EXTENSIONS AND GENERALIZATIONS ON WEAK TOPOLOGIES

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Abstract

The major motivation for this study is the desire to fully appreciate the landscape of weak topology, a key tool in functional analysis. Thus a constructive approach was taken which provided the key to open up the weak topology landscape and led to the exposure of several facts (erstwhile obscure) about weak topology and topology in general. It is proved that the well known co-finite topology on any nonempty set is a tree of many co-finite-like topologies which were called semi-co-finite topologies. The concept of reducibility of topologies was introduced and it was proved (among other things) that the discrete topology of a set X cannot be reduced in a strong sense if the cardinality of X is greater than 2. We proved that all the factor spaces are discrete if a product topology is discrete in either finite or infinite dimensional situations. We established the conditions for the inducement by a weak topology on its range topological spaces (and inheritance by a weak topology, from its range spaces) of the properties of the lower separation axioms of T_0, T_1, and T_2 (Hausdorff). We then obtained the conditions for the inducement by a weak topology on its range topological spaces (and inheritance by a weak topology, from its range spaces) of the properties of the higher separation axioms of Tychonoff, Normality, Regularity, Complete Regularity, and Complete Normality. We proved that any nontrivial weak topology is actually in the middle of a chain of pairwise strictly comparable weak topologies. We proved that every seminorm topology, which already is known to be locally convex, is actually the peak (maximum) of a sequence of pairwise strictly comparable non-locally-convex weak topologies which are generated by the given family of seminorms. We introduced the concepts of *complement of a topology*, *complement topology* and the *supra of a topology*. We introduced and defined the concept of *Exhaustive Topology* and showed that the supra of a topology cannot be discrete if the topology is not exhaustive. We also proved that no topology can exist between a topology τ and the supra τ_s of τ and have a distinct supra from τ_s. We defined *discrete weak topology* and *indiscrete weak topology* and showed that these weak topologies may not be trivial as topologies.

Contents

Chapter 1

INTRODUCTION

1.1 Background to the Study

The contributions of this thesis fall within the general area of Functional Analysis, more specifically, in the area of Topology. The major point of focus within the area is on Weak Topology, a very important aspect of Functional Analysis. Over the years weak topology has become a very important tool in the area of Functional Analysis; an area of active research interest. Topology, often called analysis situs, can be seen as an extension and generalization of Euclidean geometry; the result of generalizing the idea of nearness from distance to openness. (Kelly (1975), Benjamin Sims (1976), Angus Taylor and David Lay (1980), James Munkres (2007), and Morris (2016)) More precisely we have the following definitions.

Definition 1.1 *Let X be any non-empty set and let τ be a family of subsets of X having the property that*

1. $\emptyset \in \tau$;
2. $X \in \tau$;
3. τ *is closed under finite intersections; and*
4. τ *is closed under arbitrary unions.*

Then τ is called a topology on X and the pair (X, τ) is called a topological space.

Definition 1.2 *If (X, τ) is a topological space, then a subset G of X is called a τ-open set, or simply an open set if τ is clear from context, if $G \in \tau$.*

Definition 1.3 *If τ_1 and τ_2 are two topologies on a set X such that, say, every τ_1-open set is τ_2-open, then the topology τ_1 is said to be weaker (or coarser) than the topology τ_2. Conversely, we also say that τ_2 is stronger (or finer) than τ_1.*

Definition 1.4 *If τ_1 and τ_2 are two topologies on a set X such that τ_1 is not weaker than τ_2 and, also, τ_2 is not weaker than τ_1, then the two topologies τ_1 and τ_2 are said to be incomparable.*

Definition 1.5 *Let (X, τ) and (Y, γ) be two topological spaces. Let $f : (X, \tau) \to (Y, \gamma)$ be a function on X into Y. Then f is said to be τ-continuous, or simply continuous if τ is clear from context, if $f^{-1}(G) \in \tau$ for all $G \in \gamma$.*

Geometrically speaking, a function f is said to be continuous on an interval if its graph over the interval can be drawn without removing pencil from paper. Continuity of functions is a very important concept in analysis, as such research efforts have been devoted to finding the minimal conditions that guarantee the continuity of a function or classes of functions.

Let X be any nonempty set, let $\{(Y_\alpha, \tau_\alpha)\}$ be a family of topological spaces and let $\{\phi_\alpha\}$ be a family of functions on X into Y_α. It is of interest to determine the weakest (smallest) topology on X with respect to which each function in the family $\{\phi_\alpha\}$ is continuous. Such a topology is called the weak topology on X generated by the family of functions $\{\phi_\alpha\}$.

Theorem 1.1 *Let $\{\phi_\alpha\}_{\alpha \in \Delta}$ be any family of functions on X into Y_α, respectively, in that $\phi_\alpha : X \to Y_\alpha, \alpha \in \Delta$. For each $\alpha \in \Delta$, we consider the inverse images $\phi_\alpha^{-1}(G_\alpha)$ where $G_\alpha \in \tau_\alpha$. For simplicity, put $U_\alpha = \phi_\alpha^{-1}(G_\alpha)$ and let $U = \{U_\alpha\} \subset 2^X$. And let $B = \left\{ B_\alpha^n = \bigcap_{i=1}^{n} U_{\alpha_i} : U_{\alpha_i} \in U \right\}$, the family of sets each of which is an intersection of a finite number of sets of U. Put $\tau = \{ \bigcup_{\alpha \in \Delta} B_\alpha \}$, the collection of arbitrary unions of sets of B. Then*

1. *τ is a topology on X.*
2. *each ϕ_α is τ-continuous.*
3. *τ is the smallest topology on X with respect to which each ϕ_α is continuous.*

Proof:
To show that τ is a topology on X, we only prove that

- $\emptyset \in \tau$,
- $X \in \tau$,
- τ is closed under finite intersections, and that
- τ is closed under arbitrary unions.

