{\phi\} = 1 \tag{5.27b}$$

$$\omega = 19.9408\sqrt{\frac{EI}{\mu L^3}} \tag{5.27c}$$

The stiffness modification factors of the two segments/elements of the beam cannot be obtained with precision from Table 4.24 because there was no provision for ξ_1 = 1/3. The modification factors can however be obtained for any value of ξ_1 and ξ_2 using the attached matlab program in Appendix D. Using this matlab program program the stiffness modification factors of the two segments/elements of the bar were obtained as

For element 1: $\xi_1 = 0$, $\xi_2 = 1/3$, $\phi_1 = 1.022661$, $\phi_2 = 1.015457$, $\phi_3 = 1.242908$, $\phi_4 =$ 1.176598

For element2: $\xi_1 = 1/3$, $\xi_2 = 1$, $\phi_1 = 0.573564$, $\phi_2 = -0.312006$, $\phi_3 = 7.004386$, $\phi_4 = 1$

By applying these stiffness modification factors, the modified stiffness matrix of the bar with respect to the coordinate of the lumped mass becomes

$$k = \frac{1.18860132\ EI}{L^3} \tag{5.28}$$

By using this modified stiffness on equation (5.2) the new values of λ, natural frequency and mode shape obtained are

$$\lambda = \frac{0.00420662496832\ \mu L^3}{EI} \tag{5.29a}$$

$$\{\phi\} = 1 \tag{5.29b}$$

$$\omega = 15.4182\sqrt{\frac{EI}{\mu L^4}} \tag{5.29c}$$

These were repeated for the bars of Figures 5.1(c), 5.1(d), 5.1(e) and 5.1(f) and a summary of the obtained natural frequencies presented in Table 5.4.

Table 5.4: Comparison of the obtained natural frequencies of different lump-massed fixed-pinned beam under lateral vibration with the exact results.

	Mode No	Hamilton (Exact) (Hz)	Lagrange (Hz)	Percentage Error (%)	Lagrange with modified stiffness (Hz)	Percentage Error (%)
Figure 5.1(a)	1	15.4182	14.8131	3.92	15.4182	0
Figure 5.1 (b)	1	15.4182	19.9408	-29.33	15.4182	0
Figure 5.1 (c)	1	15.4182	15.3490	0.45	15.4182	0
	2	49.9648	45.6317	8.67	45.6569	8.62
Figure 5.1 (d)	1	15.4182	14.8844	3.46	15.4182	0
	2	49.9648	47.0247	5.88	47.3280	5.28
Figure 5.1 (e)	1	15.4182	15.4017	0.11	15.4182	0
	2	49.9648	49.0541	1.82	43.3764	13.19
	3	104.2477	91.5297	12.20	80.4687	22.81
Figure 5.1 (f)	1	15.4182	14.9382	3.11	15.4182	0
	2	49.9648	49.4204	1.09	48.4931	2.95
	3	104.2477	160.0251	-53.50	114.7123	-10.04

Table 5.3 shows the natural frequencies obtained by Hamilton's principle, the lagrange equations and the Lagrange equations with modified stiffness, it would be observed that the natural frequencies obtained from the use of Lagrange equation on

the continuous system had some measure of errors as seen from its comparison with exact results (results from the use of Hamilton's principle). However when the stiffness of the system was modified using the stiffness modification factors, the use of Lagrange equation was able to predict accurately the fundamental frequencies hence their percentage errors were zero. The modification factors generally did not improve the values of the higher frequencies hence they remained approximate.

5.2.3 Pinned-pinned beams

For the beam of Figure 5.1a (if both ends are pinned) the stiffness matrix and inertia matrix of the bar with respect to the coordinate of the lumped mass are

$$k = \frac{48EI}{L^3} \tag{5.30a}$$

$$m = \frac{1}{2}\mu L \tag{5.30b}$$

By substituting equations (5.30a) and (5.30b) into equation (5.2) and solving we obtain

$$\lambda = \frac{0.01041666667\ \mu L^4}{EI} \tag{5.31a}$$

$$\{\phi\} = 1 \tag{5.31b}$$

$$\omega = 9.7980\sqrt{\frac{EI}{\mu L^4}} \tag{5.31c}$$

From Table 4.30 the stiffness modification factors of the two segments/elements of the beam are

For element 1: $\xi_1 = 0$, $\xi_2 = 0.5$, $\phi_1 = 1.014678$, $\phi_2 = 0.796187$, $\phi_3 = 1.513519$, $\phi_4 = 1.0$

For element2: $\xi_1 = 0.5$, $\xi_2 = 1$, $\phi_1 = 1.014678$, $\phi_2 = 0.796187$, $\phi_3 = 1.513519$, $\phi_4 = 1.0$

By applying these stiffness modification factors, the modified stiffness matrix of the bar with respect to the coordinate of the lumped mass becomes

$$k = \frac{48.704544\,EI}{L^3} \tag{5.32}$$

By using this modified stiffness on equation (5.2) the new values of λ, natural frequency and mode shape obtained are

$$\lambda = \frac{0.0102659825744\ \mu L^4}{EI} \tag{5.33a}$$

$$\{\phi\} = 1 \tag{5.33b}$$

$$\omega = 9.8696\sqrt{\frac{EI}{\mu L^4}} \tag{5.33c}$$

For the beam of Figure 5.1b (if pinned at both ends) the stiffness matrix and inertia matrix of the bar with respect to the coordinate of the lumped mass are

$$k = \frac{60.75EI}{L^3} \tag{5.34a}$$

$$m = \frac{1}{2}\mu L \tag{5.34b}$$

By substituting equations (5.3a) and (5.3b) into equation (5.2) and solving we obtain

$$\lambda = \frac{0.00823045267490\ \mu L^4}{EI} \tag{5.35a}$$

$$\{\phi\} = 1 \tag{5.35b}$$

$$\omega = 11.0227\sqrt{\frac{EI}{\mu L^4}} \tag{5.35c}$$

The stiffness modification factors of the two segments/elements of the beam cannot be obtained with precision from Table 4.30 because there was no provision for ξ_1 =

1/3. The modification factors can however be obtained for any value of ξ_1 and ξ_2 using the attached matlab program in Appendix E. Using this matlab program program the stiffness modification factors of the two segments/elements of the bar were obtained as

For element 1: $\xi_1 = 0$, $\xi_2 = 1/3$, $\phi_1 = 0.954445$, $\phi_2 = 0.940740$, $\phi_3 = 1.108716$, $\phi_4 = 1.0$

For element2: $\xi_1 = 1/3$, $\xi_2 = 1$, $\phi_1 = 0.073842$, $\phi_2 = 1.365440$, $\phi_3 = 2.933404$, $\phi_4 = 1.0$

By applying these stiffness modification factors, the modified stiffness matrix of the beam with respect to the coordinate of the lumped mass becomes

$$k = \frac{48.70449256158826\ EI}{L^3} \tag{5.36}$$

By using this modified stiffness on equation (5.2) the new values of λ, natural frequency and mode shape obtained are

$$\lambda = \frac{0.01026599341668\ \mu L^4}{EI} \tag{5.37a}$$

$$\{\phi\} = 1 \tag{5.37b}$$

$$\omega = 9.8696\sqrt{\frac{EI}{\mu L^4}} \tag{5.37c}$$

These were repeated for the bars of Figures (c), (d), (e) and (f) and a summary of the obtained natural frequencies presented in Table 5.3.

Table 5.5: Comparism of the obtained natural frequencies of different lump-massed pinned-pinned beam under lateral vibration with the exact results.

	Mode No	Hamilton (Exact)	Lagrange (Hz)	Percentage Error	Lagrange with modified	Percentage Error

		(Hz)		(%)	stiffness (Hz)	(%)
Figure 5.1(a)	1	9.8696	9.7980	0.73	9.8696	0
Figure 5.1 (b)	1	9.8696	11.0227	-11.68	9.8696	0
Figure 5.1 (c)	1	9.8696	9.8590	0.11	9.8696	0
	2	39.4784	38.1838	3.28	38.9967	15.99
Figure 5.1 (d)	1	9.8696	9.7308	1.41	9.8696	0
	2	39.4784	41.9185	-6.18	41.1390	9.02
Figure 5.1 (e)	1	9.8696	9.8666	0.03	9.8696	0
	2	39.4784	39.1918	0.73	38.1782	4.26
	3	88.8264	83.2128	6.32	80.0893	20.17
Figure 5.1 (f)	1	9.8696	9.6076	2.65	9.8696	0
	2	39.4784	40.8248	-3.41	39.5599	8.73
	3	88.8264	109.7147	-23.52	104.1448	3.30

From Table 5.3, it would be observed that the natural frequencies obtained from the use of Lagrange equation on the continuous system had some measure of errors as seen from its comparison with exact results. However when the stiffness of the system was modified using the stiffness modification factors, the use of Lagrange equation was able to predict accurately the fundamental frequencies hence their percentage errors were zero.

5.2.4 **Fixed-free beam**

For the beam of Figure 5.2a the stiffness matrix and inertia matrix of the beam with respect to the coordinate of the lumped mass are

$$k = \frac{3EI}{L^3} \tag{5.38a}$$

$$m = \frac{1}{2}\mu L \tag{5.38b}$$

By substituting equations (5.38a) and (5.38b) into equation (5.2) and solving we obtain

$$\lambda = \frac{\mu L^3}{6EI} \ . \tag{5.39a}$$

$$\{\phi\} = 1 \tag{5.39b}$$

$$\omega = 2.4495\sqrt{\frac{EI}{\mu L^2}} \tag{5.39c}$$

From Table 4.36 the stiffness modification factors of the element of the bar are

$\xi_1 = 0$, $\xi_2 = 1.0$, $\phi_1 = 1.398163$, $\phi_2 = 1.283053$, $\phi_3 = 1.566491$, $\phi_4 = 1.374932$

By applying these stiffness modification factors, the modified stiffness matrix of the bar with respect to the coordinate of the lumped mass becomes

$$k = \frac{6.18117650627788 \ EI}{L^3} \tag{5.40}$$

By using this modified stiffness on equation (5.2) the new values of λ, natural frequency and mode shape obtained are

$$\lambda = \frac{0.08089074943778 \ \mu L^4}{EI} \tag{5.41a}$$

$$\{\phi\} = 1 \tag{5.41b}$$

$$\omega = 3.5160\sqrt{\frac{EI}{\mu L^4}} \tag{5.41c}$$

For the beam of Figure 5.2b the stiffness matrix and inertia matrix of the beam with respect to the coordinate of the lumped mass are

$$k = \begin{bmatrix} 109.714285714 & -34.2857142857 \\ -34.2857142857 & 13.714285714 \end{bmatrix} \frac{EI}{L^3} \tag{5.42a}$$

$$m = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \mu L \tag{5.42b}$$

By substituting equations (5.10a) and (5.10b) into equation (5.2) and solving we obtain

$$\lambda_2 = \frac{0.003783237003\ \mu L^4}{EI}, \lambda_1 = \frac{0.100383429664\ \mu L^4}{EI} \tag{5.43a}$$

$$\{\phi_2\} = \begin{Bmatrix} -0.836633177 \\ 0.5477635685 \end{Bmatrix}, \{\phi_1\} = \begin{Bmatrix} -0.3111155645 \\ -0.950372088 \end{Bmatrix} \tag{5.43b}$$

$$\omega_2 = 16.2580\sqrt{\frac{EI}{\mu L^4}}\ ,\ \omega_1 = 3.1562\sqrt{\frac{EI}{\mu L^4}} \quad . \tag{5.43c}$$

The stiffness modification factors of the two segments/elements of the bar can be obtained from Table 4.36 as

For element 1: $\xi_1 = 0$, $\xi_2 = 0.5$, $\phi_1 = 2.215575$, $\phi_2 = 2.408901$, $\phi_3 = 1.256395$, $\phi_4 = 1.200465$

For element2: $\xi_1 = 0.5$, $\xi_2 = 1$, $\phi_1 = 1.016428$, $\phi_2 = 1.005840$, $\phi_3 = 1.014903$, $\phi_4 = 1.009464$

By applying these stiffness modification factors, the modified stiffness matrix of the bar with respect to the coordinate of the lumped mass becomes

$$k = \begin{bmatrix} 150.180625308 & -488.912284903 \\ -488.912284903 & 196.9030482299 \end{bmatrix} \frac{EI}{L^3} \tag{5.44}$$

By using this modified stiffness on equation (5.2) the new values of λ, natural frequency and mode shape obtained are

$$\lambda_1 = \frac{0.0808910725\ \mu L^4}{EI}, \lambda_2 = \frac{0.00272657745\ \mu L^4}{EI} \qquad (5.45a)$$

$$\{\phi_1\} = \begin{Bmatrix} -0.321498839252 \\ -0.94691012436 \end{Bmatrix}, \{\phi_2\} = \begin{Bmatrix} -0.827292804247 \\ 0.561770964042 \end{Bmatrix}. \qquad (5.45b)$$

$$\omega_1 = 3.5160\sqrt{\frac{EI}{\mu L^4}}\ ,\ \omega_2 = 19.1510\sqrt{\frac{EI}{\mu L^4}} \qquad (5.45c)$$

These were repeated for the bars of Figures 5.2(c),5.2 (d), 5.2(e) and 5.2(f) and a summary of the obtained natural frequencies presented in Table 5.2

Table 5.6: Comparism of the obtained natural frequencies of different lump-massed fixed-free beam under lateral vibration with the exact results.

	Mode No	Hamilton (Exact) (Hz)	Lagrange (Hz)	Percentage Error (%)	Lagrange with modified stiffness (Hz)	Percentage Error (%)
Figure 5.2 (a)	1	3.5160	2.4495	30.33	3.5160	0
Figure 5.2 (b)	1	3.5160	3.1562	10.23	3.5160	0
	2	22.0345	16.2580	26.22	19.1510	13.09
Figure 5.2 (c)	1	3.5160	3.1241	11.15	3.5159	0
	2	22.0345	20.0833	8.86	20.5869	6.57
Figure 5.2 (d)	1	3.5160	3.3457	4.84	3.5159	0
	2	22.0345	18.8859	14.29	18.2887	17.00
	3	61.6972	47.0284	23.78	43.1194	30.11
Figure 5.2 (e)	1	3.5160	3.2374	7.92	3.5160	0
	2	22.0345	16.8688	23.44	18.1657	17.56
	3	61.6972	69.7577	-13.06	76.6552	-24.24

From Table 5.2 followed the same pattern as the ones before it. The natural frequencies obtained from the use of Lagrange equation on the continuous system were not exact as seen in the percentage error column. But when the stiffness of the system was modified using the stiffness modification factors, the use of Lagrange equation was able to predict accurately the fundamental frequencies hence their percentage errors were zero. However the values of the higher frequencies remained approximate, some of its predictions for higher frequencies where less accurate than that obtained without the application of the stiffness modification factors.

5.3 Systems under both longitudinal and lateral vibration

So far we have considered systems under either longitudinal or lateral vibration. But systems can be under both longitudinal and lateral vibration simultaneously. From the model for longitudinal vibration and that for lateral vibration, the stiffness equation for an element under both longitudinal and lateral vibration can be written as

$$[k] = \begin{bmatrix} \frac{EA}{L}\alpha_1 & 0 & 0 & -\frac{EA}{L}\alpha_2 & 0 & 0 \\ 0 & \frac{12EI}{L^3}\phi_1 & \frac{6EI}{L^2}\phi_2 & 0 & -\frac{12EI}{L^3}\phi_3 & \frac{6EI}{L^2}\phi_4 \\ 0 & \frac{6EI}{L^2}\phi_2 & \frac{4EI}{L}\phi_1 & 0 & -\frac{6EI}{L^2}\phi_4 & \frac{2EI}{L}\phi_3 \\ -\frac{EA}{L}\alpha_2 & 0 & 0 & \frac{EA}{L}\alpha_1 & 0 & 0 \\ 0 & -\frac{12EI}{L^3}\phi_3 & -\frac{6EI}{L^2}\phi_4 & 0 & \frac{12EI}{L^3}\phi_1 & -\frac{6EI}{L^2}\phi_2 \\ 0 & \frac{6EI}{L^2}\phi_4 & \frac{2EI}{L}\phi_3 & 0 & -\frac{6EI}{L^2}\phi_2 & \frac{4EI}{L}\phi_1 \end{bmatrix} \quad (5.46)$$

The values of the stiffness modification factors can be picked from the relevant Tables depending on the element position on the vibrating system and the system's end conditions. Consider the beam of Figure 5.1a with both ends fixed for the case of both longitudinal and lateral vibration occurring simultaneously. The lumped mass will then have two coordinates of vibration. Let the first coordinate (coordinate 1) be for the vertical vibration and coordinate 2 for the horizontal vibration, then the stiffness matrix of the system with respect to these coordinate 1 and 2 can be written as

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} \frac{4EA}{L} & 0 \\ 0 & \frac{192EI}{L^3} \end{bmatrix} \tag{5.47}$$

Recall that k_{ij} is the force in coordinate i due to a unit deformation in coordinate j. The inertia matrix can likewise be written as

$$m = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \mu L \tag{5.48}$$

By substituting equations (5.47) and (5.48) into the eigenvalue problems of equation (5.2) and solving for the natural frequencies we obtain

$$\omega_1 = 2.8284 \tag{5.49a}$$

$$\omega_2 = 19.5959 \tag{5.49b}$$

Notice that ω_1 correspond to the natural frequency obtained for the case of the beam under only longitudinal vibration only (equation (5.4c)) and ω_2 correspond to the case of the beam under only lateral vibration (equation (5.15c)). These values were not altered despite the fact that we considered both lateral and longitudinal vibration to be occurring simultaneously. We now introduce the stiffness modification factors

For the first element

$\alpha_1 = 1.233701, \alpha_2 = 1.859611$ (from Table 4.7)

$\phi_1 = 1.303552, \phi_2 = 0.663259, \phi_3 = 1.731522, \phi_4 = 1.293649$ (from Table 4.18)

These same values of stiffness modification factors apply for the second element.

After introducing the stiffness modification factors the new stiffness of the beam with respect to coordinates 1 and 2 becomes

$$K = \begin{bmatrix} \frac{4.934804\,EA}{L} & 0 \\ 0 & \frac{250.281984\,EI}{L^3} \end{bmatrix} \tag{5.50}$$

The inertia matrix will remain unchanged. By substituting equations (5.50) and (5.48) into the eigenvalue problems of equation (5.2) and solving for the natural frequencies we obtain

$$\omega_1 = 3.1416 \tag{5.51a}$$

$$\omega_2 = 22.3733 \tag{5.51b}$$

Notice that ω_1 is the exact fundamental frequency for the longitudinal vibration of fixed-fixed beam (see Table 5.1) and ω_2 is the exact value of the fundamental frequency for the lateral vibration of the fixed-fixed beam (see Table 5.3). Hence the modification factors though derived for cases of only longitudinal and lateral vibration occurring separately, can be applied when both type of vibration occur simultaneously. This can be extended to systems with more degrees of freedom.

5.4 **Frames**

The stiffness modification factors can be extended to frames. This is done by using fixed-free to represent elements fixed at one end and which are capable of undergoing rotation and translation at the other end. Pinned-free for elements pinned at one end and which are capable of undergoing rotation and translation at the other end. Free-free for elements capable of undergoing rotation and translation at both ends. For the free-free beam, the value of the nodal forces at their ends is zero. The value of the nodal forces for a fixed-free beam is also zero at the free end. With this arrangement elements are connected to each other at their free ends where their nodal forces P are zero. Recall that the stiffness coefficients were derived by equating the nodal forces and displacements of a vibrating beam with that of an equivalent lumped massed beam. To achieve equality of displacement for two dissimilar beams at their connecting ends the modification factors for one of the connecting members need to be multiplied by a constant. These were determined by trial and error. When connecting a fixed free beam with a free-free beam divide the stiffness modification factor of the fixed-free beam with the ratio of the length of the free-free beam to the fixed free beam. Likewise when connecting a pinned-free beam with a free-free beam, multiply the stiffness modification factors of the free-free beam by 1.7885.

Consider the portal frame of Figure 5.3a

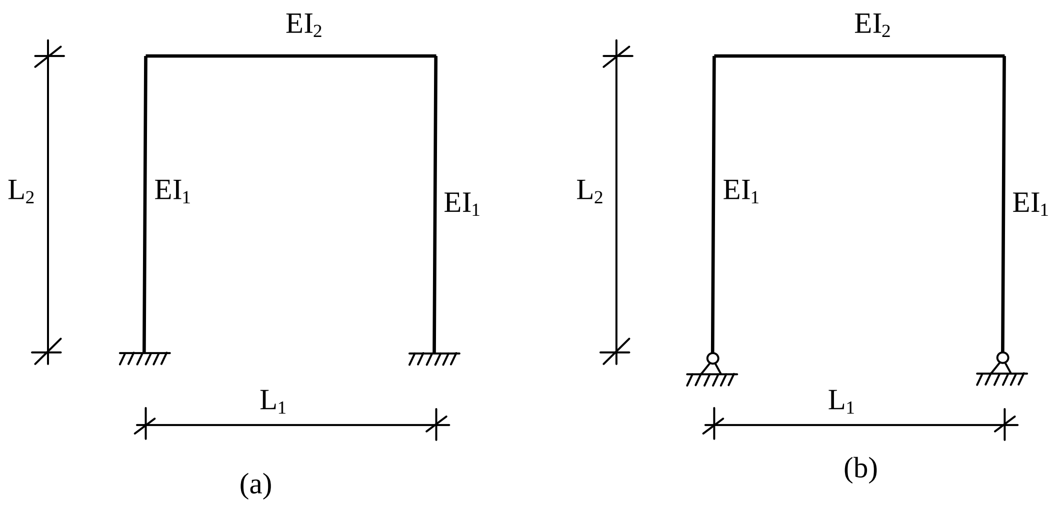
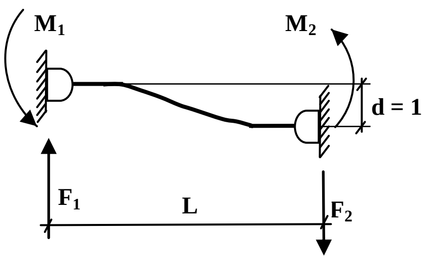
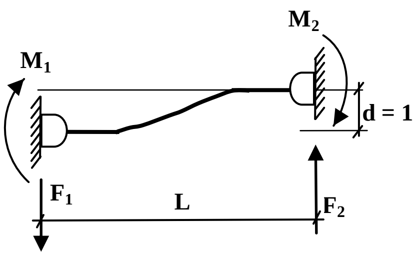
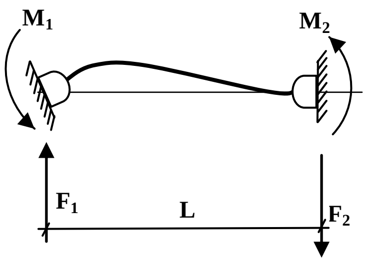
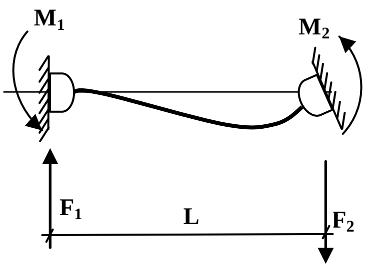
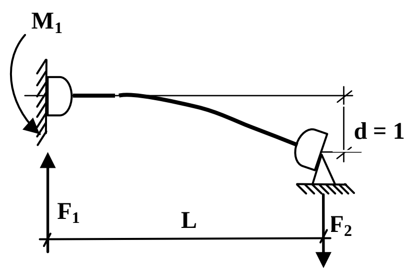
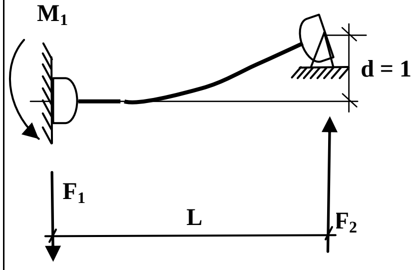
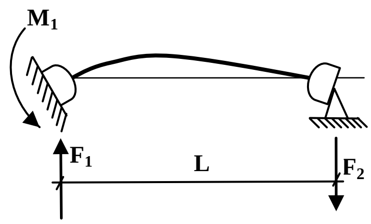


Figure 5.3: Portal frames constructed of prismatic elements.