1. - Select only the empty set $\emptyset$ from each τ_α. That is, put $\emptyset = G_\alpha \in \tau_\alpha$. Then $\phi_\alpha^{-1}(G_\alpha) = \phi_\alpha^{-1}(\emptyset) = \emptyset$. $\Longrightarrow N_\alpha = \bigcap_{i=1}^{n} U_{\alpha_i} = \bigcap_{i=1}^{n} \emptyset = \emptyset$, for such G_α. $\Longrightarrow \bigcup_{\alpha \in \Delta} N_\alpha = \emptyset$, $\Longrightarrow \emptyset = \bigcup_{\alpha \in \Delta} N_\alpha \in \tau$, $\Longrightarrow \emptyset \in \tau$.
   - Take $G_\alpha = Y_\alpha \in \tau_\alpha$, for each $\alpha \in \Delta$. Then $U_\alpha = \phi_\alpha^{-1}(Y_\alpha) = X$, for such G_α. Therefore $N_\alpha = \bigcap_{i=1}^{n} U_{\alpha_i} = X$, for such G_α. $\Longrightarrow X = \bigcup_{\alpha \in \Delta} B_\alpha \in \tau$; for such G_α. $\Longrightarrow X \in \tau$.
   - We remark that the intersection of a finite number of sets of τ is an arbitrary union of sets of B, and hence belongs to τ. That proves closure under finite intersection.[1]
   - Clearly any union, say V, of sets of τ is an arbitrary union of sets of B, proving that $V \in \tau$. Hence τ is a topology on X.
2. We observe that $\forall G_\alpha \in \tau_\alpha$, on Y_α, $\phi_\alpha^{-1}(G_\alpha) \in \tau$, on X. Thus ϕ_α is τ-continuous, $\forall \alpha \in \Delta$.
3. Suppose that γ is another topology on X with respect to which each ϕ_α is continuous. Then this implies that $\forall G_\alpha \in \tau_\alpha$, $\phi_\alpha^{-1}(G_\alpha) \in \gamma$. But $\phi_\alpha^{-1}(G_\alpha) \in \tau$. Hence $\phi_\alpha^{-1}(G_\alpha) \in \gamma \; \forall \; \phi_\alpha^{-1}(G_\alpha) \in \tau$. $\Rightarrow \tau \leq \gamma$. This implies that τ is weaker than any other topology γ with respect to which all the ϕ_α's are continuous. That is, τ is the smallest such topology.

Definition 1.6 *The topology τ on X in Theorem 1.1 is the* weak topology *on X generated or induced by the family $\{\phi_\alpha\}$ of functions.*

[1] To prove this in another way, it suffices to show that the intersection of two sets of τ is a set of τ. So let $\bigcup B_\alpha$ and $\bigcup B_\gamma$ be two sets of τ. Then $(\bigcup B_\alpha) \bigcap (\bigcup B_\gamma) = \bigcup (B_\alpha \bigcap B_\gamma)$ and this is a set of τ since it is a union of sets of B.

1.2 Statement of the Problem

1. First, we observe that $\phi_\alpha^{-1}(G_\alpha) \in \tau$ if, and only if, $G_\alpha \in \tau_\alpha$. Thus, the topology of the range spaces play critical roles in determining the resulting weak topology on X. How does the resulting weak topology change with changes in the range space topologies? More precisely, how does the resulting weak topology react to changes in the topology of one range space?

2. The term *weak topology* conveys the idea of a coarser topology. Can a weak topology on a set X be finer than a so called *strong* or *usual* topology on the set? For instance, if $X = \mathbb{R}^n$ is there a weak topology on X finer than the usual topology?

3. Let $(X, ||.||)$ be a normed linear space. The norm topology or the topology induced by the norm on X is called the strong topology on X, the topology induced on X by the family X^* (the dual of X) of all the bounded linear functionals on X is called weak topology τ_w on X, and the topology induced on X^* by the family of bounded linear maps (X^{**}) on X^* is called the weak star topology τ_{w*}. Most treatises tend to give the impression that these are the only "useful" topologies as far as a normed linear space is concerned. Are there other 'useful' weak topologies of interest on X? Can a weak topology be constructed on X which is finer than the strong topology? If answered in the affirmative, this question has interesting implications on the issue of convergence of sequences, series, nets and filters.

4. For any known class of functions, such *weak* topology, τ (say), on X is such that if $\gamma \geq \tau$, then $\phi_\alpha : (X, \gamma) \longrightarrow (Y_\alpha, \tau_\alpha)$ is continuous $\forall \alpha$. That is, $\{\emptyset, X\} \leq \tau \leq \gamma \leq 2^X$. Since anything that exists so far as weak topology is created through a constructive approach, by exploiting and analyzing the continuity property of functions, why should we stop at the weak topology or weak topologies already achieved from construction? That is, why should the experimentation stop? Do we know, for instance, if a weak topology can possibly be created which is stronger than a 'strong topology'?