In order to use the proposed model in frames it might the necessary to present it in the form of a Table of end forces due to unit deformation of prismatic members. This information is presented in Table 5.7.

Table 5.7: End forces caused by end displacement of lumped-massed elements showing the stiffness modification factors.

S/No	**Beam**	**Force**
1		$F_1 = \frac{12EI\phi_1}{L^3}, \quad F_2 = \frac{12EI\phi_3}{L^3}$ $M_1 = \frac{6EI\phi_2}{L^2}, \quad M_2 = \frac{6EI\phi_4}{L^2}$
2		$F_1 = \frac{12EI\phi_3}{L^3}, \quad F_2 = \frac{12EI\phi_1}{L^3}$ $M_1 = \frac{6EI\phi_4}{L^2}, \quad M_2 = \frac{6EI\phi_2}{L^2}$
3		$F_1 = \frac{6EI\phi_2}{L^2}, \quad F_2 = \frac{6EI\phi_4}{L^2}$ $M_1 = \frac{4EI\phi_1}{L}, \quad M_2 = \frac{2EI\phi_3}{L}$
4		$F_1 = \frac{6EI\phi_2}{L^2}, \quad F_2 = \frac{6EI\phi_2}{L^2}$ $M_1 = \frac{2EI\phi_3}{L}, \quad M_2 = \frac{4EI\phi_1}{L}$

5		$F_1 = \frac{3EI\phi_1}{L^3}, \qquad F_2 = \frac{3EI\phi_3}{L^3}$ $M_1 = \frac{3EI\phi_2}{L^2}$
6		$F_1 = \frac{3EI\phi_3}{L^3}, \qquad F_2 = \frac{3EI\phi_1}{L^3}$ $M_1 = \frac{3EI\phi_4}{L^2}, \qquad \phi_4 = 1$
7		$F_1 = \frac{3EI\phi_2}{L^2}, \qquad F_2 = \frac{3EI\phi_4}{L^2}$ $M_1 = \frac{3EI\phi_1}{L}, \qquad \phi_4 = 1$

5.4.1 Comparison of Results with that from Standard Books (Structural Engineering Handbook)

The portal frames of figure 5.4 have three degrees of freedom (ignoring axial deformation). These are illustrated in Figure 5.4.

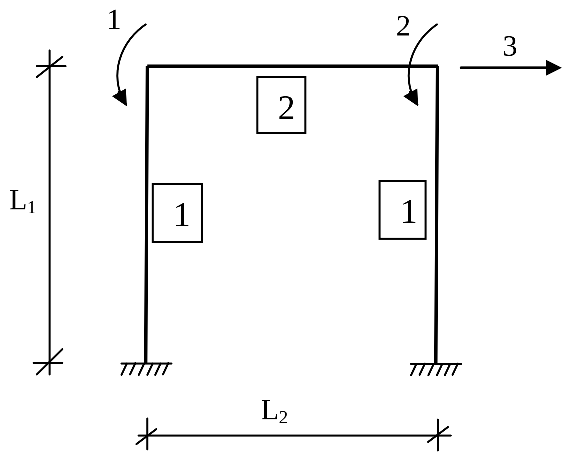


Figure 5.4: The three degrees of freedom of a planar portal frame.

Using the values of Table 5.7, the stiffness matrix of the portal frame of figure 5.4 can be formulated as

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \tag{5.52}$$

$$K_{11} = \frac{4EI_1\phi_{11}}{L_1} + \frac{4EI_2\phi_{12}}{L_2} \tag{5.53a}$$

$$K_{12} = \frac{2EI_2\phi_{32}}{L_2} \tag{5.53b}$$

$$K_{13} = \frac{6EI_1\phi_{21}}{L_1^2} \tag{5.53c}$$

$$K_{22} = \frac{4EI_2\phi_{12}}{L_2} + \frac{4EI_1\phi_{11}}{L_1} \tag{5.53d}$$

$$K_{23} = \frac{6EI_1\phi_{21}}{L_1^2} \tag{5.53e}$$

$$K_{33} = \frac{2\times12EI_1\phi_{11}}{L_1^3} \tag{5.53f}$$

From Maxwell's Reciprocal theorem or Betti's law (Das *et al*, 2011; Numayr and Al Rjoub, 2012) $K_{ij} = K_{ji}$ hence $K_{12} = K_{21}$, $K_{31} = K_{13}$ etc. ϕ_{ij} is ϕ_i for element j. Element 1 is treated as a fixed-free beam why element 2 is treated as a free-free beam.

Using the matrix transformation introduced in section 2.6.2 the stiffness with respect to coordinate 3 is obtained and together with the structure's inertia matrix the frame's natural frequency is obtained.

A comparison of the result obtained from the use of the stiffness modification factors with that obtained from structural engineering handbooks is presented in Table 5.8

Table 5.8: A Comparison of the Estimated Fundamental frequencies for the fixed portal frame with that from standards

L_2/L_1	Structural Engineering Handbook (Hz)	Without Using Stiffness Modification factors		Using Stiffness Modification factors (SMFs)	
		Frequency	% Error	Frequency	% Error
8	0.5606	0.9912	76.8	0.5693	1.6
4	1.0773	1.4771	37.1	1.0801	0.3
2	2.0295	2.1381	5.4	1.9719	2.8
1	3.5564	2.8983	-18.5	3.4155	-4.0
0.8	3.9267	3.1393	-20.1	4.0252	2.5
0.4	4.2779	3.7995	-11.2	6.4547	50.9
0.2	4.4293	4.2703	-3.6	9.8597	122.6

From the Table 5.8 it would be seen that the error in the analysis of the rigidly fixed portal frame with lumped masses is high at high values of the ratio of the portal's length to the height (L_2/L_1). The error gradually reduces with increase in the ratio L_2/L_1. By using the stiffness modification factor the error is found to be minimal at high values of the ratio L_2/L_1 and remained consistently low for values $8>L_2/L_1>0.8$ before increasing sharply. These are better observed from Figure 5.5

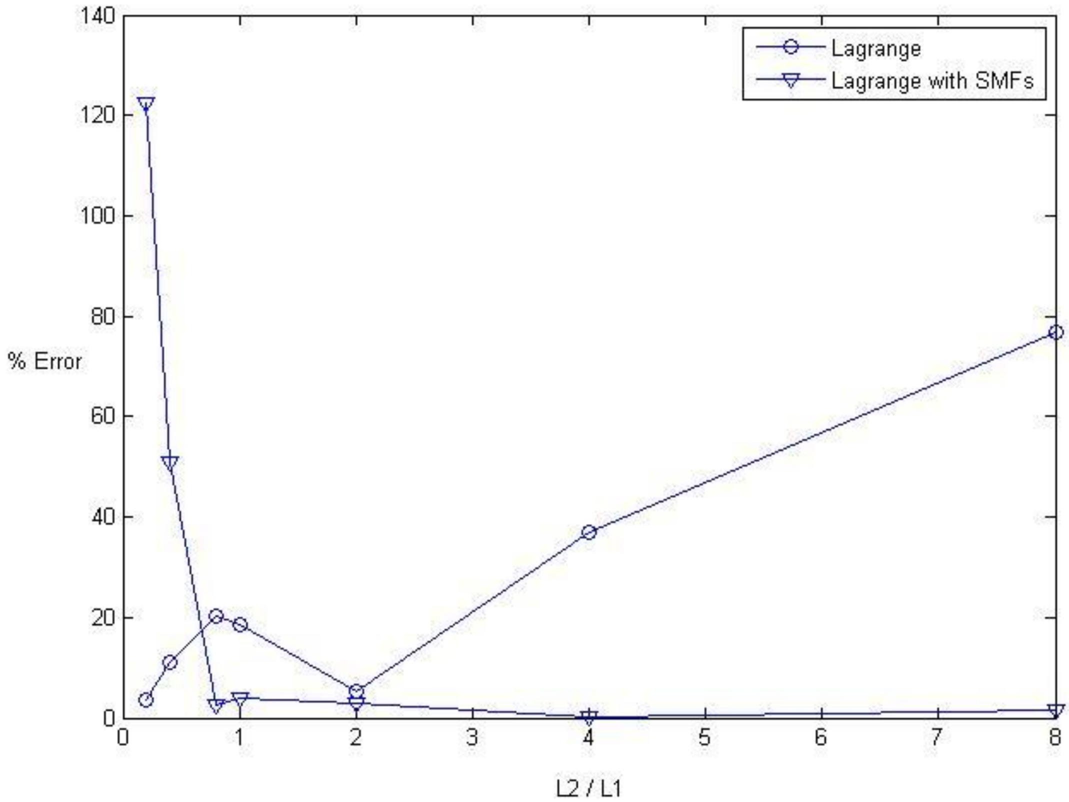


Figure 5.5: Percentage error in natural frequency against the portal's length to height ratio

Figure 5.5 above is a plot of the % error in estimated values of fundamental frequency against the portal's length to height ratio. From the graphs it would be observed that the curve for the percentage error from the use of Lagrange equations was generally high. By introducing the stiffness modification factors the percentage error became very low if not insignificant. However for L_2/L_1 values less than 0.8, the error from the use of the stiffness modification become very high and much higher than the error incurred when the stiffness modification was not used. This shows that the stiffness modification factors are not suitable in rigidly fixed portal frames of L_2/L_1 values less than 0.8. This can be extended to the hinged portal frame.

The contribution of the stiffness modification factors can also be appreciated from a statistical analysis of the results obtained from standard Tables, lumped mass without SMF and lumped mass with SMF. An F test was carried out on the results from Standard Tables and that from lumped-mass fixed portal frame without SMF using Microsoft excel software and the results are presented below

Table 5.9: F-test Analysis of the results from standard Tables and that of rigidly fixed lumped mass portal frame without SMF

For rigidly fixed portal frame

L2/L1	Standard	without SMF
8	0.5606	0.9912
4	1.0773	1.4771
2	2.0295	2.1381
1	3.5564	2.8983
0.8	3.9267	3.1393
0.4	4.2779	3.7995
0.2	4.4293	4.2703

F-Test Two-Sample for Variances

	Variable 1	*Variable 2*
Mean	2.836814	2.6734
Variance	2.541199	1.438885
Observations	7	7
df	6	6
F	1.766089	
P(F<=f) one-tail	0.253324	
F Critical one-tail	4.283866	

From the results of the F test it would be seen that F is less than Fcritical; 1.766 < 4.284. We therefore reject the null hypothesis which states that the variance of the results from standard Tables is equal to that from lumped mass frame with SMF. The results can also be obtained from the p-value which is greater than 0.05. These show that there is significant different between the results obtained by the standard Tables and that for lumped mass without SMF. By introducing the stiffness modification factors (SMF) and repeating the F-test the results below are obtained

Table 5.10: F-test Analysis of the results from standard Tables and that of rigidly fixed lumped mass portal frame with SMF

For rigidly fixed portal frame

L2/L1	**with SMF**	**Standard**
8	0.5693	0.5606
4	1.0801	1.0773
2	1.9719	2.0295
1	3.4155	3.5564
0.8	4.0252	3.9267
0.4	6.4547	4.2779
0.2	9.8597	4.4293

F-Test Two-Sample for Variances

	Variable 1	*Variable 2*
Mean	3.910914	2.836814
Variance	10.84284	2.541199
Observations	7	7
df	6	6
F	4.266823	
P(F<=f) one-tail	0.050434	
F Critical one-tail	4.283866	

From this result of Table 5.10 it would be seen also that F<Fcritical and so we reject the null hypothesis that states that the variance of the results from standard Tables and that from lumped mass with SMF are equal. This might sound like there was no benefit from the use of the SMF. But before we rule out the benefit of using the SMF it is important to note that F is only slightly lower than Fcritical unlike in Table 5.9 where the difference was much. This is further buttressed by the p-value which is 0.0504 (approximately 0.05). With these we can conclude that the use of SMF did improve the result of the analysis to almost a significant level.

Using the values of Table 5.7, the stiffness matrix of the portal frame of figure 5.3b can be formulated as

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \quad (5.54)$$

$$K_{11} = \frac{3EI_1\phi_{11}}{L_1} + \frac{4EI_2\phi_{12}}{L_2} \quad (5.55a)$$

$$K_{12} = \frac{2EI_2\phi_{32}}{L_2} \quad (5.55b)$$

$$K_{13} = \frac{3EI_1\phi_{21}}{L_1^2} \quad (5.55c)$$

$$K_{22} = \frac{4EI_2\phi_{12}}{L_2} + \frac{3EI_1\phi_{11}}{L_1} \quad (5.55d)$$

$$K_{23} = \frac{3EI_1\phi_{21}}{L_1^2} \quad (5.55e)$$

$$K_{33} = \frac{2\times 3EI_1\phi_{11}}{L_1^3} \quad (5.55f)$$

$K_{ij} = K_{ji}$ (Maxwell's theorem) hence $K_{12} = K_{21}$, $K_{31} = K_{13}$ etc. ϕ_{ij} is ϕ_i for element j. Element 1 is treated as a pinned-free beam why element 2 is treated as a free-free beam.

By applying the matrix transformation introduced in section 2.6.2 the stiffness with respect to coordinate 3 is obtained and together with the structure's inertia matrix the frame's natural frequency is obtained.

A comparism of the result obtained from the use of the stiffness modification factors with that obtained from structural engineering handbooks is presented in Table 5.11

Table 5.11: A Comparism of the Estimated Fundamental frequencies for the hinged portal frame with that from standards

L_2/L_1	Structural Engineering Handbook (Hz)	Without Using Stiffness Modification factors		Using Stiffness Modification factors (SMFs)	
		Frequency	% Error	Frequency	% Error
8	0.5525	0.3651	-33.9	0.5430	-1.7
4	1.0540	0.6325	-40.0	1.0248	-2.8
2	1.9633	1.000	-49.1	1.8513	-5.7
1	3.1416	1.4142	-55.0	3.1416	0
0.8	3.2864	1.5430	-53.0	3.6658	11.54
0.4	3.4845	1.8898	-45.8	5.6090	60.97
0.2	3.6240	2.1320	-41.2	7.8905	117.7

From the Table above it would be seen that the error in the analysis of hinged portal frames with lumped masses is very high at all values of the ratio of the portal's length to the height (L_2/L_1). The error does not show any remarkable increase or decrease with increase in the ratio L_2/L_1. By using the stiffness modification factor the error is found to be minimal at high values of the ratio L_2/L_1

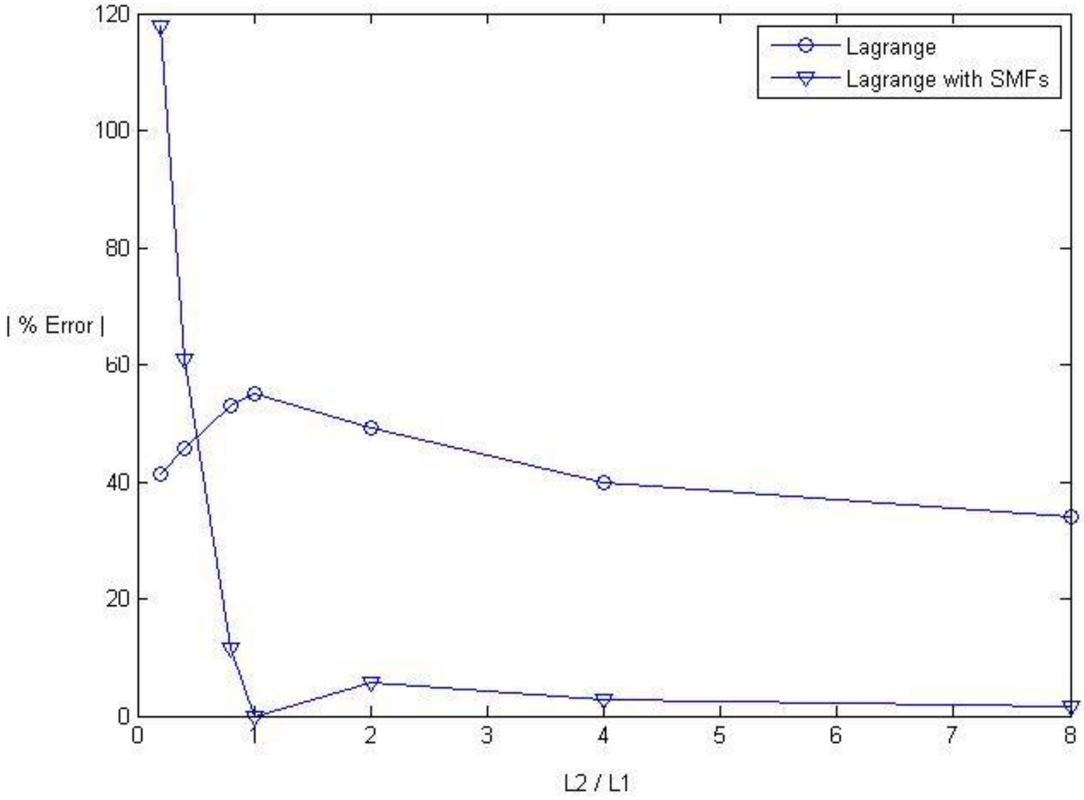


Figure 5.6: Percentage error in natural frequency against the portal's breadth to height ratio for the hinged portal frame

Figure 5.6 is a plot of the absolute % error in estimated values of fundamental frequency against the portal's breadth to height ratio. The graph shows that the percentage error from the use of Lagrange equations for hinged portal frame was generally high. By introducing the stiffness modification factors the percentage error was significantly reduced. However as observed in the case of fixed portal frame, for L_2/L_1 values less than 0.8, the error from the use of the stiffness modification become very high, and much higher than the error incurred when the stiffness modification was not used. This shows that the stiffness modification factors are not suitable in portal frames of L_2/L_1 values less than 0.8.

Likewise statistical analyses were carried out to compare the results from the use of standard Tables and that from lumped mass frame. The first was an F-test to compare the results from the use of standard tables and that for lumped mass frames without SMF and the results are presented below

Table 5.12: F-test Analysis of the results from standard tables and that of hinged lumped mass portal frame without SMF

For hinged portal frame

L2/L1	Standard	Without SMF
8	0.5525	0.3651
4	1.054	0.6325
2	1.9633	1
1	3.1416	1.4142
0.8	3.2864	1.543
0.4	3.4845	1.8898
0.2	3.624	2.132

F-Test Two-Sample for Variances

	Variable 1	*Variable 2*
Mean	2.443757	1.282371
Variance	1.568711	0.419933
Observations	7	7
df	6	6
F	3.735624	
P(F<=f) one-tail	0.066855	
F Critical one-tail	4.283866	

From the result it would be seen that F is less than the Fcritical (3.736<4.284) we therefore reject the null hypothesis that states that the variance of the two results are equal. This is further stressed by the p-value which is more than 0.05. By introducing the SMFs we obtained an improved result which is presented below

Table 5.13: F-test Analysis of the results from standard tables and that of hinged lumped mass portal frame with SMF

For hinged portal frame

L2/L1	With SMF	Standard
8	0.543	0.5525
4	1.0248	1.054
2	1.8513	1.9633
1	3.1416	3.1416

F-Test Two-Sample for Variances

	Variable 1	*Variable 2*
Mean	3.389429	2.443757
Variance	6.897234	1.568711
Observations	7	7

0.8	3.6658	3.2864	df	6	6
0.4	5.609	3.4845	F	4.396752	
0.2	7.8905	3.624	P(F<=f) one-tail	0.047249	
			F Critical one-tail	4.283866	

From the results it would be observed that F>Fcritical. Hence we uphold the null hypothesis which states that the variance of the results from standard Tables and that from lumped mass with SMFs are equal. This is further validated by the p-value of 0.0472 which is less than 0.05. The results show that the use of the SMF was able reduce the seemly significant difference between the results from use of standard tables and that from lumped mass frames.

5.4.2 Comparison of Results with Experimental and Professional Software Results

Supreeth *et al* (2015) carried out experimental studies on a three-storied aluminum frame consisting of columns and slabs. The section of the column was rectangular and measured 3 mm by 25.11 mm. There were four columns. The length of the slab was 300 mm and its width was 150 mm. The slab thickness was 12.7 mm. These slabs were attached to the column at intervals of 400 mm. Using MILDAQ Data Acquisition, accelerometers and a horizontal shake Table, the natural frequencies of vibration of the frame was obtained. They also modeled and analysed the same frame with ANSYS version 11 using SOLID187 elements (a higher order 3D, 10 node element) to obtain the frame’s natural frequencies.

By modeling the frame as a 9-degree of freedom undamped lumped mass system (see Figure 5.7) we can determine the natural frequencies of the frame with and without the stiffness modification factors.

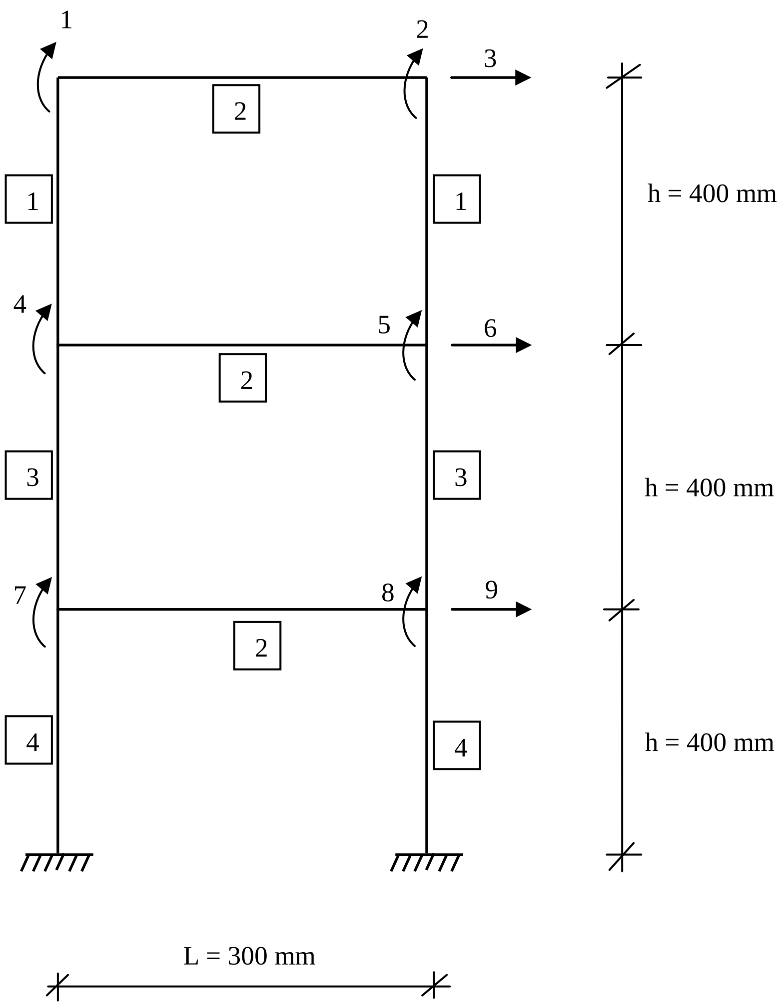


Figure 5.7: The nine degrees of freedom of the two storey aluminum frame

Using the values of Table 5.7, the stiffness matrix of the portal frame of figure 5.3b can be formulated as

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} & K_{19} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} & K_{29} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} & K_{39} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} & K_{49} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} & K_{59} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} & K_{69} \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} & K_{79} \\ K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} & K_{89} \\ K_{91} & K_{92} & K_{93} & K_{94} & K_{95} & K_{96} & K_{97} & K_{98} & K_{99} \end{bmatrix} \tag{5.56}$$

$$K_{11} = \frac{4EI_1\phi_{11}}{h} + \frac{4EI_2\phi_{12}}{L} \; , K_{12} = \frac{2EI_2\phi_{32}}{L} , K_{13} = \frac{6EI_1\phi_{31}}{L^2}$$

$K_{14} = \frac{2EI_1\phi_{31}}{h}, \; K_{16} = \frac{6EI_1\phi_{41}}{h^2}, \; K_{22} = \frac{4EI_1\phi_{11}}{h} + \frac{4EI_2\phi_{12}}{L}$

$K_{23} = \frac{-6EI_1\phi_{21}}{h^2}, K_{25} = \frac{2EI_1\phi_{21}}{h}, \; K_{26} = \frac{6EI_1\phi_{41}}{h^2}$

$K_{33} = \frac{24EI_1\phi_{11}}{h^3}, K_{34} = \frac{-6EI_1\phi_{21}}{h^2}, \; K_{35} = \frac{-6EI_1\phi_{21}}{h^2}, \; K_{36} = \frac{-24EI_1\phi_{31}}{h^3}$

$K_{44} = \frac{4EI_1\phi_{11}}{h} + \frac{4EI_3\phi_{13}}{h} + \frac{4EI_2\phi_{12}}{L}, K_{45} = \frac{2EI_2\phi_{32}}{L}, \; K_{46} = \frac{6EI_1\phi_{21}}{h^2} - \frac{6EI_1\phi_{23}}{h^2}$

$K_{47} = \frac{2EI_3\phi_{33}}{h}, \; K_{49} = \frac{6EI_3\phi_{43}}{h^2}, K_{55} = \frac{4EI_1\phi_{11}}{h} + \frac{4EI_3\phi_{13}}{h} + \frac{4EI_2\phi_{12}}{L}$

$K_{56} = \frac{6EI_1\phi_{21}}{h^2} - \frac{6EI_1\phi_{23}}{h^2}, K_{58} = \frac{2EI_3\phi_{33}}{h}, K_{59} = \frac{6EI_3\phi_{43}}{h^2}$

$K_{66} = \frac{24EI_1\phi_{11}}{h^3} + \frac{24EI_3\phi_{13}}{h^3}, K_{67} = \frac{-6EI_3\phi_{43}}{h^2}, K_{68} = \frac{-6EI_3\phi_{43}}{h^2}, K_{69} = \frac{-6EI_3\phi_{33}}{h^2}$

$K_{77} = \frac{4EI_3\phi_{13}}{h} + \frac{4EI_4\phi_{14}}{h} + \frac{4EI_2\phi_{12}}{L}, K_{78} = \frac{2EI_2\phi_{32}}{L}, \; K_{79} = \frac{6EI_3\phi_{23}}{h^2} - \frac{6EI_4\phi_{24}}{h^2}$

$K_{88} = \frac{4EI_3\phi_{13}}{h} + \frac{4EI_4\phi_{14}}{h} + \frac{4EI_2\phi_{12}}{L}, K_{79} = \frac{6EI_3\phi_{23}}{h^2} - \frac{6EI_4\phi_{24}}{h^2}$

$$K_{99} = \frac{24EI_3\phi_{13}}{h^3} + \frac{24EI_4\phi_{14}}{h^3} \tag{5.57}$$

From Maxwell's theorem $K_{ij} = K_{ji}$ hence $K_{12} = K_{21}$, $K_{31} = K_{13}$ etc. The other stiffness coefficients not listed in equation (5.57) are equal to zero. ϕ_{ij} is ϕ_i for element j. Element 4,3 and 1 are treated as segments of a fixed-free beam why element 2 is treated as a free-free beam.

From Table 4.36 or using the attached program the stiffness modification factors for the elements of the multistory frame can be obtained as

For element 1: $\phi_1 = 1.070403, \; \phi_2 = 1.068451, \; \phi_3 = 1.070402, \; \phi_4 = 1.070223$

For element 2: $\phi_1 = 1.0, \; \phi_2 = 1.0, \; \phi_3 = 1.0, \; \phi_4 = 1.0$

For element 3: $\phi_1 = 1.026731,\ \phi_2 = 1.025416,\ \phi_3 = 1.026564,\ \phi_4 = 1.025759$

For element 4: $\phi_1 = 0.980749,\ \phi_2 = 0.975733,\ \phi_3 = 1.002985,\ \phi_4 = 1.002269$

The frame's inertia matrix M is

$$M = \begin{bmatrix} \mu_1 h + \mu_2 L & 0 & 0 \\ 0 & 2\mu_1 h + \mu_2 L & 0 \\ 0 & 0 & 2\mu_1 h + \mu_2 L \end{bmatrix}$$

$$= \begin{bmatrix} 1.705762 & 0 & 0 \\ 0 & 1.868474 & 0 \\ 0 & 0 & 1.868474 \end{bmatrix} \tag{5.58}$$

where μ_1 and μ_2 are the mass per unit length of the columns and beams respectively. They can be calculated knowing that the young's modulus and density of aluminum are 6.9 x 10^9 Pa and 2700 kg/m^3 respectively.

By applying the matrix transformation introduced in section 2.6.2 the stiffness with respect to coordinates 3, 6 and 9 is obtained and together with the structure's inertia matrix obtained by lumping all masses at the respective joints (see equation 5.58) the frame's natural frequencies are obtained. In order to express the frequency in hertz the obtained values are divided by 2π. These were also done without applying the stiffness modification factors (i.e by making all the stiffness modification factors equal to one) and the results compared with the Supreeth *et al*'s experimental and ANSYS results. Table 5.14 is a summary of the results obtained.

Table 5.14: Results of the Analysis of the three-story frame

Mode	Experimental ω (Hz)	ANSYS		Lumped mass without SMF		Lumped mass with SMF	
		ω(Hz)	% Error	ω(Hz)	% Error	ω(Hz)	% Error
1	3.0	3.0473	1.577	3.3359	11.197	2.9738	0.873
2	8.30	8.6142	3.786	8.5441	3.065	6.8970	16.904
3	12.15	12.566	3.424	9.6557	20.529	8.1468	32.948

The results presented in Table 5.14 show that ANSYS was able to predict the natural frequencies to accuracy less than 3.5%. By lumping the masses at the frame's joints and solving we were able to estimate the fundamental frequency of the as 3.34Hz and with an error of 11.2%. This error was reduced to 0.873% by the introduction of the stiffness modification factors. The percentage errors were calculated with respect to the values of natural frequency obtained by experimental studies. The use of the stiffness modification factors did not improve the accuracy of the frequencies of the higher modes of vibration.

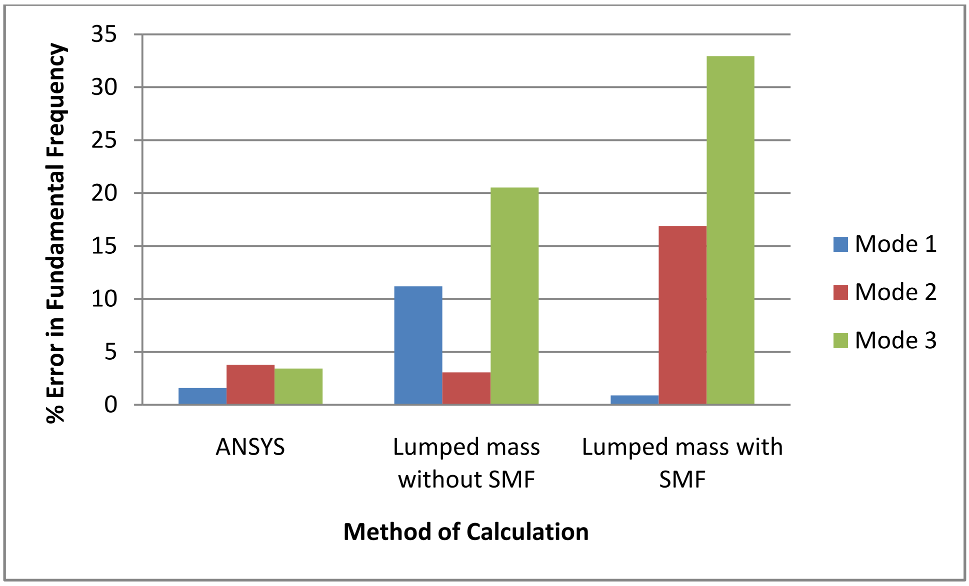


Figure 5.8: Percentage Error for the different modes of vibration

There were further illustrated by Figure 5.7. From the bar chart we observe that the use of lumped mass with stiffness modification factors gave the least percentage error of 0.87% for mode 1 (fundamental frequency). However for the higher modes of vibration (modes two and three), there was no improvement in the results from the use of the stiffness modification factors. In fact the use of the stiffness modification factors for the higher modes of vibration increased the margin of errors in the calculated results. Hence in calculating the frequency of frames for higher modes of vibration it is not advisable to use the stiffness modification factors.

We now extend the use of the stiffness modification factors to the dynamic analysis of frames including axial effects. Virgin and Lyman (2011) considered the free vibration of a plane rectangular portal frame consisting of very slender members to determine how the natural frequency of frames is influenced by the addition of masses at the corners of the frame. The experimental set up consisted of a simple portal frame composed of polycarbonate beams of cross-sectional dimensions 2.554 x 0.154 cm and density 1157 kg/m3. The bottom supports of the column were clamped. The columns measured L_1 = 15.24 cm in length and the cross beam was L_2 = 45.72 cm.

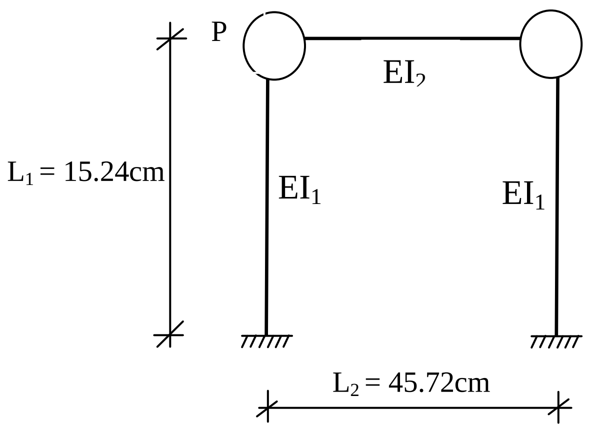


Figure 5.9: Virgin and Lyman 's experimental set up
(Virgin and Lyman (2011))

Modal data were taken using an Ometron VH300+ laser doppler vibrometer and an Endevco 2302-50 modal impact hammer. The data were recorded with a pulse data acquisition software. They also ran simulations of the frame using the commercial finite element software package ANSYS, first without considering beam column effect (effect of axial load on the geometric stiffness of axially loaded elements) and later by considering beam-column effect.

Using the portal frame of Figure 5.3, the stiffness matrix can be formulated as

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \tag{5.59}$$

The effect of axial force on the stiffness of beam can be captured by the inclusion of stability functions in the matrix of equation (5.59). The stiffness modification factors will still be represented with Ø while the Ø functions will be represented with ψ. (see Table A1 in Appendix A for a list of the stability functions).

$$K_{11} = \frac{4EI_1\phi_{11}\psi_{31}}{L_1} + \frac{4EI_2\phi_{12}}{L_2} \tag{5.60aa}$$

$$K_{12} = \frac{2EI_2\phi_{32}}{L_2} \tag{5.60b}$$

$$K_{13} = \frac{6EI_1\phi_{21}\psi_{11}}{L_1^2} \tag{5.60c}$$

$$K_{22} = \frac{4EI_2\phi_{12}\psi_{31}}{L_2} + \frac{4EI_1\phi_{11}}{L_1} \tag{5.60d}$$

$$K_{23} = \frac{6EI_1\phi_{21}\psi_{11}}{L_1^2} \tag{5.60e}$$

$$K_{33} = \frac{2\times 12EI_1\phi_{11}\psi_{11}}{L_1^3} \tag{5.60f}$$

From Betti's law $K_{ij} = K_{ji}$ hence $K_{12} = K_{21}$, $K_{31} = K_{13}$ etc. ϕ_{ij} is ϕ_i for element j while ψ_{ij} is ψ_i for element j. ψ_{ij} is equal to unity when beam column effects is ignored. Element 1 as before is treated as a fixed-free beam why element 2 is treated as a free-free beam. For element 1 the value of the stability functions from Table A (in the appendices), is given as

$$\psi_1 = \frac{z^2(1-\cos z)}{6(2-2\cos z - z\sin z)} \tag{5.61a}$$

$$\psi_3 = \frac{z(\sin z - z\cos z)}{4(2-2\cos z - z\sin z)} \tag{5.61b}$$

Where $z = \sqrt{\frac{PL_1^2}{EI_1}}$ (5.62)

P is the axial force on the vertical column of length L_1, EI_1 is the flexural rigidity of the column.

By applying the matrix transformation introduced in section 2.6.2 the stiffness with respect to coordinate 3 is obtained and together with the structure's inertia matrix the frame's natural frequency is obtained.

A comparism of the result obtained from experiment with that from ANSYS and lumped mass with and without stiffness modification factors are presented in the Table 5.15.

Table 5.15: Results of the Analysis of the loaded portal frame without any consideration for beam-column effects

Axial Load (kg)	Experiment	ANSYS		Lumped mass without SMF		Lumped mass with SMF	
	ω(Hz)	ω(Hz)	% Error	ω(Hz)	%Error	ω(Hz)	%Error
0.05	3.80	3.90	2.63	5.82	53.16	4.09	7.63
0.10	2.60	2.78	6.92	4.35	67.31	3.02	16.15
0.15	1.92	2.25	17.18	3.63	89.06	2.50	30.20
0.20	1.51	1.92	27.15	3.18	110.60	2.18	44.37
0.25	1.25	1.80	44.00	2.86	128.80	1.96	56.80
0.30	1.01	1.62	60.40	2.62	159.41	1.80	78.22
0.35	0.78	1.50	92.31	2.44	212.82	1.67	114.10

Table 5.15 shows the experimental results of the loaded portal frame together with the results of an ANSYS simulation of the model and that of a lumped mass model of the frame with and without stiffness modification factor. The results show a steady decrease in the fundamental frequency of the frame with increase in nodal mass/load. At lower nodal loads the difference in estimated fundamental between the experimental and that from ANSYS and lumped mass is lower. These increased progressively as the nodal load was increased due to beam column effects. The use of stiffness modification factors reduced significantly the errors in estimated fundamental frequencies. For instance at a nodal mass of 0.05 kg, the percentage error without stiffness modification was 53.16%. This was reduced to 7.63% with the use of stiffness modification factor. This however is higher than 2.63% obtained from the use of ANSYS. These are better observed graphically.

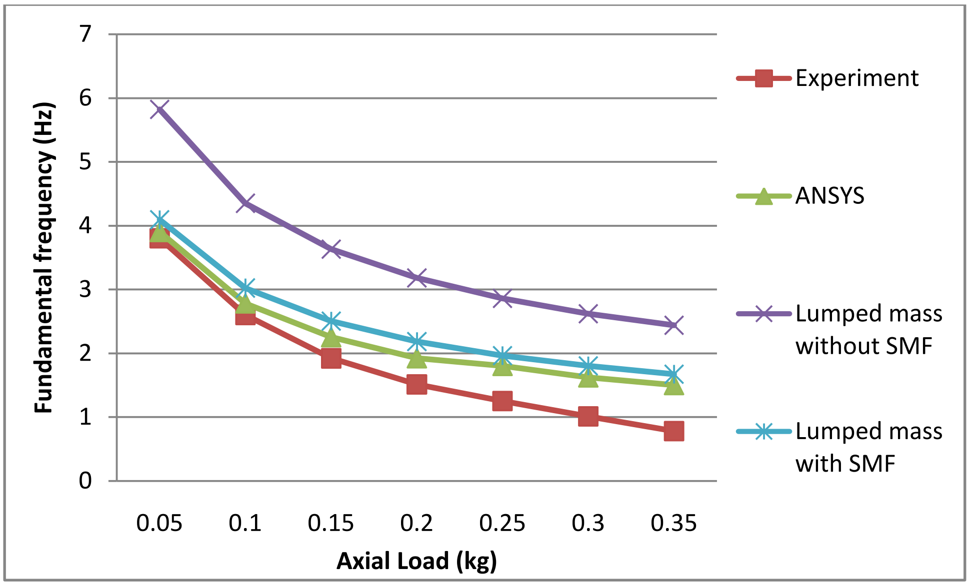


Figure 5.10: Fundamental frequency against the nodal load on frame without consideration for beam column effects

Figure 5.10 shows a graph of the fundamental frequency against the nodal loads on the frame. From the graph it would be observed that the fundamental frequencies obtained from experiment, using ANSYS, using lumped mass (without SMF) and using lumped mass (with SMF) all exhibit the same trend. The value obtained by experiment was the lowest followed by that from ANSYS and then the results from Lumped mass with SMF but finally the results from lumped mass without SMF. The stiffness modification factors (SMF) had the effect of shifting the curve for lumped mass downwards hence improving its accuracy.

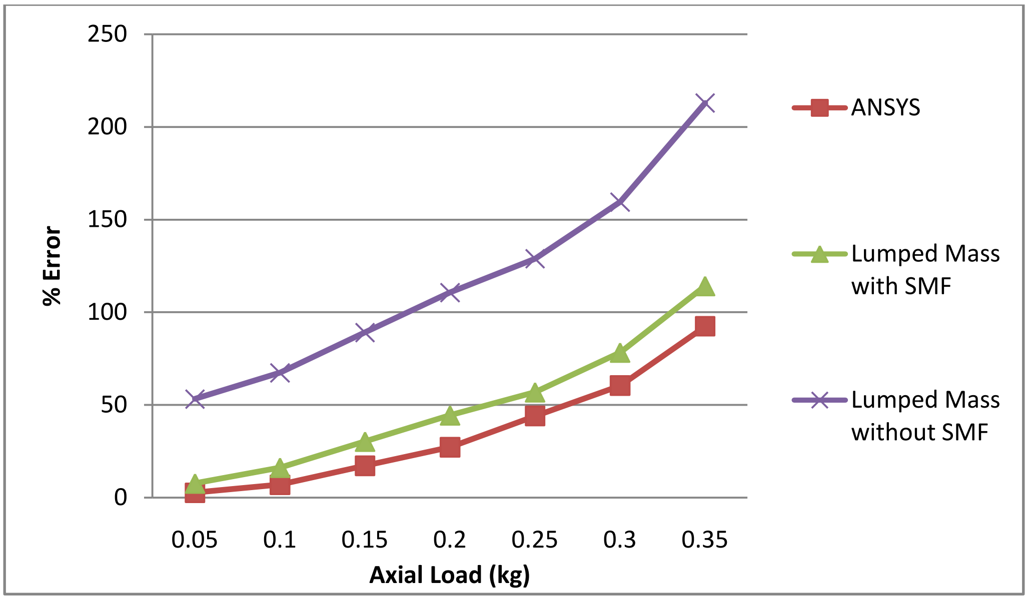


Figure 5.11: Percentage error in natural frequency against the nodal load on frame without consideration for beam column effects

This is better illustrated by Figure 5.11. From the figure it would be seen that the use of ANSYS recorded the least error for all the value of axial load. This was followed by the results from the use of lumped mass with stiffness modification factors. However at very low values of axial load the prediction from use of lumped mass with SMF and that of ANSYS are very close.

These observations were further buttressed by a statistical analysis of the results from experiment and that from lumped mass frame. Table 5.16 is the output of a t-test carried out on the results.

Table 5.16: Results of a t-test on the results of experimental studies and that of lumped mass without SMF

For rigidly fixed portal frame

Axial load	**withoutSMF**	**Experiment**
0.05	5.82	3.8
0.1	4.35	2.6
0.15	3.63	1.92
0.2	3.18	1.51
0.25	2.86	1.25
0.3	2.62	1.01
0.35	2.44	0.78

t-Test: Two-Sample Assuming Unequal Variances

	Variable 1	*Variable 2*
Mean	3.557143	1.838571
Variance	1.418157	1.115848
Observations	7	7
Hypothesized Mean Difference	0	
df	12	
t Stat	2.85636	
P(T<=t) one-tail	0.007226	
t Critical one-tail	1.782288	
P(T<=t) two-tail	0.014451	
t Critical two-tail	2.178813	

From the results it is seen that t stat is greater than t-critical hence we reject the null hypothesis which states that the mean of the two results are equal. This conclusion can also be drawn from the P-value which is 0.014 and so less than 0.05. Hence we can state that the experimental results differ significantly from the results obtained from the use of lumped mass without stiffness modification factors.

By introducing the stiffness modification factors and then comparing the results obtained with that from experimental studies we obtained the result shown in Table 5.17.

Table 5.17: Results of a t-test on the results of experimental studies and that of lumped mass with SMF

For rigidly fixed portal frame

Axial load	Experiment	withSMF
0.05	3.8	4.09
0.1	2.6	3.02
0.15	1.92	2.5
0.2	1.51	2.18
0.25	1.25	1.96
0.3	1.01	1.8
0.35	0.78	1.67

t-Test: Two-Sample Assuming Unequal Variances

	Variable 1	*Variable 2*
Mean	1.838571	2.46
Variance	1.115848	0.7267
Observations	7	7
Hypothesized Mean Difference	0	
df	11	
t Stat	-1.21124	
P(T<=t) one-tail	0.125591	
t Critical one-tail	1.795885	
P(T<=t) two-tail	0.251182	
t Critical two-tail	2.200985	

From the results of Table 5.16 it would be seen that t stat is less than t-critical hence we do not reject the null hypothesis. This means that the difference between the experimental results and that of lumped mass with SMF is not significant. This shows that the use of SMF has significantly improved the result of the analysis.

The error margin in the graph of Figure 5.11 was big because the effect of axial load on geometric stiffness was not considered. By considering beam column effect in the implementation of the calculations using ANSYS and that for lumped mass we generate a new set of fundamental frequencies that are more accurate. These are presented in Table 5.17.

Table 5.18: Results of the Analysis of the loaded portal frame with consideration for beam-column effects

Axial Load (kg)	Experiment	ANSYS		Lumped mass without SMF		Lumped mass with SMF	
	ω(Hz)	ω(Hz)	% Error	ω(Hz)	%Error	ω(Hz)	%Error
0.05	3.80	3.65	3.95	5.80	52.63	4.08	7.37
0.10	2.60	2.42	6.92	4.33	66.54	3.01	15.77
0.15	1.92	1.80	6.25	3.60	87.50	2.49	29.69
0.20	1.51	1.40	7.28	3.14	107.95	2.17	43.71
0.25	1.25	1.23	1.60	2.82	125.60	1.94	55.20
0.30	1.01	0.83	17.82	2.58	155.45	1.78	76.24
0.35	0.78	0.67	14.10	2.39	206.41	1.64	110.26

Unlike in Table 5.15, Table 5.18 contains the results of the analysis of the portal frame with consideration for beam column effect. Since the frame has slender sections the consideration of beam column effect reduced the error margin (with respect to experimental results) slightly. For the case of lumped mass the use of stiffness modification factors more than halved the error in calculated frequencies. By considering beam column effect for the case of lumped mass with SMF the error reduced slight from 7.63% to 7.37% for 0.05 kg axial load. The same trend was observed for other values of axial load.