5. It may be interesting to estimate the size of this kind of topology for a given class of functions, in order to know its relationship with other (possibly weak) topologies on X. For example, the weak topology on a normed linear space X is defined by some writers as the smallest topology on X with respect to which a particular class of bounded

linear maps, on X, are continuous. (See e.g. Hewitt (1950), Chidume (1996).) How small or big is this weak topology when compared with other possible weak topologies on the same normed linear space? It is important to construct some other weak topologies on a typical normed linear space in other to compare and contrast them.

6. Another particular example of a class of functions worth looking at here is the class of projection maps on $\mathbb{R}^2$. Take $X = \mathbb{R}^2$ and Y_1,Y_2 (in Theorem 1.1) as $\mathbb{R}_1$ and $\mathbb{R}_2$. Then the class becomes $\{p_1, p_2\}$ where $p_i : \mathbb{R}^2 \to \mathbb{R}_i$ is defined by $p_i(x, y) = x$, if $i = 1$ and $p_i(x, y) = y$, if $i = 2$. $\mathbb{R}$ is endowed with its usual topology. Then $p_i^{-1}[(x_i, y_i)]$ is an open infinite strip in the plane, for each $G_i = (x_i, y_i)$ open in $\mathbb{R}_i$, $1 \leq i \leq 2$. Their finite intersections $p_i^{-1}[(x,y_i)] \bigcap p_j^{-1}[(x_j, y_j)]$, $1 \leq i, j \leq 2$ consists of open rectangles in the plane. Certainly as a base, these open rectangles do not generate the discrete topology of $\mathbb{R}^2$. Hence the weak topology of $\mathbb{R}^2$ in terms of the projection maps is strictly coarser than the discrete topology of $\mathbb{R}^2$ (except of course the factor spaces $\mathbb{R}_i$ are endowed with discrete topologies).

7. This brings another point to light: The topology of each of the range spaces of a class of functions contributes to the nature of their common (weak) topology of continuity. This point underlines the need, again, to construct by varying the range spaces. We can, for instance, take a range space (or two) to be itself a weak topological space. If such a weak topological space is not actually constructed, it would not be clear what an open set in it would look like; and, hence, certainly it would not be known what a subbasic or a basic open set in the second inducement weak topology would look like. And so on.

8. For example, again let $X = \mathbb{R}^2$ and $\mathbb{R}_1$ (i.e. the horizontal factor space) be endowed with the discrete topology while the vertical factor space, $\mathbb{R}_2$, is still endowed with the usual topology of $\mathbb{R}$, and p_1, p_2 remain the respective projection maps. Then since singletons are open in $\mathbb{R}_1$ it makes sense to consider their inverse images $p_1^{-1}(\{x\})$; $x \in \mathbb{R}$. These inverse images are vertical infinite straight lines in the plane as shown in the appended pages of tables (fig.1). From there, we observe that the line $p_1^{-1}(\{x_0\})$ is a strong (i.e. not broken) line; where x_0 is a fixed but an arbitrarily chosen element of $\mathbb{R}$. The inverse images of open sets in $\mathbb{R}_2$ will still be $p_2^{-1}[(x, y)]$, where (x, y) is an open interval in $\mathbb{R}_2 = \mathbb{R}$. They are still represented by open infinite horizontal strips of the form shown in fig.2. The intersections $p_1^{-1}(\{x_0\}) \bigcap p_2^{-1}[(a, b)]$ of these inverse images will be finite (in terms of length) line segments with

open ends. Such lines will also be standing erect in the plane as shown in fig.3. Hence the (weak) topology on $\mathbb{R}^2$, of the projection maps, in line with Theorem 1.1, will include as open sets finite (in length) vertical straight lines with the end-points excluded. (By interchanging the roles of $\mathbb{R}_1$ and $\mathbb{R}_2$ we will see that another weak topology on $\mathbb{R}^2$ exists which includes finite horizontal straight lines, without the end-points, as open sets.) It may also be noticed that since sets of the form $(a, b], [a, b), [a, b], (a, b)$, etc. are open in $\mathbb{R}_1$, it follows that different rectangles, including the open rectangles, as shown in fig.4 are *all* open in this weak topology.

9. The last statement reveals yet another point. The discrete topology of $\mathbb{R}^2$ can be obtained as the weak topology of the projection maps by taking the inverse images $p^{-1}(G)$ of projection maps of $\mathbb{R}^2$ when the factor spaces are endowed with their own discrete topologies. Of course in the finite-dimensional case this idea is not new—yet again we are in a uniquely right position here to not only ask the question *What about infinite dimensions?* but to give the best answer so far to it. One may also ask: What about the converse? We have an important contribution with respect to its converse. This is part of the discussions in Chapter 5. (References: Goodner (1950), Hewitt (1953), Ptak (1954), Halmos (1958), Robertson (1958), Horvath (1966), Schaefer (1971), Rudin (1973), Costa and Farber (2009), Douglas and Nowak (2009), Niederkruger and Rechtman (2009), Shelluchin (2009), Jean-Paul *etal.*, (2009), Ramachandran and Wolfston (2009), Sabloff and Traynor (2009), and Leininger and Margalit (2013).)