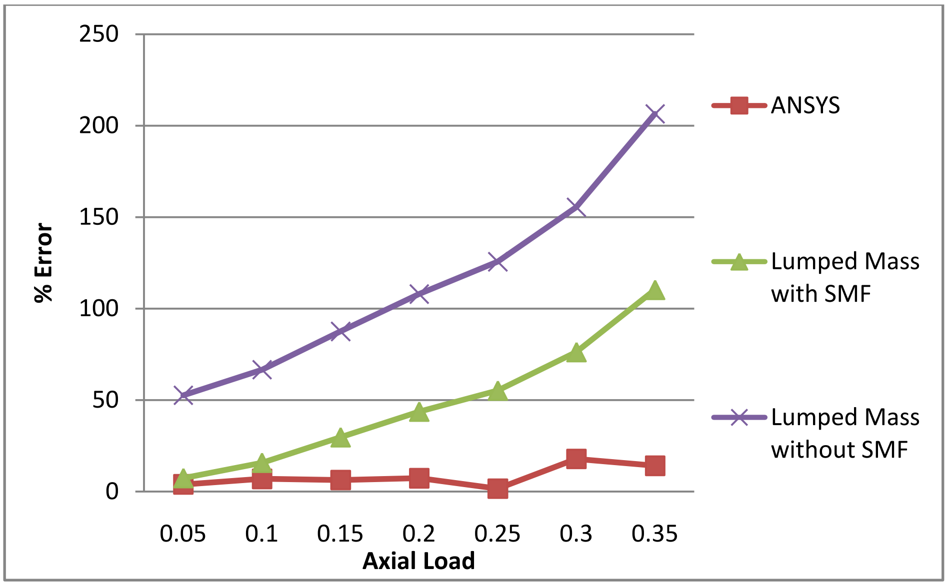


Figure 5.12: Percentage error in natural frequency against the nodal load on frame with consideration for beam column effects

From Figure 5.12 it would be observed that the percentage error in the results from ANSYS was consistently low for all values of the axial load. The result from the use of lumped mass with SMF was very low at low values of axial load but increased progressively with increase in axial load. This shows that the SMF needs some modification if it must be used for studies on bucking analysis. The same trend was shown in the use of lumped mass without stiffness modification factors. As observed earlier the use of the stiffness modification factors moved the error curve downwards and so reduced the calculated error in a uniform way.

Just as was done earlier a t-test was also carried out on the results of the analysis, first by comparing the experimental results with that of lumped mass without SMF and then with that of lumped with SMF the results are presented in Table 5.19.

Table 5.19: Results of a t-test on the results of experimental studies and that of lumped mass without and with SMF

For rigidly fixed portal frame

Axial load	withoutSMF	Experiment
0.05	5.8	3.8
0.1	4.33	2.6
0.15	3.6	1.92
0.2	3.14	1.51
0.25	2.82	1.25
0.3	2.58	1.01
0.35	2.39	0.78

t-Test: Two-Sample Assuming Unequal Variances

	Variable 1	*Variable 2*
Mean	3.522857	1.838571
Variance	1.442624	1.115848
Observations	7	7
Hypothesized Mean Difference	0	
df	12	
t Stat	2.785958	
P(T<=t) one-tail	0.008234	
t Critical one-tail	1.782288	
P(T<=t) two-tail	0.016468	
t Critical two-tail	2.178813	

For rigidly fixed portal frame

Axial load	Experiment	withSMF
0.05	3.8	4.08
0.1	2.6	3.01
0.15	1.92	2.49
0.2	1.51	2.17
0.25	1.25	1.94
0.3	1.01	1.78
0.35	0.78	1.64

t-Test: Two-Sample Assuming Unequal Variances

	Variable 1	*Variable 2*
Mean	1.838571	2.444286
Variance	1.115848	0.735895
Observations	7	7
Hypothesized Mean Difference	0	
df	12	
t Stat	-1.17768	
P(T<=t) one-tail	0.130877	
t Critical one-tail	1.782288	
P(T<=t) two-tail	0.261754	
t Critical two-tail	2.178813	

From Table 5.19 it would be seen that when the experimental results were compared with that of lumped mass without SMF that the t-stat > t-critical hence we reject the null hypothesis that states that the means of the two data are equal. This shows that there is significant difference between the results from experiment studies and that from the lumped mass portal frame. By using the SMF we generated improved results which were also compared with experimental data. The result from Table 5.19 showed that in this case t-stat < t-critical, hence we do not reject the null hypothesis. This means that the difference between the experimental results and that from the use of lumped mass with SMF is not significant. Hence we conclude that the use of the SMF significantly improved the results of the analysis.

5.4.3 Comparison of Results with that from Finite Element Analysis

a) Stiffness and mass matrices for an axial element

This is an element under longitudinal vibration. It can be taken as an element pinned at both ends. By taking the two ends of the axial member to be displaced by u_1 and u_2 and the normalized displacement at any point $\xi = \frac{x}{l}$ assumed to be a straight line. The normalized mode shapes are then

$$\varphi_1 = 1 - \xi \qquad (5.63)$$

$$\varphi_2 = \xi \qquad (5.64)$$

The expression for displacement u is the superposition of the two mode shapes

$$u = (1 - \xi)u_1 + \xi u_2 \qquad (5.65)$$

The kinetic energy of the element can be written as

$$T = \frac{1}{2}\int_0^l \dot{u}^2 \mu dx \qquad (5.66)$$

Where u is the mass per unit length of the element, $dx = ld\xi$

$$\therefore T = \frac{1}{2}\mu\int_0^1 [(1 - \xi)\dot{u}_1 + \xi\dot{u}_2]^2 ld\xi$$

$$= \frac{1}{2}\mu l\left(\frac{1}{3}\dot{u}_1^2 + \frac{1}{3}\dot{u}_1\dot{u}_2 + \frac{1}{3}\dot{u}_2^2\right) \qquad (5.67)$$

The generalized mass from the Lagrange's equation can be obtained from $\frac{d}{dt}.\frac{\partial y}{\partial \dot{u}_i}$

$$\frac{d}{dt}.\frac{\partial y}{\partial \dot{u_1}} = \frac{\mu l}{6}(2\ddot{u}_1 + \ddot{u}_2)$$

$$\frac{d}{dt}.\frac{\partial y}{\partial \dot{u_2}} = \frac{\mu l}{6}(\ddot{u}_1 + 2\ddot{u}_2) \tag{5.68}$$

Which can be written in matrix form as

$$\frac{d}{dt}.\frac{\partial y}{\partial \dot{u_i}} = \frac{\mu l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} \tag{5.69}$$

From equation (5.69) the generalized mass is obtained as

$$m = \frac{\mu l}{6}\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \tag{5.70}$$

This matrix can also be derived indirectly by substituting the shape functions φ_1 and φ_2 into the equations of kinetic energy and writing the equations for discrete elements.

The strain energy of a prismatic axial element can be written as

$$U = \int_0^l \frac{N^2}{2EA}dx = \frac{EA}{2}\int_0^l u'^2\, dx \tag{5.71}$$

Where EA is the axial rigidity and $u' = \frac{du}{dx}$

$$U = \frac{EA}{2}\int_0^l u'^2\, dx$$

$$= \frac{1}{2}\sum_i \sum_j u_i u_j \int_0^l EA\varphi_i'\varphi_j' dx$$

$$= \frac{1}{2}\sum_i \sum_j u_i u_j k_{ij} \tag{5.72}$$

From equation (5.72) k_{ij} in normalized coordinate is

$$k_{ij} = \frac{EA}{l}\int_0^1 \varphi_i'\varphi_j' d\xi \tag{5.73}$$

By substituting equations (5.63) and (5.64) into (5.73) we obtain

$$k_{11} = \frac{EA}{l}\int_0^1 \varphi_1'\varphi_1' d\xi = \frac{EA}{l}\int_0^1 (-1)^2 d\xi = \frac{EA}{l}$$

$$k_{12} = \frac{EA}{l}\int_0^1 \varphi_1'\varphi_2' d\xi = \frac{EA}{l}\int_0^1 (-1)(1) d\xi = -\frac{EA}{l}$$

$$k_{21} = k_{12} = -\frac{EA}{l}$$

$$k_{22} = \frac{EA}{l}\int_0^1 \varphi_2' \varphi_2' d\xi = \frac{EA}{l}\int_0^1 (1) d\xi = \frac{EA}{l} \tag{5.74}$$

Put in matrix form

$$\mathrm{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{5.75}$$

Equation (5.75) is the stiffness matrix of an axial element of length l.

b) <u>Stiffness and mass matrices for a beam element</u>

We consider a planar structure in which each joint will have a lateral displacement and a rotation thus each element will have four degrees of freedom.

Let the equation for deflection of the beam be a cubic polynomial expressed as

$$u = a_1 + a_2\xi + a_3\xi^2 + a_4\xi^3 \tag{5.76}$$

Where a_i = constants and $\xi = \frac{x}{l}$

By differentiating equation (5.76) we get the slope

$$lu = a_2 + 2a_3\xi + 3a_4\xi^2 \tag{5.77}$$

If we apply the boundary conditions $\xi = 0$ and $\xi = 1$ the boundary equations can be expressed in matrix form as

$$\begin{Bmatrix} u_1 \\ lu_2 \\ u_3 \\ lu_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \tag{5.78}$$

By inverting the matrix of equation (16) we obtain

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ lu_2 \\ u_3 \\ lu_4 \end{Bmatrix} \tag{5.79}$$

From equation (5.79) we can obtain the values of the constants a_i for each of the displacements equated to unity with all others equal to zero. For example for u_1 = 1 with all the other displacements equal to zero, the first column gives

$a_1 = 1, a_2 = 0, a_3 = -3 \; and \; a_4 = 2$

Substituting these into equation (5.79) gives the shape function for the first displacement u_1. The other three shape functions are obtained in a similar manner and in summary we have the following four beam shape functions

$$\varphi_1 = 1 - 3\xi^2 + 2\xi^3$$

$$\varphi_2 = l\xi - 2l\xi^2 + l\xi^3$$

$$\varphi_3 = 3\xi^2 - 2\xi^3$$

$$\varphi_4 = -l\xi^2 + l\xi^3 \qquad (5.80)$$

The displacement in general is a superposition of the four shape function and is presented as

$$u = \varphi_1 u_1 + \varphi_2 u_2 + \varphi_3 u_3 + \varphi_4 u_4 \qquad (5.81)$$

To determine the generalized mass equation (5.81) is substituted into the equation for the kinetic energy

$$T = \frac{1}{2}\int \dot{u}^2 \,\mu dx$$

$$= \frac{1}{2}\sum_i \sum_j \dot{u}_i \dot{u}_j \int \varphi_i \varphi_j \,\mu dx$$

$$= \frac{1}{2}\sum_i \sum_j m_{ij}\,\dot{u}_i \dot{u}_j \qquad (5.82)$$

The generalized mass m_{ij} which forms the elements of the mass matrix is equal to

$$m_{ij} = \int_0^l \varphi_i \varphi_j \,\mu dx = \mu l \int_0^1 \varphi_i \varphi_j \,d\xi \qquad (5.83)$$

$$m_{11} = \mu l \int_0^1 \varphi_1 \varphi_1 d\xi = \mu l \int_0^1 (1 - 3\xi^2 + 2\xi^3)^2 d\xi$$

$$= \frac{39\mu l}{105}$$

$$m_{12} = \mu l \int_0^1 \varphi_1 \varphi_2 d\xi = \mu l \int_0^1 (1 - 3\xi^2 + 2\xi^3)(l\xi - 2l\xi^2 + l\xi^3) d\xi$$

$$= \frac{11\mu l^2}{210}$$

$$m_{13} = \mu l \int_0^1 \varphi_1 \varphi_3 d\xi = \mu l \int_0^1 (1 - 3\xi^2 + 2\xi^3)(3\xi^2 - 2\xi^3) d\xi$$

$$= \frac{27\mu l}{210}$$

$$m_{14} = \mu l \int_0^1 \varphi_1\varphi_4 d\xi = \mu l \int_0^1 (1 - 3\xi^2 + 2\xi^3)(-l\xi^2 + l\xi^3)d\xi$$

$$= \frac{-13\mu l^2}{420}$$

$$m_{22} = \mu l \int_0^1 \varphi_2\varphi_2 d\xi = \mu l \int_0^1 (l\xi - 2l\xi^2 + l\xi^3)^2 d\xi$$

$$= \frac{\mu l^3}{105}$$

$$m_{23} = \mu l \int_0^1 \varphi_2\varphi_3 d\xi = \mu l \int_0^1 (l\xi - 2l\xi^2 + l\xi^3)(3\xi^2 - 2\xi^3)d\xi$$

$$= \frac{13\mu l^2}{420}$$

$$m_{24} = \mu l \int_0^1 \varphi_2\varphi_4 d\xi = \mu l \int_0^1 (l\xi - 2l\xi^2 + l\xi^3)(-l\xi^2 + l\xi^3)d\xi$$

$$= \frac{-\mu l^3}{140}$$

$$m_{33} = \mu l \int_0^1 \varphi_3\varphi_3 d\xi = \mu l \int_0^1 (3\xi^2 - 2\xi^3)^2 d\xi$$

$$= \frac{39\mu l}{105}$$

$$m_{34} = \mu l \int_0^1 \varphi_2\varphi_4 d\xi = \mu l \int_0^1 (3\xi^2 - 2\xi^3)(-l\xi^2 + l\xi^3)d\xi$$

$$= \frac{-11\mu l^2}{210}$$

$$m_{44} = \mu l \int_0^1 \varphi_4\varphi_4 d\xi = \mu l \int_0^1 (-l\xi^2 + l\xi^3)^2 d\xi$$

$$= \frac{\mu l^3}{105}$$

These give rise to the mass matrix

$$m = \frac{\mu l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{5.84}$$

This matrix is called the consistent mass because it is based on the same beam functions used for the stiffness matrix.

When a beam element is pinned at one end there is need to modify this matrix as the shape functions for such an element will no longer be the same. Because one end (the far end) is pinned. The bending moment there is zero and the shape function $\varphi_4 = 0$

Let the equation for deflection of the beam be a cubic polynomial expressed as

$$u = a_1 + a_2\xi + a_3\xi^2 \qquad (5.85)$$

Where a_i = constants and $\xi = \frac{x}{l}$

By differentiating equation (5.85) we get the slope

$$lu = a_2 + 2a_3\xi \qquad (5.86)$$

If we apply the boundary conditions $\xi = 0$ and $\xi = 1$ the boundary equations can be expressed in matrix form as

$$\begin{Bmatrix} u_1 \\ lu_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \qquad (5.87)$$

By inverting the matrix of equation (5.87) we obtain

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ lu_2 \\ u_3 \end{Bmatrix} \qquad (5.88)$$

From equation (5.88) we can obtain the values of the constants a_i for each of the displacements equated to unity with all others equal to zero. For example for $u_1 = 1$ with all the other displacements equal to zero, the first column gives

$a_1 = 1, a_2 = 0,\ and\ a_3 = -1$

Substituting these into equation (5.85) gives the shape function for the first displacement u_1. The other two shape functions are obtained in a similar manner and in summary we have the following four beam shape functions

$\varphi_1 = 1 - \xi^2$

$\varphi_2 = l\xi - l\xi^2$

$\varphi_3 = \xi^2$

$$\varphi_4 = 0 \tag{5.89}$$

The displacement in general is a superposition of the four shape functions and can be presented as

$$u = \varphi_1 u_1 + \varphi_2 u_2 + \varphi_3 u_3 + \varphi_4 u_4 \tag{5.90}$$

From equation (5.90) the generalized mass m_{ij} which forms the elements of the mass matrix is obtained as

$$m_{ij} = \int_0^l \varphi_i \varphi_j \mu dx = \mu l \int_0^1 \varphi_i \varphi_j d\xi \tag{5.91}$$

$$m_{11} = \mu l \int_0^1 \varphi_1 \varphi_1 d\xi = \mu l \int_0^1 (1-\xi^2)^2 d\xi$$

$$= \frac{8\mu l}{15}$$

$$m_{12} = \mu l \int_0^1 \varphi_1 \varphi_2 d\xi = \mu l \int_0^1 (1-\xi^2)(l\xi - l\xi^2) d\xi$$

$$= \frac{49\mu l^2}{210}$$

$$m_{13} = \mu l \int_0^1 \varphi_1 \varphi_3 d\xi = \mu l \int_0^1 (1-\xi^2)(\xi^2) d\xi$$

$$= \frac{28\mu l}{210}$$

$$m_{14} = \mu l \int_0^1 \varphi_1 \varphi_4 d\xi = \mu l \int_0^1 (1-\xi^2)(0) d\xi$$

$$= 0$$

$$m_{22} = \mu l \int_0^1 \varphi_2 \varphi_2 d\xi = \mu l \int_0^1 (l\xi - l\xi^2)^2 d\xi$$

$$= \frac{7\mu l^3}{210}$$

$$m_{23} = \mu l \int_0^1 \varphi_2 \varphi_3 d\xi = \mu l \int_0^1 (l\xi - l\xi^2)(\xi^2) d\xi$$

$$= \frac{\mu l^2}{20}$$

$$m_{24} = \mu l \int_0^1 \varphi_2 \varphi_4 d\xi = \mu l \int_0^1 (l\xi - l\xi^2)(0) d\xi$$

$$= 0$$

$$m_{33} = \mu l \int_0^1 \varphi_3 \varphi_3 d\xi = \mu l \int_0^1 (\xi^2)^2 d\xi$$

$$= \frac{\mu l}{5}$$

$$m_{34} = \mu l \int_0^1 \varphi_2 \varphi_4 d\xi = \mu l \int_0^1 (\xi^2)(0) d\xi$$

$$= 0$$

$$m_{44} = \mu l \int_0^1 \varphi_4 \varphi_4 d\xi = \mu l \int_0^1 (0)^2 d\xi$$

$$= 0$$

These give rise to the mass matrix

$$m = \frac{\mu l}{420} \begin{bmatrix} 224 & 49l & 56 & 0 \\ 49l & 14l^2 & 21l & 0 \\ 56 & 21l & 84 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.92}$$

This matrix is also a consistent mass because it is based on the same beam functions for the stiffness matrix.

c) <u>Transformation matrices</u>

In the finite element method the requirement for displacement compatibility is simplified by resolving the element displacements and forces into common coordinate system known as the global coordinate. In shorter notation we can write the transformation equations form local to global coordinates as (Leet 2002)

$$r = T\bar{r}$$

$$F = T\bar{F} \tag{5.93}$$

Where T is the transformation matrix r, F and $\bar{r}$, $\bar{F}$ are the displacement and force in the local and global coordinates respectively.

The transformation matrix for a beam element is given as

$$T = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.94}$$

Where $c = \cos\alpha$ (the cosine of angle the element makes with the global x-axis), $s = \sin\alpha$ (the sine of angle the element makes with the global x-axis).

From equations (31) and the relationship between force and displacement it can be shown that the stiffness matrix of an element in local coordinate k can be transformed to the global coordinates by the equation

$$\bar{k} = T^T kT \tag{5.95}$$

Where $\bar{k}$ is the element stiffness matrix in global coordinates, and T is the transformation matrix.

d) Finite Element Analysis of a frame

Using the Matlab software a program for the implementation of the equations developed above was written for the analysis of the portal frames of figure 5.3. The calculated results were then compared with that from the use of Lagrange equations with and without the use of stiffness modification factors. The results are present in Tables 5.20 and 5.21.

Table 5.20: A Comparison of the Estimated Fundamental frequencies for the fixed portal frame with that from the finite element method

L_2/L_1	FEM of same no of elements	Without Using Stiffness Modification factors		Using Stiffness Modification factors (SMFs)	
		Frequency(Hz)	% Error	Frequency(Hz)	% Error
8	0.4855	0.9912	-104.16	0.5693	-17.26
4	1.2339	1.4771	-19.71	1.0801	12.46
2	2.6527	2.1381	19.40	1.9719	7.78
1	4.8843	2.8983	40.66	3.4155	-0.1784
0.8	5.8092	3.1393	45.96	4.0252	-28.22
0.4	9.2651	3.7995	58.99	6.4547	-69.88
0.2	13.0333	4.2703	67.24	9.8597	-130.89

Table 5.21: A Comparison of the Estimated Fundamental frequencies for the hinged portal frame with that from the finite element method

L_2/L_1	FEM of same no of elements	Without Using Stiffness Modification factors		Using Stiffness Modification factors (SMFs)	
		Frequency(Hz)	% Error	Frequency(Hz)	% Error
8	0.2265	0.3651	-61.19	0.5430	-139.77
4	0.6553	0.6325	-1.15	1.0248	-63.89
2	1.4946	1.000	33.09	1.8513	-23.87
1	2.9948	1.4142	52.78	3.1416	-4.90
0.8	3.5915	1.5430	57.04	3.6658	-2.07

0.4	5.5575	1.8898	66.00	5.6090	-0.93
0.2	72585	2.1320	70.63	7.8905	-8.71

From tables 5.20 and 5.21 it would be seen that the results of the analysis improved with the use of the stiffness modification factors though not to the extent expected.

The results might appear to conflict with the already established norm. This is so because the results from the finite element analysis improve with increase in the number of elements. The subdivision of each member (beam or column) into a number of elements greatly enhances the results. Treating each beam or column as a single element often does not lead to accurate results.

In the case above the number of elements where made the same as that for the lumped mass frame hence accuracy of results was not emphasized. The results from the use of the Lagrange equations with modified stiffness differ from that obtained from the application of the finite element method.

e) Convergence of Results from finite element Analysis

By comparing the results from the use of Lagrange equations on lumped masses with that from the finite element method for the portal frame of figure 5.3(a) for different number of elements, table 5.22 was generated.

Table 5.22: Calculated values of the first three natural frequencies for a rigidly fixed portal frame (putting axial deformation into consideration). EA=1, EA=1, $L_1 = L_2$

	First Mode		Second Mode		Third Mode	
Number of Elements per member	Finite Element Method (Hz)	Lagrange Equations on lumped masses (Hz)	Finite Element Method (Hz)	Lagrange Equations on lumped masses (Hz)	Finite Element Method (Hz)	Lagrange Equations on lumped masses (Hz)
One	1.3064	0.9401	1.6463	1.2030	3.4251	1.6570
Two	1.2647	1.0063	1.5891	1.2427	3.0056	1.8979
Three	1.2558	1.0197	1.5762	1.2515	2.9004	1.9488
Four	1.2526	1.0245	1.5713	1.2546	2.8628	1.9673
Five	1.2510	1.0268	1.5689	1.2561	2.8454	1.9759
Six	1.2501	1.0280	1.5675	1.2570	2.8360	1.9806
Seven	1.2495	1.0287	1.5667	1.2575	2.8303	1.9835

Having already established that the results of the finite element method converge to the exact solution, when applied in frames without the use of stiffness modification factors it is observed from table 3.22 that the obtained values vary widely from that obtained from Lagrange's equation with lumped mass. This was due to the fact the stiffness matrices were generated with respect to both translation and rotation at the element ends and were never converted into translations only in the selected joints seeing that it is difficult to determine the rotational inertia of the lumped masses. This further necessitates the use of the stiffness modification factors. They help in cushioning the effect of the lumping and the neglect of the rotational inertia of the lumped masses.

Chapter 6

CONCLUSION AND RECOMMENDATION

6.1 Conclusion

Structural systems have continuous distribution of masses and are best modeled and analyzed as systems with infinite number of masses and hence infinite degrees of freedom. This can be achieved using the Hamilton's principle which allows for the analysis of structures as systems with a continuous distribution of mass. But this method has a limitation. Using the Hamilton's principle, it is very difficult to formulate the necessary differential equations for complex structural systems. This problem is partly offset by another energy theorem, the Lagrange equations. This energy theorem allows structural systems to be modeled as an assemblage of discrete masses connected by mass-less elements. The method makes it easier to formulate the relevant equations for complex structural systems as attention is now focused on only the positions of the discrete masses hence reducing the system to one of finite degrees of freedom. The solution presented by the Lagrange equations is exact for such systems, but when a continuous system is modeled as having discrete masses connected by mass-less elements the results becomes approximate. The result can however be improved by the addition of more discrete masses and by maintaining even spacing between the masses but this places a limitation on the free selection of points for lumping of masses. The core parameters of a dynamical system are the mass distribution (inertia matrix) and the structure stiffness (stiffness matrix). Any misrepresentation in either of them introduces an error in the result of the analysis. Mass discretization as seen in the use of Lagrange equations for the analysis of continuous systems introduces an error in the mass distribution. There is need to make a corresponding modification in the systems' stiffness matrix and this was the crux of this work. To achieve this, the force equilibrium equations of discrete elements of the beam had to be formulated for such systems under free vibration (using the Hamilton's principle and the principle of virtual work) and the inherent forces causing vibration obtained. This is then equated to the corresponding equation of motion of the system (knowing that the equation is actually a force equilibrium equation) and the stiffness matrix of the system necessary for such equality obtained. This was used to generate Tables of stiffness modification factors for segments of beams of different end conditions both for lateral and longitudinal vibration. By employing the Lagrange equations to lumped massed beams using these modification factors, we were able to predict accurately the fundamental frequency of the beams irrespective of the position or number of lumped mass introduced.

This has been extended to frames by treating frames as structural systems consisting of restrained beams connected together by beams. The fundamental frequencies obtained from the use of stiffness modification factors on frames when compared with that from experimental studies and from the finite element software ANSYS were observed to be approximate. They however gave a significantly improved prediction of the fundamental frequency than obtained without the use of the stiffness modification factors.

6.1.1 Contribution to Knowledge

1. In order to obtain an accurate dynamic response from lumped massed beam there must of necessity be a modification in the stiffness composition of the structure.

2. The project shows that no linear/simple modification of the stiffness distribution of the lumped massed beams can cause them to be dynamically equivalent to continuous beams.

3. The modification factors alter the equilibrium condition of each element and is dependent on the end conditions of the structural system.

4. Tables of stiffness modification factors for segments of beams of different end conditions for both longitudinal and lateral vibration are presented.

5. An extension of the use of the Tables to lumped massed frames significantly improved the results of the analysis.

6.2 Recommendation

The use of the stiffness modification factors as a compliment on continuous systems modeled as 'lumped masses connected by mass-less elements' has great potentials. It combines the accuracy of Hamilton's principle with the simplicity of the Lagrange's equations. So far accuracy was achieved in its application for lumped massed beams but results for frames are still approximate. This might be due to the rotational inertia of elements of connecting beams which were not put into consideration in calculating the stiffness modification factors. The deformation due to shearing force might also be responsible.

There is need to extend the work by incorporating the effect of shear and axial forces in the stiffness of the beam elements.

REFERENCES

Adekola A. O.(1988), *Mechanics of Statically Indeterminate Structures*. Macmillan Nigeria Publishers Ltd Ibadan, Nigeria

Ahmadian H., Friswell M. I. and Mottershead J. E. (1998), *Minimization of the discretization error in mass and stiffness formulations by an inverse method*, International Journal for Numerical Methods in Engineering, Vol. 41, pp 371 – 387.

Argyris J. H. and Kelsey S. (1960), *Energy Theorems and Structural Analysis*, Butterworths, London

Barber J. R., (1992), *Solid Mechanics & its Application Volume 12: Elasticity*. Kluwer Academic Publishers. The Netherlands

Bathe K. J. (2006), *Finite Element Procedures*, Prentice-Hall, Pearson Education Inc. USA

Boresi A. P. Schmidt R. J (2003) *Advanced Mechanics of Materials*, John Wiley and Sons Inc. USA

Boyajian D. M (2005), *The Elusive Straight Line: A Structural Engineering Example*, World Transaction on Engineering and Technology Education, Vol 4, No 1, pp 75 – 78.

Britannica (2014),Hooke's law, Physic,

www.britannica.com/EBchecked/topic/271336/Hookes-law 5th September 2013

Britannica (2014), Building construction

www.britannica.com/EBchecked/topic/83859/building_construction/60128/Manufactured-building materials#ref260135 5th September 2013

Case J., Lord Chilver, Carl T. F. Ross (1999). *Strength of Materials and Structures*. 4th Edition, Hodder Headline Group, London

Chapra S. C and Canale R. P (2007), Numerical Methods for Engineers, Fifth Edition, Tata McGraw-Hill, India.

Chen G. and Zhou J. (1993), *Vibration and Damping in distributed Systems*, Vol. 1, CRC Press Inc. USA

Chiauzzi L, Mazi A, Mucciavelli M, Cassidy J. F.,Kutyn K, Traber J, Ventura C, and Yao F (2012), *Estimation of Fundamental Period of Reinforced Concrete Buildings: Code provision vs experiment measures in Victora and Vancouver (BC, Canada)*, 15 WCEE, Lisboa.

Christian M. (2009), Explicit *local buckling analysis of stiffened composite plates accounting for periodic boundary conditions and stiffener-plate interaction*, Composite Structures, Elsevier, Vol. 91, pp 249 – 265.

Cinithia A, Umesha P. A, Nagesh R. I. (2012), *A Rational Approach for Fundamental Period of Low and Medium Rise Building frames*, International Journal of Modern Engineering Research Vol 2, Issue 5, pp 3340 – 3346

Clive L, Dym P. E., Harry E. W (2007), *Estimating Fundamental Frequencies of Tall Buildings*, Journal of Structural Engineering (ASCE), Oct. pp. 1 - 5

Clough R. W.,(1971), *Analysis of Structural Vibrations and Dynamic Response* Recent Advances in Matrix Methods of Structural Analysis and Design, University of Alabama Press, Huntsville pp 441 – 482

Crowley H., Pinho R. (2004), *Period-height Relationship for Existing European Reinforced Concrete Buildings*, Journal of Earthquake Engineering Vol. 8, Issue 1, pp 93 - 119

Das M. M., Das B. M. and Saikia M. D. (2011), Structural Analysis, PHI Learning Private Limited Delhi

Denke P. H. (1954), *A matrix method of structural analysis*, Proceedings of 2nd U.S. National Congress on Applied Mechanics, ASCE, pp.445-457,

Donaldson B. K. (2006), *Introduction to Structural dynamics,* Cambridge University Press, United Kingdom.

Duncan W. J. and Collar R. A (1934), A *method for the solution of oscillations* *problems by matrices*, Philosophical Magazine Series 7, Vol. 17, pp. 865

Duncan W. J. and Collar R. A (1935), *Matrices applied to the motions of damped systems*, Philosophical Magazine Series 7, Vol. 19, pp. 197

Eurocode 8-CEN(2004), Design of Structures for Earthquake Resistance Part 1: General Rules, seismic actions and rules for buildings, EN 1998 (1), Brussels

Faroughi S. and Ahmadian H. (2010), *Shape functions associated with the inverse element formulations*, Journal of Mechanical Engineering Science, Vol 225, pp 304 – 311.

Faroughi S., Ahmadian H. and Ghareghani A., (2011), *Shape functions associated with super-convergent mass matrix*, Inverse Problems in Science and Engineering, Taylor and Francis Group, Vol. 19, No. 8, pp 1087 – 1095

Frazer R. A. Duncan W. J. and Collar R. A (1938), *Elementary Matrices and some Applications to Dynamics and Differential Equations*, 1st edition Cambridge University Press, UK

Fried I and Chavez M (2005), *Superaccurate finite element eigenvalue computation*, Journal of Sound and Vibration, Elsevier Vol 275, pp 415 – 422.

Fried I and Leong K., (2005), *Superaccurate finite element eigenvalue via a Rayleigh* *quotient correction*, Journal of Sound and Vibration, Elsevier Vol 288, pp 375 – 386.

Gere, J. M., Goodno B. J (2012). *Mechanics of Materials*. Eighth Edition, Cengage Learning, United States of America

Gere, J. M., Timoshenko, S. P.,(1961). *Theory of Elastic Stability*. Second Edition, McGraw-Hill Kogakusha Ltd, Tokyo

Ghali A, Neville A. M., Brown T. G (2003) *Structural Analysis: A Unified Classical and Matrix Approach*. 5th Edition, Spon Press London

Graig R. R. and Kurdila A. J. (2006), *Fundamentals of Structural Dynamics*, Wiley and Sons Publishers New Jersy

Harrison H. R., Nettleton T.(1994). *Principles of Engineering Mechanics*. Second Edition, Edward Arnold, a division of Hodder Headline Plc London

Hesameddin K. P, Irfanoglu A and Hacker T. J (2015), *Effective Viscous Damping ratio in Seismic Response of Reinforced Concrete Structures*, 6th International Conference on Advances in Experimental Engineering, University of Illinois USA pg 1 - 8

Hibbeler, R. C.(2006). *Structural Analysis*. Sixth Edition, Pearson Prentice Hall, New Jersey

Houmat A. (2009), *A sector elliptic p-element applied to membrane vibrations*, Thin-Walled Structures, Elsevier, Vol 47, pp 172 – 177

Hsiao J. K. (2009), Computation of Fundamental Periods of Moment Frames using a Hand-Calculated Approach, Electronic Journal of Structural Engineering, Vol. 9, pp 16 - 28

Hutton D. V., (2004), *Fundamentals of Finite Element Analysis*, McGraw-Hill, Singapore.

International Conference of Building Officials (1997), *Universal Building Code (UBC) Volume 2, Structural Engineering Design Provisions*, International Conference of Building Officials Whittier California

Iyer R. (1993), *Error Estimation and Adaptive Refinements for Finite Element Analysis of Structures*, PhD Thesis, Indian Institute of Science, Bangalore

Iyer R., Pilani G. S and Rao T. V (2003), *Influence of Mass Representation Schemes on Vibration Characteristics of Structures*, Technical Journal: Aerospace Engineering, The Institute of Engineers Vol 84, pp 19 - 26

Jenkins, W. M.,(1990). *Structural Analysis using computers*. First edition, Longman Group Limited, Hong Kong

Kaljevic I, Patnaik S. N., Hopkins D. A., (1996), *Three-dimensional structural* *analysis by the integrated force method*, Computers and Structures. 58: 869-886

Kaneko I, Lawo H, Thierauf G (1983), *On computational procedures for the force method*. International Journal for Numerical Methods in Engineering Vol. 18 pp 1469-1495

Key S. W. and Beisinger (1971), *Transient Dynamic Analysis of Thin Shells by Finite Element Methods.* Proceedings of Third Conference Matrix Methods in Structural Mechanics, Wright-Patterson Air Force Base, Ohio

Kim K. I., (1993), *A Review of mass matrices for eigenproblems*, Computers and Structures, Vol 46, pp 1041 – 1048.

Kose M. M.(2009), Parameters affecting the Fundamental Period of RC building with Infill Walls, Engineering Structures Vol. 31, pp 93 - 102

Langefors B. (1952), *Analysis of Elastic structures by matrix coefficients, with special regard to Semimonocoque Structures*, Journal of Aeronautical Sciences, Vol 19, pp. 451–458

Leet, K. M., Uang, C.,(2005), *Fundamental of Structural Analysis*. 4th Ed. McGraw-Hill New York

Levy S. (1947), *Computation of influence coefficients for aircraft structures with* *discontinuities and sweepback*, Journal of Aeronautical Sciences, Vol 14, pp. 547–560,

Masi A. and Vona M. (2009), Experimental and Numerical Evaluation of the Fundamental period of undamaged and damaged RC framed buildings, Bull Earthquake Engineering Vol. 8, pp 643 - 656

McGuire, W., Gallagher R. H., Ziemian, R. D.(2000). *Matrix Structural Analysis*. Second Edition, John Wiley & Sons, Inc. New York

Megson T. H. G.,(2005). *Structural and Stress Analysis*, 2nd Edition, Elsevier Butterworth-Heinemann, Oxford

Meriam J. L., Kraige L. G.,(2002). *Engineering Mechanics Volume 1: Statics*, 5th Edition, John Wiley & Sons Inc. USA

Milsted M. G. and Hutchinson J. R. (1974), *Use of trigonometric terms in the finite element method with application to vibration of membranes*, Journal of Sound and Vibration, Elsevier, Vol 32, pp 327 – 346.

Nash, W. A, Potter M. C.(2014). *Schaum's Outline of Theory and Problems of Strength of Materials*. Sixth Edition, McGraw-Hill Companies, New York

Nassani D. E. (2013), A Simple Model for Calculating the Fundamental Period of Vibration in Steel Structures, APCBEE Procedia 9, Elsevier , pp 339 - 346

Nandi S. K. and Bosu S (2004), Effect of Mass Matrix Formulation Schemes on Dynamics of Structures, 2004 International ANSYS Conference Proceeding, www.ansys.com, paper 50

Numayr S. K. and Al Rjoub S. Y. (2012), Comments of the Understanding of the Virtual Work Method, International Journal of Civil and Structural Engineering, Vol. 2, Issue 3, pp 763 – 774

Okonkwo V. O. (2012), Analysis of Multi-storey steel frames, Unpublished MEng Thesis, Nnamdi Azikwe University, Awka

Onyeyili I. O. (2003), Analysis of Statically Determinate Structure, El'Demak Publishers, Enugu

Onyeyili I. O and Okonkwo V. O. (2012), *Formulation of the Equilibrium Equations of transversely loaded Elements taking shear deformation into consideration*, International Journal of Engineering Science and Technology, Vol 4, Issue 4, pp 1812 – 1827

Osadebe N. N. (1999), *An Improved MDOF Model Simulating some System with Distributed Mass*, Journal of University of Science and Technology Kumasi, Vol 19, pp 56 – 61

Osadebe N. N (2011), *Lecture Notes on Stability of Structures*, M. Eng Degree programme, Nnamdi Azikwe University, Awka

Padmanabhan S. and Rajasekaran S. (1989), *Equations of Curved Beams*, Journal of Engineering Mechanics, Vol 115, Issue 5

Patnaik S N (1986) *The variational energy formulation for the integrated force method*. AIAA Journal Vol 24 pp 129-137

Patnaik S. N., Hopkins D. A and Coroneos R. (1996), *Structural optimization with approximate sensitivities*. Computers and Structures 58: 407-418

Patnaik S N, Yadagiri S (1976), *Frequency analysis of structures*. Computer Methods in Applied Mechanics and Engineering Vol 9 pp 245-265

Pinto F. and Prato C. A (2006), *Three Dimensional Indirect Boundry Element Method Formulation for Dynamic analysis of frames buried in semi-infinite Elastic Media*, Journal of Engineering Mechanics, Vol. 132, No 9, pp 967-978

Rajasekaran S. (2009), *Structural Dynamics of Earthquake Engineering, Theory and Application Using Mathematica and Matlab*, Woodhead Publishing Limited Cambridge, UK

Rajasekaran S and Sankarasubramanian (2001), *Computational Structural Mechanics*, PHI Learning Pvt Ltd, New Delhi, India

Rand T. (1951) *An approximate method for computation of stresses in sweptback wings*, Journal of Aeronautical Sciences, Vol 18, pp.61–63

Rao S. S. (2005), *The Finite Element Method in Engineering*, 4th Ed., Elsevier Butterworth-Heinemann U.S.A

Robinson J, Haggenmacher G. W. (1971) *Some new developments in matrix force analysis*. Recent Advances in matrix methods of structural analysis and design (University of Alabama) pp. 183-228

Saikat, B., (2001). *Tips and Tricks for Computer-Aided Structural Analysis*, Ensel Software, India

Samuelsson A., Zienkiewicz O. C.(2006), Review: History of the Stiffness Method. *International Journal for Numerical Methods in Engineering*. Vol. 67: 149 – 157

Segal and Val (2006), *Energy Evaluation for Ramberg-Osgood Hysteretic Model*, Journal of Engineering Mechanics, Vol 132 No. 9 pp 907 – 913.

Shigley J. E. and Mischke C. R. and Budynas R. G (2004), *Mechanical Engineering Design*, McGraw-Hill Professional pp 192

Stroud K. A., Booth D. J,(2007). *Engineering Mathematics*. Fourth Edition, Palgrave Macmillan Press Ltd, London

Sule S. (2011), *Approximate Method for the determination of natural frequencies of a*

Multi-degree of freedom beam system, Nigerian Journal of Technology, Vol 30, No 2

Supreeth A. R, Rekha B, Prema K. W. P, Shijina K. (2015), *Experimental and Numerical Studies on Free Vibration Characteristics of a Three-storied Building Frame*, International Journal of Engineering Research and Technology, Vol. 4, Issue 5, pp 241 - 245

Tauchert T. R. (1974), *Energy Principles in Structural Mechanics*, International Student Edition, McGraw-Hill Kogakusha Ltd Tokyo

Thomson W. T. and Dahleh M. D. (1998), *Theory of Vibrations with Applications*, 5th Edition, Prentice Hall New Jersey

Timoshenko S. P.(1953) *History of Strength of Materials*, McGraw-Hill, New York

Topou A. (1979), *A contribution to the systematic analysis of finite element structures using the force method*. Unpublished PhD dissertation, Essen University, Essen, Germany

Turner M. J. (1959) *The direct stiffness method of structural analysis*, Structural and Materials Panel Paper, AGARD Meeting, Aachen, Germany.

Turner M. J, Martin H. C and Weikel R. C (1964) *Further development and* *applications of the stiffness method, AGARDograph 72: Matrix Methods of Structural Analysis*, ed. by B. M. Fraeijs de Veubeke, Pergamon Press, Oxford, pp. 203–266, 1964.

Virgin L. N. and Lyman T. C. (2011), *Dynamics of Civil Structures*, Conference Proceedings of the Society for Experimental Mechanics, The Society for Experimental Mechanics, Vol. 4, Series 13, pp 119-124

Vitor Dias da Silva,(2006). *Mechanics and Strength of Material*. Springer-Verlag Heidelberg New York

Washizu K. (1971), *Some Remarks on Basic Theory for Finite Element Method*. Recent Advances in Matrix Methods of Structural Analysis and Design, University of Alabama Press, Huntsville pp 25 – 46

Wehle L. B. and Lansing W. (1952) *A method for reducing the analysis of complex redundant structures to a routine procedure*, Journal of Aeronautical Sciences, Vol 19, pp. 677-684,

Wells M.,(2010). *Engineers: A history of Engineering and Structural Design*. Routledge New York

Wikipedia (2014), History of Structural Engineering, en.wikipedia.org/wiki/History_of_structural_engineering 17th May 2014

Wikipedia (2014), Euler-Bernoulli beam theory, en.wikipedia.org/wiki/Euler-Bernoulli_beam_theory 17th May 2014

Wilson, E. L.,(2002). *Three Dimensional Static and Dynamic Analysis of Structures: A Physical Approach with Emphasis on Earthquake Engineering*. Third Edition, Computers and Structures Inc. California, USA

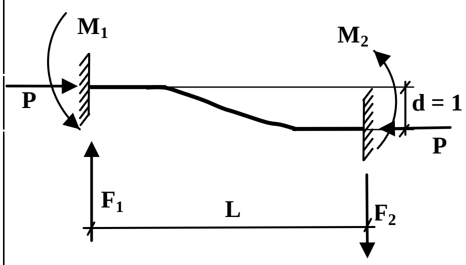
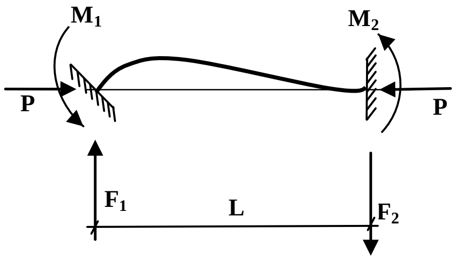
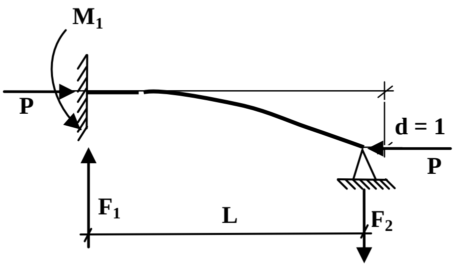
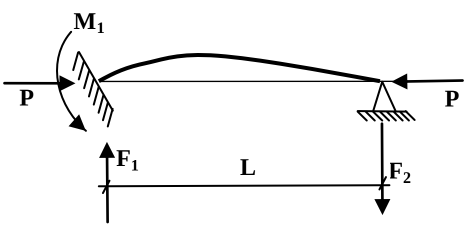
Williams M. (2016), *Structural Dynamics*, C. R. C Press Inc. USA

Wong K. K. F and Zhao D. (2007), *Uncoupling of Potential Energy in Non-linear Seismic Analysis of framed structures*, Journal of Engineering Mechanics, Vol 133, No. 10 pp 1061-1071

Zienkiwicz O. C., Taylor R. L. and Zhu Z., J. (2013), *The Finite Element Method: Its Basics and Fundamentals,* 7th edition, Butterworth Heinemann, New York.

APPENDIX A

Table A1: End forces caused by unit end displacement of prismatic element under axial forces.

S/No	Beam	Force
1		$F_1 = \frac{12EI\psi_2}{L^3}$, $F_2 = \frac{12EI\psi_2}{L^3}$ $M_1 = \frac{6EI\psi_1}{L^2}$, $M_2 = \frac{6EI\psi_1}{L^2}$
2		$F_1 = \frac{6EI\psi_1}{L^2}$, $F_2 = \frac{6EI\psi_1}{L^2}$ $M_1 = \frac{4EI\psi_3}{L}$, $M_2 = \frac{2EI\psi_4}{L}$
3		$F_1 = \frac{3EI\psi_6}{L^3}$, $F_2 = \frac{3EI\psi_6}{L^3}$ $M_1 = \frac{3EI\psi_5}{L^2}$
4		$F_1 = \frac{3EI\psi_5}{L^2}$, $F_2 = \frac{3EI\psi_5}{L^2}$ $M_1 = \frac{3EI\psi_5}{L}$
$\psi_1 = \frac{z^2(1-\cos z)}{6(2-2\cos z - z\sin z)}$ $\psi_2 = \frac{z^3 \sin z}{12(2-2\cos z - z\sin z)}$ $\psi_3 = \frac{z(\sin z - z\cos z)}{4(2-2\cos z - z\sin z)}$ $\psi_2 = \frac{z(z-\sin z)}{2(2-2\cos z - z\sin z)}$		$\psi_5 = \frac{z^2 \sin z}{3(\sin z - z\cos z)}$ $\psi_6 = \frac{z^3 \cos z}{3(\sin z - z\cos z)}$ $z = kL$ $k = \sqrt{\frac{P}{EI}}$

Source: Osadede (2011)

APPENDIX B

```
%Find the stiffness factor for a segment of a fixed fixed bar under free
%longitudinal vibration

e1 = 0
e2 = 0.9

for j=1:1:9
    Aj=2*(2-2*(-1)^j)/(j*j*j*pi*pi*pi);
    F1j= Aj*(j*pi*e2*cos(j*pi*e1) - j*pi*e1*cos(j*pi*e1) -...
        sin(j*pi*e2)+sin(j*pi*e1))/(e2-e1);
    F2j= Aj*(-j*pi*cos(j*pi*e2) + j*pi*cos(j*pi*e1)- (j*pi*e2*cos(j*pi*e1)-
...
        j*pi*e1*cos(j*pi*e1) - sin(j*pi*e2)+sin(j*pi*e1))/(e2-e1));

    u1j = Aj*sin(j*pi*e1);
    u2j = Aj*sin(j*pi*e2);
    if j==1
        u11 = u1j;
        u21 = u2j;
    end

    col1(j,1)=Aj;
    col2(j,1)=F1j;
    col3(j,1)=F2j;
    col4(j,1)=u1j;
    col5(j,1)=u2j;


end
Col1 = sum(col1);
Col2 = sum(col2);
Col3 = sum(col3);
Col4 = sum(col4);
Col5 = sum(col5);


if e1==0 && e2<1

    format long;
    Table=[col1 col2 col3 col4 col5 ;Col1 Col2 Col3 Col4 Col5];
    F1 =- Col2;
    F2 = -Col3;
    u1 = Col4;
    u2 = Col5;

    P1 = F1+(u1-u2)/(e2-e1);
    P2 = F2+(-u1+u2)/(e2-e1);
    Q1 = ((e2-e1)*u21*(P2+pi*pi*(e2-e1)*u21/2) - ...
         (e2-e1)*u11*(P1+pi*pi*(e2-e1)*u11/2))/(u21*u21-u11*u11)
    Q2 = ((e2-e1)*u11*(P2+pi*pi*(e2-e1)*u21/2) - ...
         (e2-e1)*u21*(P1+pi*pi*(e2-e1)*u11/2))/(u21*u21-u11*u11)
    format short;

elseif e1>0 && e2<1
    format long;
    Table=[col1 col2 col3 col4 col5 ;Col1 Col2 Col3 Col4 Col5 ];
```

```
    F1 = -Col2;
    F2 = -Col3;
    u1 = Col4;
    u2 = Col5;
    P1 = F1+(u1-u2)/(e2-e1);
    P2 = F2+(-u1+u2)/(e2-e1);
    Q1 = ((e2-e1)*u21*(P2+pi*pi*(e2-e1)*u21/2) - ...
        (e2-e1)*u11*(P1+pi*pi*(e2-e1)*u11/2))/(u21*u21-u11*u11)
    Q2 = ((e2-e1)*u11*(P2+pi*pi*(e2-e1)*u21/2) -...
        (e2-e1)*u21*(P1+pi*pi*(e2-e1)*u11/2))/(u21*u21-u11*u11)

    format short;

else
    format long;
    Table=[col1 col2 col3 col4 col5;Col1 Col2 Col3 Col4 Col5];
    F1 = -Col2;
    F2 = -Col3;
    u1 = Col4;
    u2 = Col5;
    P1 = F1+(u1-u2)/(e2-e1);
    P2 = F2+(-u1+u2)/(e2-e1);
    Q1 = ((e2-e1)*u21*(P2+pi*pi*(e2-e1)*u21/2) - ...
        (e2-e1)*u11*(P1+pi*pi*(e2-e1)*u11/2))/(u21*u21-u11*u11)
    Q2 = ((e2-e1)*u11*(P2+pi*pi*(e2-e1)*u21/2) - ...
        (e2-e1)*u21*(P1+pi*pi*(e2-e1)*u11/2))/(u21*u21-u11*u11)

    format short;
end
```