1.3 Aim and Objectives

The broad aim of this study is to explore deeply, and thus more fully appreciate the vast landscape of Weak Topology by taking a constructive approach and asking questions (maybe outside the box). More specifically, the objectives of the study are to:

1. apply a geometric/constructive approach in studying weak topology.

2. determine how the resulting weak topology on a set X changes with respect to changes in the topologies of the range spaces.

3. determine the process of inheritance or inducement of discreteness property between a weak topology and the topologies of the range spaces in a weak topological system.

4. determine the process of inheritance and inducement of the lower separation axioms T_0, T_1, T_2 (Hausdorff), and $T_{2\frac{1}{2}}$ (Tychonoff) properties between a weak topology and the topologies of the range spaces in a weak topological system. The same searchlight of inquiry is beamed on the higher separation axioms such as regularity, normality, complete regularity, and perfect normality.

5. investigate whether a fixed family of functions will always (or not always) generate different weak topologies on a set X—even as the range topologies change—and to characterize the scenarios in general weak topological systems.

1.4 Study Area and Scope

This study focuses on Weak Topology. Weak topology is a very important aspect of Functional Analysis. Our focus is on extending and generalizing various concepts relating to weak topology.

1.5 Some Historical Preliminaries

1.5.1 On Discreteness and the Separation Axioms

Rene Descartes (1596–1650) was the first man to come up with the idea of a set whose elements are ordered *coordinate points*. In his work *La Geometrie*, published in 1637, he visualized this kind of set to exist in the form of what has come to be known as the *Cartesian plane*, after his name. Since most geometrical shapes and figures idealized by the geometer, Euclid, can easily be depicted pictorially on the Cartesian plane, some people also refer to this plane as the Euclidean plane. The idea of Cartesian product sets has, since the time of Descartes, also been extended beyond the plane—to arbitrary product sets.

The first person to think of topologizing a (Cartesian) product set was Tieze, who lived through the late nineteenth century to the early twentieth century. His idea gave rise to the *box topology*; also called Tieze topology by some people. Soon after his contemporary, Tychonoff, constructed a topology now commonly known as the *product topology* (or Tychonoff product topology) on Cartesian product sets. (Angus Taylor and David Lay (1980))

Several topological results have since early twentieth century been established in respect of (particularly) the product topology on Cartesian product sets. Yet all is never done. Among our contributions in this work is the statement and proof (in section 4.5) that if a product topology is perfectly normal, then all the factor spaces would be perfectly normal and, hence, completely normal. Also we proved (in that section) that if a product topology is completely regular then all the factor spaces will be completely regular. And we prove in section 4.4 that if a product topology is discrete, then all the factor spaces are discrete. (Nachbin (1948), Nachbin (1954), Collins (1955), Kakutani and Klee (1963), Seymour Lipschutz (1965), Johnson (1967), Benjamin Sims (1976), Sheldon Davis (2005), D'Aristotile and Fiorenza (2006), Buhovsky (2009), Yan Song (2013), Schirmer (2013), Greenwood and Lockyer (2013), Matkowski (2013), Abbas Mujaheed *etal.,* (2013), Ikegami *etal.,* (2013), Yukinobu Toda (2013), Bernard *etal.,* (2013), and Morris (2016).)

1.5.2 Complement Topology: What is it?

It is known that a topology τ on a set X is the collection of all the *open* subsets of X. Hence, a topology τ on a set X is a collection of subsets of X which satisfy the *axiom of openness*; the standard four conditions. Openness of a subset is therefore relative to the topology under consideration. Some sets which are considered closed in one topology are open in another topology and vice-versa. A question of interest is *Can all those sets considered closed with respect to a topology on a set X be precisely the ones considered open with respect to another topology on X?* Of course, we are excluding the trivial cases of the discrete and indiscrete topologies on X. This seemingly academic but rather interesting question is the main motivation for the inquiry that led to the discovery of what we are inclined to call Complement Topology. (Myers (1950), Bartle *etal.,* (1955), Kelly (1955), Wada (1961), Barov and Dijkstra (2009), Zhao and Liang (2011), Carmago *etal.,* (2013), Elias and Wilson (2013), and Delaroche (2013).)

Chapter 2

LITERATURE REVIEW

2.1 The Usual Weak Topologies

Definition 2.1 *Two linear spaces A and B, over the same scalar field, K, are said to be in duality if A maps B into K via linear maps $\varphi_a : B \to K$, where $\varphi_a(b) = \langle a, b\rangle$, and $a \in A$ are chosen arbitrarily and fixed; and B maps A into K via linear maps $\varphi_b : A \to K$ where $\varphi_b(a) = \langle a, b\rangle$, and $b \in B$ are chosen arbitrarily and fixed.*

Remark
The full meaning of the definition is that the pairing maps φ_a on B into K are linear, and the pairing maps φ_b on A into K are linear. One may also better understand this idea by looking at the following alternative definition.

Definition 2.2 *By a dual system we mean a pair (A, B) of two linear spaces over the same scalar field K, together with a bilinear form $\langle ., .\rangle : A \times B \to K$ on the product of A and B into K.*

Definition 2.3 *The duality (A, B) is said to be separated in B if the maps φ_a are one-to-one, and separated in A if the maps φ_b are one-to-one. If the duality is separated in both A and B then the dual system is simply said to be separated.*

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Definition 2.4 *Let (X, X^*) be a dual system over K, where X^* is the topological dual of X. Let A be a finite subset of X. Then the polar A° of A is a subset of X^* defined by*

$$A^\circ = \{f \in X^* : |\langle x, f\rangle| \leq 1, \forall x \in A\}$$

Theorem 2.1 *The collection* $\tau = \{A^\circ : A \text{ is a finite subset of } X\}$ *is a topology on* X^**, called the weak star topology.*[1]

Definition 2.5 *If we interchange the roles of* X *and* X^* *in theorem 2.1 above we get a topology on* X*, called the weak topology on* X*.*[2]

Definition 2.6 *If we collect the polars of compact subsets of* X^* *we get the Mackey topology on* X*.*

Definition 2.7 *If we collect the polars of bounded subsets of* X^* *we get the strong topology on* X*; and the strong topology coincides with the norm topology.*[3]

NOTE
Some writers (though not many) also call *the* weak topology *the weakened topology* when they compare it with what they call *the initial topology.* Ambiguity arises then in this area if we also note that some writers see *the weak topology* as *the initial topology* itself. (See for instance Edwards (1995), page 89; and some internet references below.)

The process of Theorem 1.1 (which we have to reiterate is not our innovation) in this work results in the formulation of several weak topologies depending on functions, their domain of definition and range spaces. Many people have used the process to obtain some weak topologies, but certainly not many people (to say the least) have used this process to *construct* weak topologies. For example, using the process, one can easily believe that a weak topology would be obtained on the Cartesian plane R^2 if the projection maps are the family of functions—resulting in a product topology—and the range spaces are endowed with the cofinite topology. However, one should not and cannot just believe any statement about the shape or structure of an open set in such a weak topology if actual construction of the weak topology has never been done. (And one might still ask: What *is* actual construction here and when might it really be necessary. Answer: Actual construction here means simply taking cofinite-topology-open sets (or generally speaking,

[1] It is our opinion that this is not the only topology on X^* that is worth being called *weak star* topology. Any family of functions on X^* can be used to construct a *weak star* topology on X^*. The acronym *star* is just an indication that the weak topology (in a context) on X^* is actually obtained on X^*. We also express similar thoughts and opinion about what has definitively been called *the weak topology* on X.

[2] See Taylor and Lay (1980), page 157 on this; and Edwards (1995), pages 88-89.

[3] (Taylor and Lay (1980), page 168)

open sets) in the range spaces and first collecting their inverse images under the projection maps before finally building up the weak topology according to Theorem 1.1. This actual construction might be needed when, for instance, we want to *obtain* or *construct* another weak topology on any other set using this *cofinite topology induced* weak topology as one of the range topological spaces.) Strictly speaking, different ways of constructing weak topology result in different weak topologies being obtained; since a change of class of functions, a change of range spaces, or a change of domain space will necessarily change the substance of the weak topology obtained. In all these cases *still* it is only by constructing a weak topology that we can practically know the geometrical properties of its open sets. In their discussion of *Weak Topologies* some authors assume throughout the existence of bounded linear maps of different sorts; in fact, from the outset some explain this topic as the study of the relationship between a linear space and its conjugate or dual space (Halperin (1953), Kolmogorov and Fomin (1957), Yosida (1968), Angus Taylor and David Lay (1980), Rudin (1991), and Edwards (1995)). We have to state here, though, that weak topologies do not have to do with only dual systems; nor does it have to be defined among linear spaces which are known to be in duality. In short, the existence (or otherwise) of linear structure on a set does not make (or mar) the possibility of a weak topology being obtained on the set. In fact, in some important applications of topology—like in space-time physics—you cannot have linear structure on a set before you get a topology on it. (What for instance would be $5Paris$ or where will $3London + 7NewYork$ be?)

The somewhat narrow focus of research on weak topologies, the somewhat even fewer results so far existing in this area, and the actual dearth of intensive and extensive research activities on weak topologies in the recent times have even made some mathematicians to come to a conclusion that nothing new can come from any research effort entirely devoted to weak topology! We are however not discouraged or deterred by this really dangerous conclusion; if anything, we are rather encouraged and motivated by them. We *re*-viewed the concept of weak topology here from the prism of constructive approach, and the result is not discouraging.

One impeccable fact is this: Whenever a family of functions is changed, or a range space is changed, or the domain space is changed, a different weak topology is obtained. Accepting this fact and taking it for granted is not enough. It is important that we go further to use it to construct as many weak topologies as we can, in order to see what more we can see, and possibly say more as we can about weak topologies.

The following definition of *the weak topology* and *the weak star topology* can also be found in existing literature, and no publication (past or present) has opined that something more is needed—on the basis of the kind of insight that practical construction can give.