APPENDIX C

```
%Find the stiffness mod. factor for a segment of a fixed free bar under
free
%vibration

e1 = 0
e2 = 1

for j=1:1:9
    i = j*2-1;
    Aj = 16/(i*i*i*pi*pi*pi);

    F1j= Aj*(i*pi*e2/2*cos(i*pi*e1/2) - i*pi*e1/2*cos(i*pi*e1/2) ...
        - sin(i*pi*e2/2)+sin(i*pi*e1/2))/(e2-e1);
    F2j= Aj*(-i*pi/2*cos(i*pi*e2/2) + i*pi/2*cos(i*pi*e1/2)- ...
        (i*pi*e2/2*cos(i*pi*e1/2) - i*pi*e1/2*cos(i*pi*e1/2) -...
        sin(i*pi*e2/2)+sin(i*pi*e1/2))/(e2-e1));

    u1j = Aj*sin(i*pi*e1/2);
    u2j = Aj*sin(i*pi*e2/2);
    if j==1
        u11 = u1j;
        u22 = u2j;
    end

    col1(j,1)=Aj;
    col2(j,1)=F1j;
    col3(j,1)=F2j;
    col4(j,1)=u1j;
    col5(j,1)=u2j;

end
Col1 = sum(col1);
Col2 = sum(col2);
Col3 = sum(col3);
Col4 = sum(col4);
Col5 = sum(col5);

if e1==0 && e2<1

    format long;
    Table=[col1 col2 col3 col4 col5;Col1 Col2 Col3 Col4 Col5];
    F1 =- Col2;
    F2 = -Col3;
    u1 = Col4;
    u2 = Col5 ;
    P1 = F1+(u1-u2)/(e2-e1);
    P2 = F2+(-u1+u2)/(e2-e1);
    Q1 = ((e2-e1)*u22*(P2+pi*pi*(e2-e1)*u22/8) - ...
        (e2-e1)*u11*(P1+pi*pi*(e2-e1)*u11/8))/(u22*u22-u11*u11)
    Q2 = ((e2-e1)*u11*(P2+pi*pi*(e2-e1)*u22/8) - ...
        (e2-e1)*u22*(P1+pi*pi*(e2-e1)*u11/8))/(u22*u22-u11*u11)
    format short;

elseif e1>0 && e2<1
    format long;
    Table=[col1 col2 col3 col4 col5 ;Col1 Col2 Col3 Col4 Col5 ];
    F1 = -Col2;
```

```
    F2 = -Col3;
    u1 = Col4;
    u2 = Col5;
    P1 = F1+(u1-u2)/(e2-e1);
    P2 = F2+(-u1+u2)/(e2-e1);
    Q1 = ((e2-e1)*u22*(P2+pi*pi*(e2-e1)*u22/8) - ...
        (e2-e1)*u11*(P1+pi*pi*(e2-e1)*u11/8))/(u22*u22-u11*u11)
    Q2 = ((e2-e1)*u11*(P2+pi*pi*(e2-e1)*u22/8) - ...
        (e2-e1)*u22*(P1+pi*pi*(e2-e1)*u11/8))/(u22*u22-u11*u11)
    format short;

else
    format long;
    Table=[col1 col2 col3 col4 col5;Col1 Col2 Col3 Col4 Col5];
    F1 = -Col2;
    F2 = -Col3;
    u1 = Col4;
    u2 = Col5 ;
    P1 = F1+(u1-u2)/(e2-e1);
    P2 = F2+(-u1+u2)/(e2-e1);
    Q1 = ((e2-e1)*u22*(P2+pi*pi*(e2-e1)*u22/8) - ...
        (e2-e1)*u11*(P1+pi*pi*(e2-e1)*u11/8))/(u22*u22-u11*u11)
    Q2 = ((e2-e1)*u11*(P2+pi*pi*(e2-e1)*u22/8) - ...
        (e2-e1)*u22*(P1+pi*pi*(e2-e1)*u11/8))/(u22*u22-u11*u11)
    format short;
end
```

APPENDIX D

```
%Find the stiffness mod. factor for a segment of a fixed fixed beam under
% free vibration

e1 = 0.2;
e2 = 0.6;
BL(1) = 4.7300408;
BL(2) = 7.8532047;
BL(3) = 10.99560784;
BL(4) = 14.1371655;
BL(5) = 17.27875965739948;
BL(6) = 20.42035224562606;
BL(7) = 23.56194490204046;
BL(8) = 26.70353755550819;
BL(9) = 29.84513020910326;
BL(10) = 32.98672286269282;
BL(11) = 36.12831551628263;
BL(12) = 39.26990816987242;
w = BL(1)*BL(1);
for j=1:1:7
    Bl=BL(j);
    pj=Bl*Bl;
    aj=(cos(Bl) - cosh(Bl))/(sinh(Bl) - sin(Bl));

    C1 = 1;
    C2 = (cos(Bl)-cosh(Bl))/(sinh(Bl)-sin(Bl));
    C3 = -1;
    C4 = (cosh(Bl)-cos(Bl))/(sinh(Bl)-sin(Bl));
    Mj = 0.5*(C1*C1*(1+sinh(2*Bl)/(2*Bl)) + C2*C2*(sinh(2*Bl)/(2*Bl)-1)
+...
        C3*C3*(1+sin(2*Bl)/(2*Bl)) + C4*C4*(1-sin(2*Bl)/(2*Bl)) + ...
        2*C1*C2*(sinh(Bl)*sinh(Bl)/Bl) + 2*C1*C3*(cosh(Bl)*sin(Bl)+...
        sinh(Bl)*cos(Bl))/Bl...
        + 2*C1*C4*(1-cos(Bl)*cosh(Bl)+sin(Bl)*sinh(Bl))/Bl + ...
        2*C2*C3*(sinh(Bl)*sin(Bl)+cosh(Bl)*cos(Bl)-1)/Bl...
        + 2*C2*C4*(cosh(Bl)*sin(Bl)-sinh(Bl)*cos(Bl))/Bl + ...
        2*C3*C4*(sin(Bl)*sin(Bl)/Bl));
    Aj= -2*(sinh(Bl)+sin(Bl))/(Bl*Bl*Bl) + 12*(cosh(Bl)-...
        cos(Bl))/(Bl*Bl*Bl*Bl) - 24*(sinh(Bl)-sin(Bl))/(Bl*Bl*Bl*Bl*Bl) +
...
        aj*(-2*(cosh(Bl)-cos(Bl))/(Bl*Bl*Bl) + 12*(sinh(Bl)-...
        sin(Bl))/(Bl*Bl*Bl*Bl) - 24*(cosh(Bl)+cos(Bl)-2)/(Bl*Bl*Bl*Bl*Bl));
    Aj = Aj/Mj;

    W1 = Bl*Bl*Bl*((e2+e1)/2*(e2*e2-e1*e1)-2*(e2*e2*e2-e1*e1*e1)/3)*...
         (-sinh(Bl*e1)-C2*cosh(Bl*e1)+sin(Bl*e1)+ C4*cos(Bl*e1))+...
        C2*(Bl*(e1-e2)*cosh(Bl*e2)+2*sinh(Bl*e2)-Bl*(e2-e1)*cosh(Bl*e1)...
        -2*sinh(Bl*e1)) - ...
        C4*(-Bl*(e1-e2)*cos(Bl*e2)-2*sin(Bl*e2)+Bl*(e2-e1)*cos(Bl*e1)+...
        2*sin(Bl*e1)) + Bl*(e1-e2)*sinh(Bl*e2) -...
        Bl*(e2-e1)*sinh(Bl*e1)+Bl*((e1-e2))*sin(Bl*e2)-Bl*(e2-e1)*...
        sin(Bl*e1)+2*cosh(Bl*e2)-2*cosh(Bl*e1)-2*cos(Bl*e2)+2*cos(Bl*e1);

    F1j = -6*Aj*W1/(e2-e1)^3;

    W2 = Bl*Bl*Bl*((2*e2+e1)/2*(e2*e2-e1*e1)-(e2*e2*e2-e1*e1*e1))*...
         (-sinh(Bl*e1)-C2*cosh(Bl*e1)+sin(Bl*e1)+ C4*cos(Bl*e1))+...
        Bl*Bl*Bl*e1*(e2-e1)*(e2-e1)/2*(sinh(Bl*e1)+C2*cosh(Bl*e1)...
        -sin(Bl*e1)-C4*cos(Bl*e1)) - ...
```

```
        Bl*Bl*(e2-e1)*(e2-e1)/2*(cosh(Bl*e1)+C2*sinh(Bl*e1)+...
        cos(Bl*e1)-C4*sin(Bl*e1)) + ...
        Bl*(e1-e2)*sinh(Bl*e2)-2*Bl*(e2-e1)*sinh(Bl*e1)+...
        Bl*(e1-e2)*sin(Bl*e2) - ...
        2*Bl*(e2-e1)*sin(Bl*e1)-3*cos(Bl*e2)+3*cos(Bl*e1)+...
        3*cosh(Bl*e2)-3*cosh(Bl*e1) + ...
        C2*(Bl*(e1-e2)*cosh(Bl*e2)-2*Bl*(e2-e1)*cosh(Bl*e1)+...
        3*sinh(Bl*e2)-3*sinh(Bl*e1)) - ...
        C4*(-Bl*(e1-e2)*cos(Bl*e2)+2*Bl*(e2-e1)*cos(Bl*e1)-3*sin(Bl*e2)+...
        3*sin(Bl*e1));

    F2j = -2*Aj*W2/(e2-e1)^2;

    F3j = Aj/(e2-e1)^3*(6*W1 + Bl*Bl*Bl*(e2-e1)^3*(sinh(Bl*e2)-sinh(Bl*e1)-
...
        sin(Bl*e2)+sin(Bl*e1) + C2*(cosh(Bl*e2)-cosh(Bl*e1)) - ...
        C4*(cos(Bl*e2)-cos(Bl*e1))));

    F4j = Aj/(e2-e1)^2*(-6*W1 + 2*W2 - Bl*Bl*(e2-e1)^2*(Bl*e2*...
        (-sinh(Bl*e1)-C2*cosh(Bl*e1)+sin(Bl*e1)+C4*cos(Bl*e1)) +...
        Bl*e1*(sinh(Bl*e1)+C2*cosh(Bl*e1)-sin(Bl*e1)-C4*cos(Bl*e1)) + ...
        (cosh(Bl*e2)+C2*sinh(Bl*e2)+cos(Bl*e2)-C4*sin(Bl*e2)) -...
        (cosh(Bl*e1)+C2*sinh(Bl*e1)+cos(Bl*e1)-C4*sin(Bl*e1))));

    u1j = Aj*(cosh(Bl*e1)+C2*sinh(Bl*e1)-cos(Bl*e1)+C4*sin(Bl*e1));
    u2j = Aj*Bl*(sinh(Bl*e1)+C2*cosh(Bl*e1)+sin(Bl*e1)+C4*cos(Bl*e1));
    u3j = Aj*(cosh(Bl*e2)+C2*sinh(Bl*e2)-cos(Bl*e2)+C4*sin(Bl*e2));
    u4j = Aj*Bl*(sinh(Bl*e2)+C2*cosh(Bl*e2)+sin(Bl*e2)+C4*cos(Bl*e2));
    if j==1
        u11 = u1j;
        u22 = u2j;
        u33 = u3j;
        u44 = u4j;
    end

    col1(j,1)=Aj;
    col2(j,1)=F1j;
    col3(j,1)=F2j;
    col4(j,1)=F3j;
    col5(j,1)=F4j;
    col6(j,1)=u1j;
    col7(j,1)=u2j;
    col8(j,1)=u3j;
    col9(j,1)=u4j;


end
Col1 = sum(col1);
Col2 = sum(col2);
Col3 = sum(col3);
Col4 = sum(col4);
Col5 = sum(col5);
Col6 = sum(col6);
Col7 = sum(col7);
Col8 = sum(col8);
Col9 = sum(col9);

if e1>=0 && e2>=0
```

```
    format long;
    Table=[col1 col2 col3 col4 col5 col6 col7 col8 col9;Col1 Col2 Col3 ...
        Col4 Col5 Col6 Col7 Col8 Col9];
    F1 = -Col2;
    F2 = -Col3;
    F3 = -Col4;
    F4 = -Col5;
    u1 = Col6;
    u2 = Col7;
    u3 = Col8;
    u4 = Col9;
    P1 = F1 + 12*u1/(e2-e1)^3 + 6*u2/(e2-e1)^2 - 12*u3/(e2-e1)^3 +...
        6*u4/(e2-e1)^2;
    P2 = F2 + 6*u1/(e2-e1)^2 + 4*u2/(e2-e1) - 6*u3/(e2-e1)^2 + 2*u4/(e2-
e1);
    P3 = F3 - 12*u1/(e2-e1)^3 - 6*u2/(e2-e1)^2 + 12*u3/(e2-e1)^3 -...
        6*u4/(e2-e1)^2;
    P4 = F4 + 6*u1/(e2-e1)^2 + 2*u2/(e2-e1) - 6*u3/(e2-e1)^2 + 4*u4/(e2-
e1);


    A = [12*u11/(e2-e1)^3 6*u22/(e2-e1)^2 -12*u33/(e2-e1)^3 6*u44/(e2-
e1)^2; ...
        4*u22/(e2-e1) 6*u11/(e2-e1)^2 2*u44/(e2-e1) -6*u33/(e2-e1)^2;...
        12*u33/(e2-e1)^3 -6*u44/(e2-e1)^2 -12*u11/(e2-e1)^3 -6*u22/(e2-
e1)^2;...
        4*u44/(e2-e1) -6*u33/(e2-e1)^2 2*u22/(e2-e1) 6*u11/(e2-e1)^2];

    P = [P1+0.5*(e2-e1)*w*w*u11; P2; P3+0.5*(e2-e1)*w*w*u33; P4];
    Q = A\P

    format short;
end
```

APPENDIX E

```
%Find the stiffness factor for a segment of a fixed pinned beam under free
%vibration

e1 = 0.95
e2 = 1
BL(1) = 3.92660232;
BL(2) = 7.06858275;
BL(3) = 10.210176123;
BL(4) = 13.35176878;
BL(5) = 16.49336143135;
BL(6) = 19.634954084937;
BL(7) = 22.77654673853;
BL(8) = 25.91813939211580;
BL(9) = 29.05973204570559;
w = BL(1)*BL(1);
for j=1:1:7
    Bl=BL(j);
    pj=Bl*Bl;
    aj=(cos(Bl) - cosh(Bl))/(sinh(Bl) - sin(Bl));

    C1 = 1;
    C2 = (cos(Bl)-cosh(Bl))/(sinh(Bl)-sin(Bl));
    C3 = -1;
    C4 = (cosh(Bl)-cos(Bl))/(sinh(Bl)-sin(Bl));
    Mj = 0.5*(C1*C1*(1+sinh(2*Bl)/(2*Bl)) + C2*C2*(sinh(2*Bl)/(2*Bl)-1)
+...
        C3*C3*(1+sin(2*Bl)/(2*Bl)) + C4*C4*(1-sin(2*Bl)/(2*Bl)) + ...
        2*C1*C2*(sinh(Bl)*sinh(Bl)/Bl) + 2*C1*C3*(cosh(Bl)*sin(Bl)+...
        sinh(Bl)*cos(Bl))/Bl...
        + 2*C1*C4*(1-cos(Bl)*cosh(Bl)+sin(Bl)*sinh(Bl))/Bl + ...
        2*C2*C3*(sinh(Bl)*sin(Bl)+cosh(Bl)*cos(Bl)-1)/Bl...
        + 2*C2*C4*(cosh(Bl)*sin(Bl)-sinh(Bl)*cos(Bl))/Bl + ...
        2*C3*C4*(sin(Bl)*sin(Bl)/Bl));
    Aj= -(cosh(Bl)+cos(Bl))/(Bl*Bl) + 18*(cosh(Bl)-
cos(Bl))/(Bl*Bl*Bl*Bl)...
        - 48*(sinh(Bl)-sin(Bl))/(Bl*Bl*Bl*Bl*Bl) + ...
        aj*(-(sinh(Bl)+sin(Bl))/(Bl*Bl) + 18*(sinh(Bl)-...
        sin(Bl))/(Bl*Bl*Bl*Bl) - 48*(cosh(Bl)+cos(Bl)-2)/(Bl*Bl*Bl*Bl*Bl));
    Aj = Aj/Mj;

    W1 = Bl*Bl*Bl*((e2+e1)/2*(e2*e2-e1*e1)-2*(e2*e2*e2-e1*e1*e1)/3)*...
         (-sinh(Bl*e1)-C2*cosh(Bl*e1)+sin(Bl*e1)+ C4*cos(Bl*e1))+...
        C2*(Bl*(e1-e2)*cosh(Bl*e2)+2*sinh(Bl*e2)-Bl*(e2-e1)*...
        cosh(Bl*e1)-2*sinh(Bl*e1)) - ...
        C4*(-Bl*(e1-e2)*cos(Bl*e2)-2*sin(Bl*e2)+Bl*(e2-e1)*cos(Bl*e1)+...
        2*sin(Bl*e1)) + Bl*(e1-e2)*sinh(Bl*e2) -...
        Bl*(e2-e1)*sinh(Bl*e1)+Bl*((e1-e2))*sin(Bl*e2)-Bl*(e2-e1)*...
        sin(Bl*e1)+2*cosh(Bl*e2)-2*cosh(Bl*e1)-2*cos(Bl*e2)+2*cos(Bl*e1);

    F1j = -6*Aj*W1/(e2-e1)^3;

    W2 = Bl*Bl*Bl*((2*e2+e1)/2*(e2*e2-e1*e1)-(e2*e2*e2-e1*e1*e1))*...
         (-sinh(Bl*e1)-C2*cosh(Bl*e1)+sin(Bl*e1)+ C4*cos(Bl*e1))+...
        Bl*Bl*Bl*e1*(e2-e1)*(e2-e1)/2*(sinh(Bl*e1)+C2*cosh(Bl*e1)-...
        sin(Bl*e1)-C4*cos(Bl*e1)) - ...
        Bl*Bl*(e2-e1)*(e2-e1)/2*(cosh(Bl*e1)+C2*sinh(Bl*e1)+cos(Bl*e1)-...
        C4*sin(Bl*e1)) + ...
        Bl*(e1-e2)*sinh(Bl*e2)-2*Bl*(e2-e1)*sinh(Bl*e1)+Bl*...
         (e1-e2)*sin(Bl*e2) - ...
```

```
        2*Bl*(e2-e1)*sin(Bl*e1)-3*cos(Bl*e2)+3*cos(Bl*e1)+...
        3*cosh(Bl*e2)-3*cosh(Bl*e1) + ...
        C2*(Bl*(e1-e2)*cosh(Bl*e2)-2*Bl*(e2-e1)*cosh(Bl*e1)+...
        3*sinh(Bl*e2)-3*sinh(Bl*e1)) - ...
        C4*(-Bl*(e1-e2)*cos(Bl*e2)+2*Bl*(e2-e1)*cos(Bl*e1)-...
        3*sin(Bl*e2)+3*sin(Bl*e1));

    F2j = -2*Aj*W2/(e2-e1)^2;

    F3j = Aj/(e2-e1)^3*(6*W1 + Bl*Bl*Bl*(e2-e1)^3*(sinh(Bl*e2)-...
        sinh(Bl*e1)-sin(Bl*e2)+sin(Bl*e1) + C2*(cosh(Bl*e2)-cosh(Bl*e1)) -
...
        C4*(cos(Bl*e2)-cos(Bl*e1))));
    F4j = Aj/(e2-e1)^2*(-6*W1 + 2*W2 - Bl*Bl*(e2-e1)^2*(Bl*e2*...
        (-sinh(Bl*e1)-C2*cosh(Bl*e1)+sin(Bl*e1)+C4*cos(Bl*e1)) +...
        Bl*e1*(sinh(Bl*e1)+C2*cosh(Bl*e1)-sin(Bl*e1)-C4*cos(Bl*e1)) + ...
        (cosh(Bl*e2)+C2*sinh(Bl*e2)+cos(Bl*e2)-C4*sin(Bl*e2)) - ...
        (cosh(Bl*e1)+C2*sinh(Bl*e1)+cos(Bl*e1)-C4*sin(Bl*e1))));

    u1j = Aj*(cosh(Bl*e1)+C2*sinh(Bl*e1)-cos(Bl*e1)+C4*sin(Bl*e1));
    u2j = Aj*Bl*(sinh(Bl*e1)+C2*cosh(Bl*e1)+sin(Bl*e1)+C4*cos(Bl*e1));
    u3j = Aj*(cosh(Bl*e2)+C2*sinh(Bl*e2)-cos(Bl*e2)+C4*sin(Bl*e2));
    u4j = Aj*Bl*(sinh(Bl*e2)+C2*cosh(Bl*e2)+sin(Bl*e2)+C4*cos(Bl*e2));
    if j==1
        u11 = u1j;
        u22 = u2j;
        u33 = u3j;
        u44 = u4j;
    end

    col1(j,1)=Aj;
    col2(j,1)=F1j;
    col3(j,1)=F2j;
    col4(j,1)=F3j;
    col5(j,1)=F4j;
    col6(j,1)=u1j;
    col7(j,1)=u2j;
    col8(j,1)=u3j;
    col9(j,1)=u4j;

    end
Col1 = sum(col1);
Col2 = sum(col2);
Col3 = sum(col3);
Col4 = sum(col4);
Col5 = sum(col5);
Col6 = sum(col6);
Col7 = sum(col7);
Col8 = sum(col8);
Col9 = sum(col9);

if e1 >-1 && e2<1

    format long;
    Table=[col1 col2 col3 col4 col5 col6 col7 col8 col9;Col1 Col2 Col3...
        Col4 Col5 Col6 Col7 Col8 Col9];
    F1 = -Col2;
    F2 = -Col3;
    F3 = -Col4;
    F4 = -Col5;
```

```
    u1 = Col6;
    u2 = Col7;
    u3 = Col8;
    u4 = Col9;
    P1 = F1 + 12*u1/(e2-e1)^3 + 6*u2/(e2-e1)^2 - 12*u3/(e2-e1)^3 ...
        + 6*u4/(e2-e1)^2;
    P2 = F2 + 6*u1/(e2-e1)^2 + 4*u2/(e2-e1) - 6*u3/(e2-e1)^2 + 2*u4/(e2-
e1);
    P3 = F3 - 12*u1/(e2-e1)^3 - 6*u2/(e2-e1)^2 + 12*u3/(e2-e1)^3 ...
        - 6*u4/(e2-e1)^2;
    P4 = F4 + 6*u1/(e2-e1)^2 + 2*u2/(e2-e1) - 6*u3/(e2-e1)^2 + 4*u4/(e2-
e1);

    %A = [12*u1/(e2-e1)^3 6*u2/(e2-e1)^2 -12*u3/(e2-e1)^3 6*u4/(e2-e1)^2;
...
    %    4*u2/(e2-e1) 6*u1/(e2-e1)^2 2*u4/(e2-e1) -6*u3/(e2-e1)^2;...
    %    12*u3/(e2-e1)^3 -6*u4/(e2-e1)^2 -12*u1/(e2-e1)^3 -6*u2/(e2-
e1)^2;...
    %    4*u4/(e2-e1) -6*u3/(e2-e1)^2 2*u2/(e2-e1) 6*u1/(e2-e1)^2];

    A = [12*u11/(e2-e1)^3 6*u22/(e2-e1)^2 -12*u33/(e2-e1)^3 6*u44/(e2-
e1)^2; ...
        4*u22/(e2-e1) 6*u11/(e2-e1)^2 2*u44/(e2-e1) -6*u33/(e2-e1)^2;...
        12*u33/(e2-e1)^3 -6*u44/(e2-e1)^2 -12*u11/(e2-e1)^3 -6*u22/(e2-
e1)^2;...
        4*u44/(e2-e1) -6*u33/(e2-e1)^2 2*u22/(e2-e1) 6*u11/(e2-e1)^2];

    P = [P1+0.5*(e2-e1)*w*w*u11; P2; P3+0.5*(e2-e1)*w*w*u33; P4];
    Q = A\P
    format short;
elseif e2==1
    format long;
    Table=[col1 col2 col3 col4 col5 col6 col7 col8 col9;Col1 Col2...
        Col3 Col4 Col5 Col6 Col7 Col8 Col9];
    F1 = -Col2;
    F2 = -Col3;
    F3 = -Col4;
    F4 = -Col5;
    u1 = Col6;
    u2 = Col7;
    u3 = Col8;
    u4 = Col9;
    P1 = F1 + 12*u1/(e2-e1)^3 + 6*u2/(e2-e1)^2 - 12*u3/(e2-e1)^3 +...
        6*u4/(e2-e1)^2;
    P2 = F2 + 6*u1/(e2-e1)^2 + 4*u2/(e2-e1) - 6*u3/(e2-e1)^2 + 2*u4/(e2-
e1);
    P3 = F3 - 12*u1/(e2-e1)^3 - 6*u2/(e2-e1)^2 + 12*u3/(e2-e1)^3 -...
        6*u4/(e2-e1)^2;
    P4 = F4 + 6*u1/(e2-e1)^2 + 2*u2/(e2-e1) - 6*u3/(e2-e1)^2 + 4*u4/(e2-
e1);

    A = [3*u11/(e2-e1)^3 3*u22/(e2-e1)^2 -3*u33/(e2-e1)^3; ...
        3*u22/(e2-e1) 3*u11/(e2-e1)^2 0;...
        3*u33/(e2-e1)^3 0 -3*u11/(e2-e1)^3];
    P = [P1+0.5*(e2-e1)*w*w*u11; P2-3*u33/(e2-e1)^2; P3+...
        0.5*(e2-e1)*w*w*u33+3*u22/(e2-e1)^2];
    Q = A\P
end

format short;
```