Definition 2.8 *Let E be a normed linear space over $K(= R$ or $C)$ and let E^* be the space of all bounded linear functionals on E. For each $x \in E$, fixed but chosen arbitrarily, let φ_x be a function on E^*, $\varphi_x : E^* \to K$ defined by $\varphi_x(f) = \langle x, f\rangle$, $\forall f \in E^*$. Then the coarsest topology on E^* relative to which all the functions φ_x are continuous is known as the weak topology on E^*. This is the topology often referred to as weak star topology, and it corresponds to the one given in terms of* polars*. If we interchange the roles played by E and E^* and have (linear) functionals defined on E, in terms of elements of E^*, $\varphi_f : E \to K$, defined by $\varphi_f(x) = \langle x, f\rangle = f(x)$, where $f \in E^*$, we obtain what is often (only) known as* the *weak topology on the linear space E.*

We have to state here that some mathematicians have defined *the weak star topology*, on E^*, as the coarsest topology on E^* with respect to which elements of a subspace of the bidual E^{**} (precisely the range $J(E)$ of the canonical map of E) are continuous. They then defined *the weak topology* on E^* as the coarsest topology, on E^*, with respect to which *all* elements of the bidual E^{**} are continuous. Question: If we mean by the bidual E^{**} the topological dual of E^* what do we call the topology on E^* generated by the algebraic dual of E^* (or a subspace thereof); and vice versa? The same reasoning and final question applies to whatever we may have, before now, called *the* weak topology on E. The fact is that the use of the article the in the phrase: *the weak topology on X* is more or less an error (when either (a) it is used to imply that only one fixed family of functions can induce what may be called weak or weak star topology on a set, or (b) it is used to mean that any given family of functions can induce only one weak or weak star topology on a set X). And surprisingly (as will be seen in the excerpts at the end of this section) many authors in this area of mathematical research have maintained the use of the article *the* in their definition of weak topology, without due attention to the numerosity of weak topologies that can be constructed on a set.[4] This is why for instance some authors have stated some theorems on weak topology as if only one weak topology is in existence. For example,

[4]Even a fixed family of functions can be used to obtain different weak topologies on one set—by varying the topology of the range spaces. So, when we say *the weak topology* or *the weak star topology* the next question would be, or should be: which one?

in Proposition 6.9 on page 124 of Chidume (1996), it is explicitly stated that: *"The weak topology is Hausdorff."* Part of our contribution to knowledge in this thesis is the exposition that some weak topologies are NOT Hausdorff. (See subsection 4.1.13.) One topological property of one weak topology may not be shared by other weak topologies.
And even if in the end it is accepted to define weak topology as one unique topological object on a set, it is equally acceptable that the method (of Theorem 1.1) used in obtaining such a topology on a set can be extended in use to obtain other topologies, on the same set, which would deserve a name no less than 'weak topologies'.

REMARK

1. It may be observed that E here is not taken to be a Banach space as is done in some writings. Completeness of E is not necessary in this definition; in fact there are cases in which E is just a linear space (not normed). And in this work many of the constructions do not require a linear structure on a set to be made possible.

2. We may observe that this (weak star) topology is usually the only other weak topology that some mathematicians often talk about.

3. The functions φ_x are bounded and linear; and if we denote this weak topology by τ, then even though the dual system (E, E^*) is separated in E^* we cannot say that all bounded linear maps (i.e. not only those of the form φ_x) are continuous with respect to this topology τ. One simple reason for this is that the maps φ_x are *defined on* E^* whereas the usual elements f of E^* are defined on E. There are two classes of bounded linear maps in focus here; those of the form ϕ_x on E^* and those of the form $\phi_f, f \in E^*$ on E.

Let us now take some excerpts from literature on how different authors have approached the concept of weak topology in terms of its definition.

1. *"Let E be a set and let $\{Y_i\}_{i \in I}$ be a family of topological spaces. For each $i \in I$ we shall associate a map $\varphi_i : E \to Y_i$. Our problem of interest is to find how to endow E with a* smallest *topology such that the maps $\varphi_i, i \in I$ are continuous."* Quoted from Chidume (1996), page 121.

2. *"Let E be a Banach space and $f \in E^*$. For each $f \in E^*$ we associate a map $\varphi_f : E \to R$ defined by $\varphi_f(x) = \langle f, x \rangle = f(x)$ for all $x \in E$. As*

f ranges over E^* we obtain a family $\{\varphi_f\}_{f \in E^*}$ of maps of E into R.
Definition 6.6
The weak topology on E (denoted by ω) is the smallest topology on E which makes the maps φ_f continuous." Quoted from Chidume (1996), page 124.

3. *"Suppose that E is a vector space, no topology on E being involved. Let L be a vector subspace of E, and let E^* be the algebraic dual of E. Amongst those topologies on E^* relative to which each function $x^* \to \langle x, x^* \rangle$ is continuous, x being fixed but arbitrary in L, there is a weakest. This topology is denoted by $\sigma(E^*, L)$ and is termed the weak topology on E^* generated by L."* Quoted from Edwards R.E. (1995), page 88, Subsection 1.11.1

4. *"Suppose again that E is a vector space with algebraic dual E^*. We let M be a vector subspace of E^*. The method of Subsection 1.11.1 leads then to the topology $\sigma(E^{**}, M)$. Using the device of injecting E into E^{**} (Subsection 1.4.7), we obtain the induced topology $\sigma(E, M)$, spoken of as the weak topology on E generated by M.*
Naturally, $\sigma(E, M)$ can equally well be defined without intermediate reference to E^{**} and the injection of E into E^{**}. It is simply the weakest topology on E relative to which each of the linear forms $x \to \langle x, x^* \rangle$, x^* being fixed but arbitrary, is continuous." Quoted from Edwards R.E (1995), page 88, Subsection 1.11.2

One may now wonder if the excerpts and quotations above cannot have more modern, recent and even present-day updates, since the excerpts seem to come from only (more or less) very old works. The answer to this is that what one may call the latest ideas and updates on weak topology are not radically different from the old ones. The following excerpts from some internet publications will illustrate this point.

SOME INTERNET REFERENCES

1. *"A normed vector space $(X, \|.\|_X)$ automatically generates a topology, known as the* norm topology *or* strong topology *on X, generated by the open balls. . . . . . However, in some cases, it is useful to work in topologies on vector spaces that are weaker than a norm topology. . . . . . Two basic weak topologies for this purpose are the weak topology on a normed vector space X, and the weak* topology on a dual vector space*

X^*." Culled from page 1 of 13 of Internet publication titled *The strong and weak topologies: What's new*, by Terence Tao, 21 February, 2009, via the link http://terrytao.wordpress.com/2009/02/21/245b-notes-11. Terence Tao (2009)

2. ***"Definition 2 (Weak and** weak* **topologies)***
Let V be a topological vector space, and let V^* be its dual.

   - *The weak topology on V is the topology generated by the seminorms $\|x\|_\lambda = |\lambda(x)|$ for all $\lambda \in V^*$.*
   - *The weak* topology on V^* is the topology generated by the seminorms $\|\lambda\|_x = |\lambda(x)|$ for all $x \in V$".* Culled from page 3 of 13 of Internet publication titled *The strong and weak topologies: What's new*, by Terence Tao, 21 February, 2009, via the link http://terrytao.wordpress.com/2009/02/21/245b-notes-11.

3. *"Let X be a topological vector apace over K. We may define a possibly different topology on X using the continuous (or topological) dual space X^*. The dual space consists of all linear functions from X into the base field K which are continuous with respect to the given topology. The weak topology on X is the initial topology with respect to X^*. In other words, it is the coarsest topology such that each element of X^* is a continuous function. A subbase for the weak topology is the collection of sets of the form $\varphi^{-1}(U)$ where $\varphi \in X^*$ and U is an open subset of the base field K."* Excerpted from *Weak topology, on Wikipedia, the free encyclopedia, 2013, page 2.* While the author of this internet publication did not make his/her name known, references were made to works by Yosida Kosaku (1980), Gert Pedersen (1989), Walter Rudin (1991), John Conway (1994),and Willard Stephen (2004).

4. *"More generally, if F is a subset of the algebraic dual space, then the initial topology of X with respect to F , denoted by $\sigma(X, F)$, is the weak topology with respect to X."* Excerpted from *Weak topology, on Wikipedia, the free encyclopedia, 2013, page 2.* While the author of this internet publication did not make his/her name known, references were made to works by Yosida Kosaku (1980), Gert Pedersen (1989), Walter Rudin (1991), John Conway (1994),and Willard Stephen (2004).

5. *"A space X can be embedded into X^{**} by $x \mapsto T_x$, where $T_x(\phi) = \phi(x)$. Thus $T : X \to X^{**}$ is an injective linear mapping, though it*

is not surjective unless X is reflexive. The weak* topology on X^* is the weak topology induced by the image of T. In other words, it is the coarsest topology such that the maps T_x from X^* to the base field R or C remain continuous." Excerpted from *Weak topology, on Wikipedia, the free encyclopedia, 2013, page 3*. While the author of this internet publication did not make his/her name known, references were made to works by Yosida Kosaku (1980), Gert Pedersen (1989), Walter Rudin (1991), John B. Conway (1994),and Willard Stephen (2004).

6. Jawad Y. Abuhlail in his Ph.D. Dissertation (on *On the Linear Weak Topology and Dual Pairings Over Rings*) in 2001, talked about *linear weak topology* which has to do with *R-pairing*, *R-modules*, *dual pairings*, *dense pairings* and *locally projective modules*. In this internet publication, the auther (in 2003) made references to Berning (1994), Bourbaki (1974), Brszinski (2003), Garfinked (1976), Kothe (1966), Kelly and Namioka (1976), Lambe and Radford (1997), Ohm and Rush (1972), Radford (1973), Wisbaner (1991), and Zimmermann-Huisgen (1976).

From the foregoing, it may already be clear that even the so-called latest writings on weak topology have not really brought any change to how the concept is *conceptualized*: no blame is intended here. Certainly they seem interested or involved in a discussion of the topic as a flash in the pan of what they actually set out to do—use any definition of weak topology available to them to achieve any goal they set out to achieve; not to use the definitions to construct weak topologies in order to have a more practical feel of the concept. The usual approach to the concept of weak topology can best be described as *theory on weak topologies* while the approach to the concept adopted here can be described as *practice on weak topologies*.

The existence of many versions and (even) variants of *analytic* definition of weak topology again highlights the need to clarify this concept further, and the *constructive* approach (adopted in this thesis) to formulating weak topologies does not only come in handy in this regard: it also made more discoveries about topology in general possible as revealed hereafter.

Chapter 3

METHODOLOGY

3.1 Introduction

Necessarily in this work, we have applied the scientific method of research in carrying out our study—encompassing hypothesis, observation, experimentation, new theorems (when available) and their proof, and conclusions. Let us briefly state below how these apply in the context of this thesis.

1. **Hypothesis:** The hypotheses are the basic definitions of topology, open or closed sets, topological space, weaker and stronger topologies, weak topology, continuity of functions, etc. These are outlined in the background to the study.

2. **Observations:** We observed the variants and versions of weak topology (in terms of the definition) in literature and we raised some questions concerning these various definitions.