APPENDIX F

```
%Find the stiffness factor for a segment of a pinned pinned beam under free
%vibration

e1 = 0
e2 = 1/3
BL(1) = 3.141592654;
BL(2) = 6.2831854;
BL(3) = 9.424777961;
BL(4) = 12.5663706144;
BL(5) = 15.70796326795;
BL(6) = 18.8495559215388;
BL(7) = 21.99114857513;
BL(8) = 25.1327412287835;
BL(9) = 28.27433388230814;

w = BL(1)*BL(1);
for j=1:1:7
    Bl=BL(j);
    pj=Bl*Bl;
    aj= -sinh(Bl)/sin(Bl);

    C1 = 0;
    C2 = 1;
    C3 = 0;
    C4 = -sinh(Bl)/sin(Bl);
    Mj = 0.5*(C1*C1*(1+sinh(2*Bl)/(2*Bl)) + C2*C2*(sinh(2*Bl)/(2*Bl)-1)
+...
        C3*C3*(1+sin(2*Bl)/(2*Bl)) + C4*C4*(1-sin(2*Bl)/(2*Bl)) + ...
        2*C1*C2*(sinh(Bl)*sinh(Bl)/Bl) + 2*C1*C3*(cosh(Bl)...
        *sin(Bl)+sinh(Bl)*cos(Bl))/Bl...
        + 2*C1*C4*(1-cos(Bl)*cosh(Bl)+sin(Bl)*sinh(Bl))/Bl + ...
        2*C2*C3*(sinh(Bl)*sin(Bl)+cosh(Bl)*cos(Bl)-1)/Bl...
        + 2*C2*C4*(cosh(Bl)*sin(Bl)-sinh(Bl)*cos(Bl))/Bl + ...
        2*C3*C4*(sin(Bl)*sin(Bl)/Bl));

    Aj= -sinh(Bl)/(Bl*Bl) + 12*sinh(Bl)/(Bl*Bl*Bl*Bl)  -...
        24*(cosh(Bl)-1)/(Bl*Bl*Bl*Bl*Bl) + ...
        aj*(-sin(Bl)/(Bl*Bl) - 12*sin(Bl)/(Bl*Bl*Bl*Bl)-...
        24*(cos(Bl)-1)/(Bl*Bl*Bl*Bl*Bl));
    Aj = Aj/Mj;


    W1 = Bl*Bl*Bl*((e2+e1)/2*(e2*e2-e1*e1)-2*(e2*e2*e2-e1*e1*e1)/3)*...
         (-sinh(Bl*e1)-C2*cosh(Bl*e1)+sin(Bl*e1)+ C4*cos(Bl*e1))+...
        C2*(Bl*(e1-e2)*cosh(Bl*e2)+2*sinh(Bl*e2)-Bl*(e2-e1)*...
        cosh(Bl*e1)-2*sinh(Bl*e1)) - ...
        C4*(-Bl*(e1-e2)*cos(Bl*e2)-2*sin(Bl*e2)+Bl*(e2-e1)*cos(Bl*e1)+...
        2*sin(Bl*e1)) + Bl*(e1-e2)*sinh(Bl*e2) -...
        Bl*(e2-e1)*sinh(Bl*e1)+Bl*((e1-e2))*sin(Bl*e2)-Bl*(e2-e1)*...
        sin(Bl*e1)+2*cosh(Bl*e2)-2*cosh(Bl*e1)-2*cos(Bl*e2)+2*cos(Bl*e1);
    F1j = -6*Aj*W1/(e2-e1)^3;


    W2 = Bl*Bl*Bl*((2*e2+e1)/2*(e2*e2-e1*e1)-(e2*e2*e2-e1*e1*e1))*...
         (-sinh(Bl*e1)-C2*cosh(Bl*e1)+sin(Bl*e1)+ C4*cos(Bl*e1))+...
        Bl*Bl*Bl*e1*(e2-e1)*(e2-e1)/2*(sinh(Bl*e1)+C2*cosh(Bl*e1)-...
```

```
        sin(Bl*e1)-C4*cos(Bl*e1)) - ...
        Bl*Bl*(e2-e1)*(e2-e1)/2*(cosh(Bl*e1)+C2*sinh(Bl*e1)+...
        cos(Bl*e1)-C4*sin(Bl*e1)) + ...
        Bl*(e1-e2)*sinh(Bl*e2)-2*Bl*(e2-e1)*sinh(Bl*e1)+...
        Bl*(e1-e2)*sin(Bl*e2) - ...
        2*Bl*(e2-e1)*sin(Bl*e1)-3*cos(Bl*e2)+3*cos(Bl*e1)+...
        3*cosh(Bl*e2)-3*cosh(Bl*e1) + ...
        C2*(Bl*(e1-e2)*cosh(Bl*e2)-2*Bl*(e2-e1)*cosh(Bl*e1)+...
        3*sinh(Bl*e2)-3*sinh(Bl*e1)) - ...
        C4*(-Bl*(e1-e2)*cos(Bl*e2)+2*Bl*(e2-e1)*cos(Bl*e1)-...
        3*sin(Bl*e2)+3*sin(Bl*e1));

    F2j = -2*Aj*W2/(e2-e1)^2;

    F3j = Aj/(e2-e1)^3*(6*W1 + Bl*Bl*Bl*(e2-e1)^3*...
        (C2*(cosh(Bl*e2)-cosh(Bl*e1)) - ...
        C4*(cos(Bl*e2)-cos(Bl*e1))));

    F4j = Aj/(e2-e1)^2*(-6*W1 + 2*W2 - Bl*Bl*(e2-e1)^2*(Bl*e2*...
        (-C2*cosh(Bl*e1)+C4*cos(Bl*e1)) +...
        Bl*e1*(C2*cosh(Bl*e1)-C4*cos(Bl*e1)) + ...
        (C2*sinh(Bl*e2)-C4*sin(Bl*e2)) - (C2*sinh(Bl*e1)-C4*sin(Bl*e1))));

    u1j = Aj*(C2*sinh(Bl*e1)+C4*sin(Bl*e1));
    u2j = Aj*Bl*(C2*cosh(Bl*e1)+C4*cos(Bl*e1));
    u3j = Aj*(C2*sinh(Bl*e2)+C4*sin(Bl*e2));
    u4j = Aj*Bl*(C2*cosh(Bl*e2)+C4*cos(Bl*e2));
    if j==1
        u11 = u1j;
        u22 = u2j;
        u33 = u3j;
        u44 = u4j;
    end

    col1(j,1)=Aj;
    col2(j,1)=F1j;
    col3(j,1)=F2j;
    col4(j,1)=F3j;
    col5(j,1)=F4j;
    col6(j,1)=u1j;
    col7(j,1)=u2j;
    col8(j,1)=u3j;
    col9(j,1)=u4j;


end
Col1 = sum(col1);
Col2 = sum(col2);
Col3 = sum(col3);
Col4 = sum(col4);
Col5 = sum(col5);
Col6 = sum(col6);
Col7 = sum(col7);
Col8 = sum(col8);
Col9 = sum(col9);

if e1 <1 && e2<1 && e1~=0 && e2~=0

    format long;
```

```
    Table=[col1 col2 col3 col4 col5 col6 col7 col8 col9;Col1 Col2...
        Col3 Col4 Col5 Col6 Col7 Col8 Col9];
    F1 = -Col2;
    F2 = -Col3;
    F3 = -Col4;
    F4 = -Col5;
    u1 = Col6;
    u2 = Col7;
    u3 = Col8;
    u4 = Col9;
    P1 = F1 + 12*u1/(e2-e1)^3 + 6*u2/(e2-e1)^2 - 12*u3/(e2-e1)^3 +...
        6*u4/(e2-e1)^2;
    P2 = F2 + 6*u1/(e2-e1)^2 + 4*u2/(e2-e1) - 6*u3/(e2-e1)^2 + 2*u4/(e2-
e1);
    P3 = F3 - 12*u1/(e2-e1)^3 - 6*u2/(e2-e1)^2 + 12*u3/(e2-e1)^3 - ...
        6*u4/(e2-e1)^2;
    P4 = F4 + 6*u1/(e2-e1)^2 + 2*u2/(e2-e1) - 6*u3/(e2-e1)^2 + 4*u4/(e2-
e1);

    A = [12*u11/(e2-e1)^3 6*u22/(e2-e1)^2 -12*u33/(e2-e1)^3 6*u44/(e2-
e1)^2; ...
        4*u22/(e2-e1) 6*u11/(e2-e1)^2 2*u44/(e2-e1) -6*u33/(e2-e1)^2;...
        12*u33/(e2-e1)^3 -6*u44/(e2-e1)^2 -12*u11/(e2-e1)^3 -6*u22/(e2-
e1)^2;...
        4*u44/(e2-e1) -6*u33/(e2-e1)^2 2*u22/(e2-e1) 6*u11/(e2-e1)^2];

    P = [P1+0.5*(e2-e1)*w*w*u11; P2; P3+0.5*(e2-e1)*w*w*u33; P4];
    Q = A\P
    format short;

elseif e1> 0 && e2==1
    format long;
    Table=[col1 col2 col3 col4 col5 col6 col7 col8 col9;Col1 ...
        Col2 Col3 Col4 Col5 Col6 Col7 Col8 Col9];
    F1 = -Col2;
    F2 = -Col3;
    F3 = -Col4;
    F4 = -Col5;
    u1 = Col6;
    u2 = Col7;
    u3 = Col8;
    u4 = Col9;
    P1 = F1 + 12*u1/(e2-e1)^3 + 6*u2/(e2-e1)^2 - 12*u3/(e2-e1)^3 ...
        + 6*u4/(e2-e1)^2;
    P2 = F2 + 6*u1/(e2-e1)^2 + 4*u2/(e2-e1) - 6*u3/(e2-e1)^2 + 2*u4/(e2-
e1);
    P3 = F3 - 12*u1/(e2-e1)^3 - 6*u2/(e2-e1)^2 + 12*u3/(e2-e1)^3...
    - 6*u4/(e2-e1)^2;
    P4 = F4 + 6*u1/(e2-e1)^2 + 2*u2/(e2-e1) - 6*u3/(e2-e1)^2 + 4*u4/(e2-
e1);

    A = [3*u11/(e2-e1)^3 3*u22/(e2-e1)^2 -3*u33/(e2-e1)^3; ...
        3*u22/(e2-e1) 3*u11/(e2-e1)^2 0;...
        3*u33/(e2-e1)^3 0 -3*u11/(e2-e1)^3];
    P = [P1+0.5*(e2-e1)*w*w*u11; P2-3*u33/(e2-e1)^2; P3+0.5*(e2-e1)*w*w*...
        u33+3*u22/(e2-e1)^2];
    Q = A\P

elseif e1==0 && e2<1
   format long;
```

```
    Table=[col1 col2 col3 col4 col5 col6 col7 col8 col9;Col1 Col2 Col3...
        Col4 Col5 Col6 Col7 Col8 Col9];
    F1 = -Col2;
    F2 = -Col3;
    F3 = -Col4;
    F4 = -Col5;
    u1 = Col6;
    u2 = Col7;
    u3 = Col8;
    u4 = Col9;
    P1 = F1 + 12*u1/(e2-e1)^3 + 6*u2/(e2-e1)^2 - 12*u3/(e2-e1)^3 ...
        + 6*u4/(e2-e1)^2;
    P2 = F2 + 6*u1/(e2-e1)^2 + 4*u2/(e2-e1) - 6*u3/(e2-e1)^2 + 2*u4/(e2-
e1);
    P3 = F3 - 12*u1/(e2-e1)^3 - 6*u2/(e2-e1)^2 + 12*u3/(e2-e1)^3...
    - 6*u4/(e2-e1)^2;
    P4 = F4 + 6*u1/(e2-e1)^2 + 2*u2/(e2-e1) - 6*u3/(e2-e1)^2 + 4*u4/(e2-
e1);

    A = [3*u11/(e2-e1)^3 0 -3*u33/(e2-e1)^3; ...
        3*u33/(e2-e1)^3 -3*u44/(e2-e1)^2 -3*u11/(e2-e1)^3;...
        3*u44/(e2-e1) -3*u33/(e2-e1)^2 0];

    P = [P1+0.5*(e2-e1)*w*w*u11-3*u4/(e2-e1)^2; P3+0.5*(e2-e1)*w*w*u33; P4-
3*u11/(e2-e1)^2];
    Q = A\P


format short;
end
```

APPENDIX G

```
%Find the stiffness factor for a segment of a fixed fREE beam under free
%vibration

e1 = 0;
e2 = 1;

%BL(1) = 4.7300408;

%BL(1) = 0.000001;
BL(1) = 4.7300408;
BL(2) = 7.8532047;
BL(3) = 10.99560784;
BL(4) = 14.1371655;
BL(5) = 17.27875965739948;
BL(6) = 20.42035224562606;
BL(7) = 23.56194490204046;
BL(8) = 26.70353755550819;
BL(9) = 29.84513020910326;
BL(10) = 32.98672286269282;
BL(11) = 36.12831551628263;
BL(12) = 39.26990816987242;
w = BL(1)*BL(1);
for j=1:2:7
    Bl=BL(j);
    pj=Bl*Bl;
    aj=(cos(Bl) - cosh(Bl))/(sinh(Bl) - sin(Bl));

    C1 = 1;
    C2 = (cosh(Bl)-cos(Bl))/(sin(Bl)-sinh(Bl));
    C3 = 1;
    C4 = (cosh(Bl)-cos(Bl))/(sin(Bl)-sinh(Bl));
    Mj = 0.5*(C1*C1*(1+sinh(2*Bl)/(2*Bl)) + C2*C2*(sinh(2*Bl)/(2*Bl)-1)
+...
        C3*C3*(1+sin(2*Bl)/(2*Bl)) + C4*C4*(1-sin(2*Bl)/(2*Bl)) + ...
        2*C1*C2*(sinh(Bl)*sinh(Bl)/Bl) + 2*C1*C3*(cosh(Bl)*sin(Bl)+...
        sinh(Bl)*cos(Bl))/Bl...
        + 2*C1*C4*(1-cos(Bl)*cosh(Bl)+sin(Bl)*sinh(Bl))/Bl +...
        2*C2*C3*(sinh(Bl)*sin(Bl)+cosh(Bl)*cos(Bl)-1)/Bl...
        + 2*C2*C4*(cosh(Bl)*sin(Bl)-sinh(Bl)*cos(Bl))/Bl +...
        2*C3*C4*(sin(Bl)*sin(Bl)/Bl));
    Bl1=Bl/2;
    Aj= -12*(sinh(Bl1)-sin(Bl1))/(Bl1*Bl1*Bl1) + 24*(cosh(Bl1)+cos(Bl1))...
        /(Bl1*Bl1*Bl1*Bl1) - 24*(sinh(Bl1)+sin(Bl1))/(Bl1*Bl1*Bl1*Bl1*Bl1)
- ...
        C2*(12*(cosh(Bl1)+cos(Bl1))/(Bl1*Bl1*Bl1) -
24*(sinh(Bl1)+sin(Bl1))...
        /(Bl1*Bl1*Bl1*Bl1) + 24*(cosh(Bl1)-
cos(Bl1))/(Bl1*Bl1*Bl1*Bl1*Bl1));
    Aj = Aj/(2*Mj);

    W1 = Bl*Bl*Bl*((e2+e1)/2*(e2*e2-e1*e1)-2*(e2*e2*e2-e1*e1*e1)/3)*...
        (-sinh(Bl*e1)-C2*cosh(Bl*e1)-sin(Bl*e1)+ C4*cos(Bl*e1))+...
        C2*(Bl*(e1-e2)*cosh(Bl*e2)+2*sinh(Bl*e2)-Bl*(e2-e1)*cosh(Bl*e1)...
        -2*sinh(Bl*e1)) - ...
        C4*(-Bl*(e1-e2)*cos(Bl*e2)-2*sin(Bl*e2)+Bl*(e2-e1)*cos(Bl*e1)+...
        2*sin(Bl*e1)) + Bl*(e1-e2)*sinh(Bl*e2) -...
        Bl*(e2-e1)*sinh(Bl*e1)-Bl*(e1-e2)*sin(Bl*e2)+Bl*(e2-e1)*...
        sin(Bl*e1)+2*cosh(Bl*e2)-2*cosh(Bl*e1)+2*cos(Bl*e2)-2*cos(Bl*e1);
```

```
    F1j = -6*Aj*W1/(e2-e1)^3;

    W2 = Bl*Bl*Bl*((2*e2+e1)/2*(e2*e2-e1*e1)-(e2*e2*e2-e1*e1*e1))*...
         (-sinh(Bl*e1)-C2*cosh(Bl*e1)-sin(Bl*e1)+ C4*cos(Bl*e1))+...
        Bl*Bl*Bl*e1*(e2-e1)*(e2-e1)/2*(sinh(Bl*e1)+C2*cosh(Bl*e1)+...
        sin(Bl*e1)-C4*cos(Bl*e1)) - ...
        Bl*Bl*(e2-e1)*(e2-e1)/2*(cosh(Bl*e1)+C2*sinh(Bl*e1)-cos(Bl*e1)...
        -C4*sin(Bl*e1)) + ...
        Bl*(e1-e2)*sinh(Bl*e2)-2*Bl*(e2-e1)*sinh(Bl*e1)-Bl*(e1-
e2)*sin(Bl*e2) + ...
        2*Bl*(e2-e1)*sin(Bl*e1)+3*cos(Bl*e2)-3*cos(Bl*e1)+3*cosh(Bl*e2)...
        -3*cosh(Bl*e1) + ...
        C2*(Bl*(e1-e2)*cosh(Bl*e2)-2*Bl*(e2-
e1)*cosh(Bl*e1)+3*sinh(Bl*e2)...
        -3*sinh(Bl*e1)) - ...
        C4*(-Bl*(e1-e2)*cos(Bl*e2)+2*Bl*(e2-e1)*cos(Bl*e1)-3*sin(Bl*e2)...
        +3*sin(Bl*e1));



    F2j = -2*Aj*W2/(e2-e1)^2;

    F3j = Aj/(e2-e1)^3*(6*W1 + Bl*Bl*Bl*(e2-e1)^3*(sinh(Bl*e2)-
sinh(Bl*e1)...
        +sin(Bl*e2)-sin(Bl*e1) + C2*(cosh(Bl*e2)-cosh(Bl*e1)) - ...
        C4*(cos(Bl*e2)-cos(Bl*e1))));

    F4j = Aj/(e2-e1)^2*(-6*W1 + 2*W2 - Bl*Bl*(e2-e1)^2*(Bl*e2*...
         (-sinh(Bl*e1)-C2*cosh(Bl*e1)-sin(Bl*e1)+C4*cos(Bl*e1)) +...
        Bl*e1*(sinh(Bl*e1)+C2*cosh(Bl*e1)+sin(Bl*e1)-C4*cos(Bl*e1)) + ...
         (cosh(Bl*e2)+C2*sinh(Bl*e2)-cos(Bl*e2)-C4*sin(Bl*e2)) - ...
         (cosh(Bl*e1)+C2*sinh(Bl*e1)-cos(Bl*e1)-C4*sin(Bl*e1))));



    u1j = Aj*(cosh(Bl*e1)+C2*sinh(Bl*e1)+cos(Bl*e1)+C4*sin(Bl*e1));
    u2j = Aj*Bl*(sinh(Bl*e1)+C2*cosh(Bl*e1)-sin(Bl*e1)+C4*cos(Bl*e1));
    u3j = Aj*(cosh(Bl*e2)+C2*sinh(Bl*e2)+cos(Bl*e2)+C4*sin(Bl*e2));
    u4j = Aj*Bl*(sinh(Bl*e2)+C2*cosh(Bl*e2)-sin(Bl*e2)+C4*cos(Bl*e2));
    if j==1
        u11 = u1j;
        u22 = u2j;
        u33 = u3j;
        u44 = u4j;
    end

    col1(j,1)=Aj;
    col2(j,1)=F1j;
    col3(j,1)=F2j;
    col4(j,1)=F3j;
    col5(j,1)=F4j;
    col6(j,1)=u1j;
    col7(j,1)=u2j;
    col8(j,1)=u3j;
    col9(j,1)=u4j;


end
Col1 = sum(col1);
Col2 = sum(col2);
```

```
Col3 = sum(col3);
Col4 = sum(col4);
Col5 = sum(col5);
Col6 = sum(col6);
Col7 = sum(col7);
Col8 = sum(col8);
Col9 = sum(col9);

if e1>=0 && e2>=0

    format long;
    Table=[col1 col2 col3 col4 col5 col6 col7 col8 col9;Col1 Col2 Col3 ...
        Col4 Col5 Col6 Col7 Col8 Col9];
    F1 = -Col2;
    F2 = -Col3;
    F3 = -Col4;
    F4 = -Col5;
    u1 = Col6
    u2 = Col7
    u3 = Col8
    u4 = Col9
    P1 = F1 + 12*u1/(e2-e1)^3 + 6*u2/(e2-e1)^2 - 12*u3/(e2-e1)^3 +...
        6*u4/(e2-e1)^2;
    P2 = F2 + 6*u1/(e2-e1)^2 + 4*u2/(e2-e1) - 6*u3/(e2-e1)^2 + 2*u4/(e2-
e1);
    P3 = F3 - 12*u1/(e2-e1)^3 - 6*u2/(e2-e1)^2 + 12*u3/(e2-e1)^3 -...
        6*u4/(e2-e1)^2;
    P4 = F4 + 6*u1/(e2-e1)^2 + 2*u2/(e2-e1) - 6*u3/(e2-e1)^2 + 4*u4/(e2-
e1);

    %A = [12*u1/(e2-e1)^3 6*u2/(e2-e1)^2 -12*u3/(e2-e1)^3 6*u4/(e2-e1)^2;
...
    %    4*u2/(e2-e1) 6*u1/(e2-e1)^2 2*u4/(e2-e1) -6*u3/(e2-e1)^2;...
    %    12*u3/(e2-e1)^3 -6*u4/(e2-e1)^2 -12*u1/(e2-e1)^3 -6*u2/(e2-
e1)^2;...
    %    4*u4/(e2-e1) -6*u3/(e2-e1)^2 2*u2/(e2-e1) 6*u1/(e2-e1)^2];

    A = [12*u11/(e2-e1)^3 6*u22/(e2-e1)^2 -12*u33/(e2-e1)^3 6*u44/(e2-
e1)^2; ...
        4*u22/(e2-e1) 6*u11/(e2-e1)^2 2*u44/(e2-e1) -6*u33/(e2-e1)^2;...
        12*u33/(e2-e1)^3 -6*u44/(e2-e1)^2 -12*u11/(e2-e1)^3 -6*u22/(e2-
e1)^2;...
        4*u44/(e2-e1) -6*u33/(e2-e1)^2 2*u22/(e2-e1) 6*u11/(e2-e1)^2];

    P = [P1+0.5*(e2-e1)*w*w*u11; P2; P3+0.5*(e2-e1)*w*w*u33; P4];
    Q = A\P

    format short;
end
```

Appendix H

Below is an empirical model of a single degree of freedom lumped mass beam under longitudinal vibration.

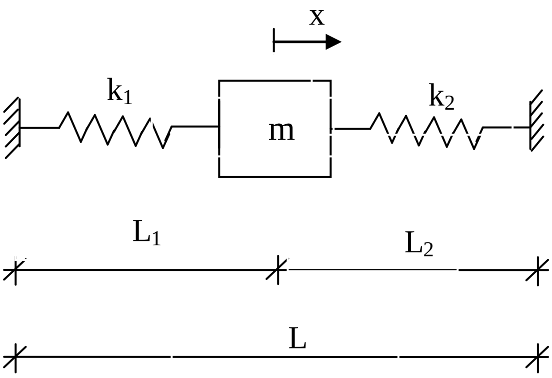


Figure H1 (a): One degree of freedom longitudinally vibrating system

The free body diagram can be represented as

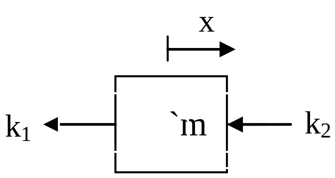


Figure H1 (b): An isolated one degree of freedom longitudinally vibrating mass

From the equilibrium of forces

$$m\ddot{x} = -k_1 x - k_2 x \tag{H1}$$

Let $x = A \sin \omega t$ (H2)

By substituting equation (H2) into (H1) and simplifying

$$\omega = \sqrt{\frac{k_1+k_2}{m}} \tag{H3}$$

But $k_1 = \frac{EA}{L_1}$, $k_2 = \frac{EA}{L_2}$ and $m = \frac{\mu L}{2}$ (H4)

Where EA is the axial rigidity of the modeled bar and μ is the mass per unit length

By substituting equations (H4) into (H3) we obtain

$$\omega = \sqrt{\frac{2EA}{L_1 L_2 \mu}} \qquad \text{(H5)}$$

With equation (H5) the natural frequencies of the bar at different positions of the lumped mass can be obtained. This is done by varying the values of L_1 and L_2. The calculated values are presented in Table H1. In the Table the value of EA and u were taken as unity.

Table H1: Calculated values of natural frequency of the longitudinally vibrating bar at different values of L_1 and L_2

L_1(m)	**$L_2 = 1 - L_1$ (m)**	**ω (Hz)**
0.1	0.9	4.714
0.2	0.8	3.536
0.3	0.7	3.086
0.4	0.6	2.887
0.5	0.5	2.828
0.6	0.4	2.887
0.7	0.3	3.086
0.8	0.2	3.536
0.9	0.1	4.714

We observe from Table H1 that the calculated natural frequencies vary with the position of the lumped mass and are not constant as will be expected for a continuous bar (the natural frequency of a continuous bar fixed at both ends under longitudinal vibration ω_o is 3.142). The variation is better observed in a plot of natural frequency against the L_1.

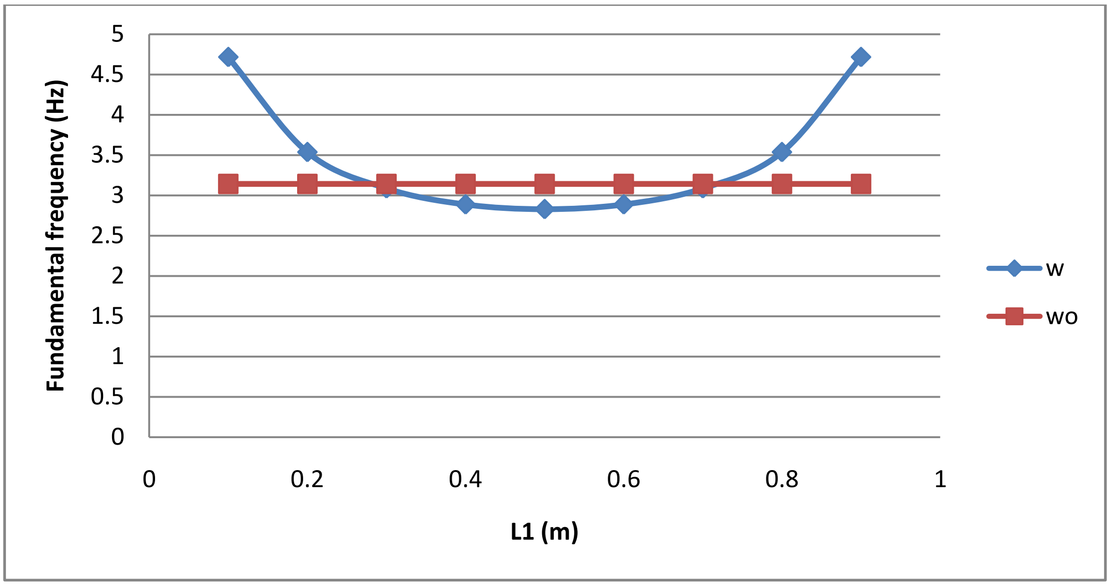


Figure H2: Fundamental frequency against L_1

From the graph it would be seen that the fundamental frequency for the SOF lumped mass beam under longitudinal vibration varied with the position of the lumped. The value at lower values of L_1 was greater that the exact natural frequency. But as the value of L_1 increase it reduced and finally became lower that the exact value before increasing again. The variation in the calculated natural frequency is due to the discretisation of the mass distribution by lumping. This can also be observed in a case of a two-degree of freedom (2DOF) lumped mass beam under longitudinal vibration.

Below is an empirical model of a two-degree of freedom lumped mass beam under longitudinal vibration.

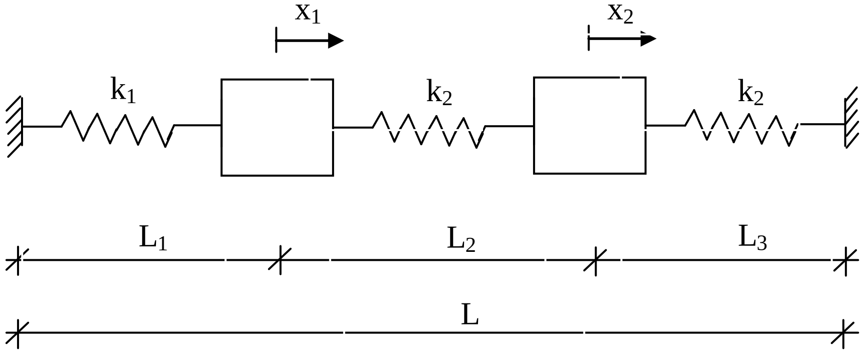


Figure H2a: Two degrees of freedom longitudinally vibrating system

The free body diagram can be represented as

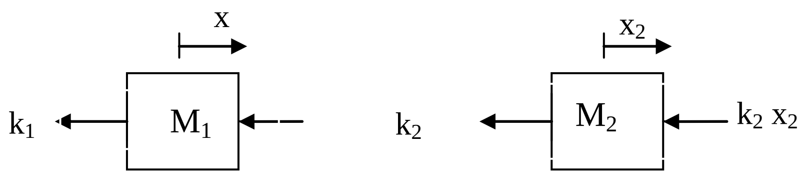


Figure H2a: Two degrees of freedom longitudinally vibrating free body mass system

From the equilibrium of forces

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_2 - x_1) \qquad \text{(H6a)}$$

$$m_2\ddot{x}_2 = k_2(x_1 - x_2) - k_3x_2 \qquad \text{(H6b)}$$

Let

$$x_1 = A_1 \sin \omega t \qquad \text{(H7a)}$$

$$x_2 = A_2 \sin \omega t \qquad \text{(H7b)}$$

By substituting equations (H7a) and (H7b) into (H6a) and (H6b) and simplifying we have

$$(k_1 + k_2 - m_1\omega^2)A_1 - k_2A_2 = 0 \qquad \text{(H8a)}$$

$$k_2A_1 + (m_2\omega^2 - k_2 - k_3)A_2 = 0 \qquad \text{(H8b)}$$

$$\begin{bmatrix} k_1 + k_2 - m_1\omega^2 & k_2 \\ k_2 & m_2\omega^2 - k_2 - k_3 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(H9)}$$

For a non- trivial solution the determinant of the coefficient has to be zero, hence

$$(k_1 + k_2 - m_1\omega^2)(m_2\omega^2 - k_2 - k_3) + k_2^2 = 0$$

$$\omega^2 = \frac{k_1 + k_2(m_1 + m_2) + k_3m_1 \pm \sqrt{(k_1 + k_2(m_1 + m_2) + k_3m_1)^2 - 4m_1m_2(k_1k_2 + k_1k_3 + k_2k_3)}}{2m_1m_2}$$

(H10)

From the principles of strength of materials and Hooke’s law

$$k_1 = \frac{EA}{L_1}, k_2 = \frac{EA}{L_2} \text{ and } m_1 = \frac{\mu(L_1+L_2)}{2}, m_2 = \frac{\mu(L_3+L_4)}{2} \quad \text{(H11)}$$

Using equation (H10) the natural frequencies of vibration of the model of figure 2 were calculated for different values of L_1, L_2 and L_3. There values were compared with the exact value of a corresponding continuous bar. See Tables H2 and H3.

Table H2: Calculation of the natural frequency of the longitudinally vibrating bar at $L_1 = L_2$, L=1, EA = 1 and μ = 1

L_1(m)	L_2(m)	L(m)	m_1(kg)	m_2(kg)	ω_1(Hz)	ω_o(Hz)
0.05	0.0	0.9	0.0	0.47	29.02	3.14
0.1	0.1	0.8	0.1	0.45	14.92	3.14
0.1	0.1	0.7	0.1	0.42	10.27	3.14
0.2	0.2	0.5	0.2	0.40	8.004	3.14
0.2	0.2	0.4	0.2	0.37	6.721	3.14
0.3	0.3	0.4	0.3	0.35	5.982	3.14
0.3	0.3	0.3	0.3	0.32	5.654	3.14
0.4	0.4	0.2	0.4	0.30	5.814	3.14
0.4	0.4	0.1	0.4	0.27	7.099	3.14

Table H3: Calculation of the natural frequency for $L_1 = L_3$, L=1, EA = 1 and μ = 1

L_1(m)	L_2(m)	L_3(m)	m_1(kg)	m_2(kg)	ω_1(Hz)	ω_o(Hz)
0.0	0.9	0.0	0.47	0.47	7.64	3.14
0.1	0.8	0.1	0.45	0.45	6.01	3.14
0.1	0.7	0.1	0.42	0.42	5.43	3.14
0.2	0.5	0.2	0.40	0.40	5.21	3.14
0.2	0.4	0.2	0.37	0.37	5.22	3.14
0.3	0.4	0.3	0.35	0.35	5.43	3.14
0.3	0.3	0.3	0.32	0.32	5.91	3.14
0.4	0.2	0.4	0.30	0.30	6.88	3.14
0.4	0.1	0.4	0.27	0.27	9.30	3.14

ω_1 and ω_2 are the calculated natural frequencies obtained from equation (H10) while ω_o is the exact natural frequency for a corresponding continuous bar (obtained using the Hamilton's principle).

From Tables H2 and H3 it would be seen that the values of calculated natural frequencies vary with the position of the lumped mass. However at some positions the calculation was very close to that of the exact solution. This is better appreciated in a plot of the calculated natural frequencies against L1. The plot is presented below.

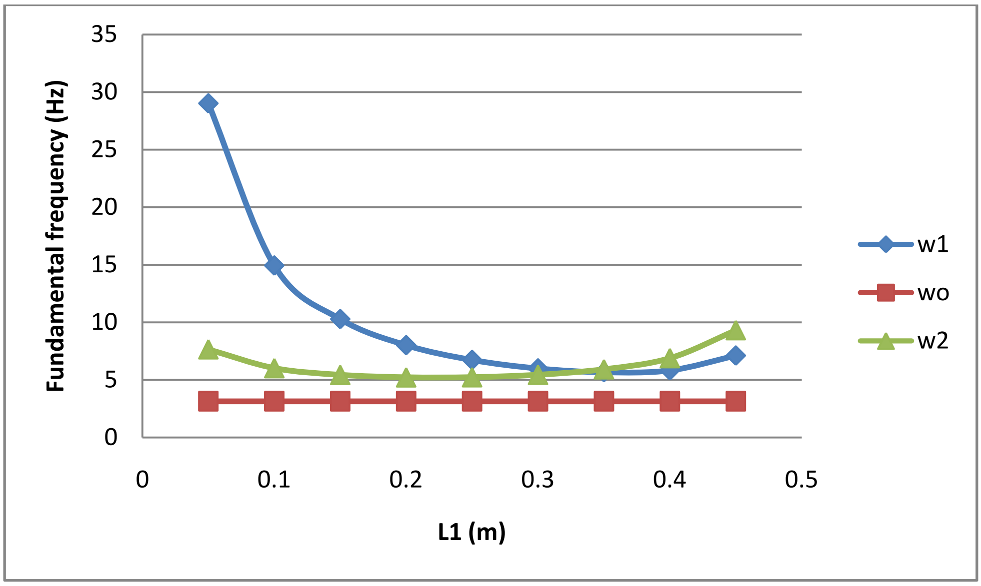


Figure H3: Calculated Fundamental frequencies against L_1

From the graph it would be seen that the calculated natural frequencies were higher than the exact frequencies for all the chosen position of lumped mass. The values were seen to be closer L_1 values of 0.3m to 0.35m. These are values at which the spacing of the lumps is fairly uniform.

So far we have only looked at longitudinal vibrations. The same observation can be made for lateral vibrations.

Below is an empirical model of a single degree of freedom lumped mass beam under lateral vibration.

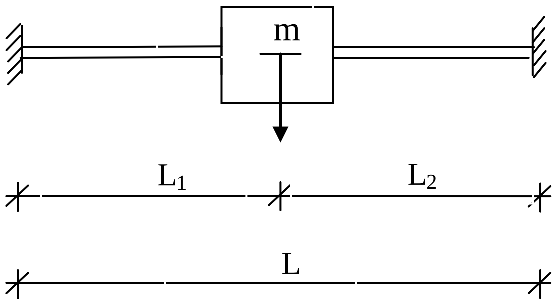


Figure H4: One degree of freedom laterally

From the equation of motion for free undamped vibration

$$m\ddot{x} + kx = 0 \tag{H12}$$

By substituting equation (H2) into (H12) and simplifying

$$\omega = \sqrt{\frac{k}{m}} \tag{H13}$$

From the principle of virtual work the deflection at the position of the lumped mass can be obtained as

$$\Delta = \frac{PL_1^3 L_2^3}{3EIL^3} \tag{H14}$$

Where P is the weight of the lumped mass

But k is the beam stiffness and can be obtained as

$$k = \frac{P}{\Delta} = \frac{3EIL^3}{L_1^3 L_2^3} \tag{H15}$$

and $m = \frac{\mu L}{2}$ as stated earlier.

By substituting equation (H15) into (H13)

$$\omega = \sqrt{\frac{6EIL^2}{L_1^3 L_2^3 \mu}} \tag{H16}$$

Where EI is the flexural rigidity of the modeled beam and μ is the mass per unit length

Equation (H16) was used to generate a range of possible values of natural frequencies at different position of the lumped mass. These were presented in Table 4 below.

Table H4: Calculated values of natural frequency of the laterally vibrating beam at different values of L_1 and L_2

L_1(m)	**$L_2 = 1 – L_1$(m)**	**ω (Hz)**	**ω_o (Hz)**
0.1	0.9	90.722	22.373
0.2	0.8	38.273	22.373
0.3	0.7	25.454	22.373
0.4	0.6	20.833	22.373
0.5	0.5	19.596	22.373
0.6	0.4	20.833	22.373
0.7	0.3	25.454	22.373
0.8	0.2	38.273	22.373
0.9	0.1	90.722	22.373

From Table H4 above it can be seen that the calculated values of natural frequencies varied with the position of the lumped mass. The pattern of variation can be appreciated in a plot of the natural frequencies against L_1 presented in Figure H5.

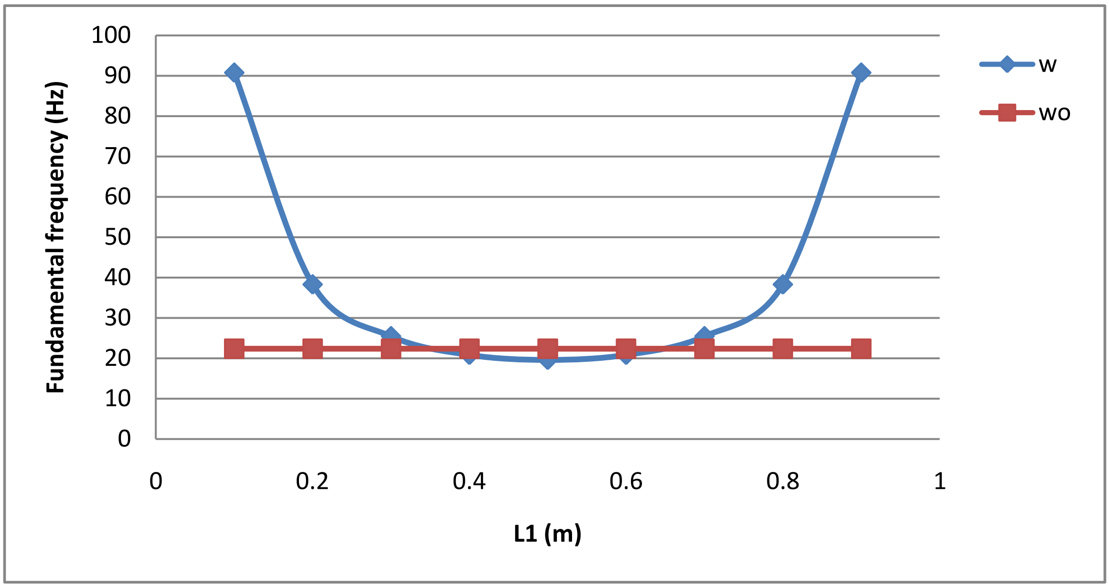


Figure H5: Calculated natural frequencies against lump position L_1

From the graph above it is observed that the calculated frequency varied from the natural frequency but was close at values of L_1 = 0.35m and 0.65m. At very high and very low values of L_1 the error in the calculated frequency is magnified.

APPENDIX J

A mathematical proof that the axial forces (both applied and reactive forces) can be represented as transverse forces and analysed as shearing forces (knowing fully well that they are axial forces) without jeopardizing the accuracy of obtained results, provided that there was no bending moment on the axially loaded bar.

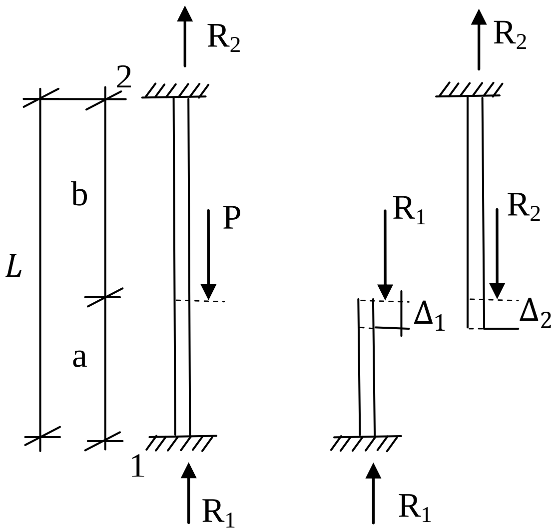


Figure J(a)
A vertical bar with a concentrated load P acting at a distance *a* from support 1 shown in the whole and decomposed state

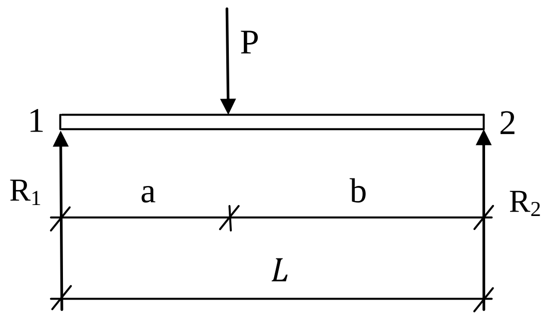


Figure J(b)
The vertical bar represented as a horizontal bar with a concentrated load P acting at a distance *a* from support 1

Figure J(a) shows an axially loaded bar with a concentrated load P at a distance *a* from one of the supports. The bar was decomposed into its constituent parts, showing the lower part (below the concentrated load) and the upper part (above the concentrated load).

From the principle of vertical equilibrium of forces we can establish from figure J (a) that

$$R_1 + R_2 = P \quad (1)$$

i.e. the sum of all vertical forces are equal.

Hence in its decomposed state the condition of vertical equilibrium was also maintained.

From Hooke's law, the compression of the lower section of the bar Δ_1 due to the compressive force R_1 is

$$\Delta_1 = \frac{R_1 a}{EA} \tag{2}$$

where EA is the axial rigidity of the bar.

In like manner the elongation Δ_2 of the upper section of the bar due to R_2 can be stated as

$$\Delta_2 = \frac{R_2 b}{EA} \tag{3}$$

Since the upper section and the lower section are part of the same bar, the elongation of the upper section will be equal to the compression of the lower section (compatibility condition).

$$\Delta_1 = \Delta_2 \tag{4}$$

By substituting equations (2) and (3) into equation (4) and simplifying we obtain

$$\boldsymbol{\frac{R_1}{b} = \frac{R_2}{a}} \tag{5}$$

Equation (5) holds irrespective of the value of the concentrated load P, provided the bar is of a uniform material (Young's modulus E is constant) and constant cross section (prismatic).

By representing the vertical bar as a horizontal bar with the concentrated load P acting transversely (see Figure J(b)) an equation similar to equation (5) can be obtained.

First, by taking moments about the two supports 1 and 2 we obtain the reactions R_1 and R_2 as

$$R_1 = \frac{Pb}{L} \tag{6}$$

$$R_2 = \frac{Pa}{L} \tag{7}$$

From equation (6) by making P/L the subject of the formula we obtain

$$\frac{P}{L} = \frac{R_1}{b} \tag{8}$$

Likewise from equation (7)

$$\frac{P}{L} = \frac{R_2}{a} \tag{9}$$

By equating equation (8) and (9) we obtain

$$\frac{\boldsymbol{R_1}}{\boldsymbol{b}} = \frac{\boldsymbol{R_2}}{\boldsymbol{a}} \qquad (10)$$

Notice that equation (10) is the same as the equation (5) we obtained for the vertical bar.