3. **Experimentation:** We did not just raise the questions. We tried to find answers to the questions by constructiong weak topologies. And, in doing this, we strictly and rigidly followed the procedure already laid down in Theorem 1.1.

4. **New Theorems:** The experiments we made with construction resulted in new revelations coming up. Some of these new revelations are stated and proved as new results.

5. **Conclusions:** Each section ends with a list of conclusive assertions which are carried forward to the chapter on summaries, conclusions and suggestions. These sectional summaries, conclusions and suggestions altogether form chapter five.

"A good research method should lead to:

- *Originality/Novelty*
- *Contribution*
- *Significance*
- *Technical Soundness; and*
- *Critical Assessment of Existing Work*

(Igbokwe (2009))

3.2 The Hyperplane-Open Weak Topologies of $\mathbb{R}^n$

In this section we *practically* construct weak topologies, compare or contrast them with other (weak) topologies, and describe, depict and sketch the geometrical shapes and properties of open sets in such (weak) topological spaces. In particular, we start off by bringing forth constructible and easy to visualize examples of weak topologies such as

1. line-open topology in $\mathbb{R}^n$; $n \geq 2$
2. plane-open topology in $\mathbb{R}^n$; $n \geq 3$
3. hyperplane-open topology on $\mathbb{R}^n$, generally.

Throughout the remainder of this work, $\mathbb{R}$ will denote the set of real numbers, $\mathbb{N}$ the set of natural numbers and $\mathbb{R}^n$ the product of n-copies of $\mathbb{R}$, for each $n \in \mathbb{N}$. (For references see Bochner and Taylor (1938), Lefschetz (1949), Williamson (1954), Koethe (1969), Porter (1996), Balacheff (2009), Stephen (2004), and Klein *etal.,* (2009).)

We know that in general, lines (curved or not) are not open in the usual topology of the Cartesian plane $\mathbb{R}^2$. It is also known that lines, curved or straight, are open subsets of $\mathbb{R}^2$ when the discrete topology is assumed. Is there another topology on $\mathbb{R}^2$ in which lines are open? If there are, are they weak topologies and what is their landscape. We are here set to introduce another topology on $\mathbb{R}^2$, and indeed on $\mathbb{R}^n$, with respect to which lines (even if only straight ones) are open. We recall or observe that the usual topology of $\mathbb{R}^2$ is generated (in line with theorem 1.1) by the projection maps when the factor spaces of $\mathbb{R}^2$ are themselves endowed with their usual topologies.

Construction 3.1 *Consider $\mathbb{R}^2$. Let the horizontal factor space $\mathbb{R}_1$ be endowed with the discrete topology $(\mathbb{R}_1, D)$ and the vertical factor space with the usual topology $(\mathbb{R}_2, u)$. Then the coarsest topology on $\mathbb{R}^2$ with respect to which the projection maps p_1 and p_2 are continuous is called the* vertical line open topology *because vertical lines are among the basic open sets in this topology.*

Singletons $\{x_0\}$ are open in the horizontal factor space $(\mathbb{R}_1)$. So $p_1^{-1}(\{x_0\})$ is a subbasic open set in the resulting weak topology on $\mathbb{R}^2$. Such a set is an infinite (in length) vertical line passing through the point $(x_0, 0)$ in the plane. That is

$$p_1^{-1}(\{x_0\}) = \{(x_0, y) : y \in \mathbb{R}\}.$$

Hence the basic open sets in this weak topology on $\mathbb{R}^2$ include vertical lines with finite lengths; that is, vertical lines with finite lengths without the end points. To see this, we recall or observe that a sub-basic open set emanating from the vertical factor space $\mathbb{R}_2$, where the sets (a, b) are open, is of the form

$$p_2^{-1}\{(a, b)\} = \{(x, y) : a < y < b\},$$

an infinite horizontal open strip. Its intersection with vertical lines will result to finite vertical lines (without the endpoints) as open sets.

The result is that when the base for this weak topology is formed *all* vertical lines in the plane, no matter their length, will turn out to be basic elements and, as such, open in this resultant topology. (See fig.3) This topology will include vertical lines of all lengths because the second factor space, the vertical factor space, with usual topology still has intervals of various lengths as open sets and the inverse images of such open intervals under the second projection map, that is

$$p_2^{-1}\{(a, b)\} = \{(x, y) : a < y < b\}$$

will still be infinite horizontal strips. Their intersection with infinite vertical lines will result to vertical lines of all lengths as open sets.

At this juncture one may still ask: How many topologies on $\mathbb{R}^2$ are all vertical lines open with respect to? For now we know there are two topologies (this Verical line open topology and the discrete topology) on $\mathbb{R}^2$ with respect to which all vertical lines are open. We may still define the open vertical line topology as the weaker topology on $\mathbb{R}^2$ which makes all vertical lines open.

Vertical lines are not the only open sets in this topology on $\mathbb{R}^2$. In the discrete topological factor space, the horizontal axis, every other type of set (apart from singletons) is still open. In particular, subintervals of $\mathbb{R}$ of the form $(a,b], [a,b), (a,b), [a,b]$ are all open. Hence the usual open rectangles, vertically half-open, half-closed, and vertically closed rectangles are all open in this open vertical line topology of $\mathbb{R}^2$. It then follows that this topology is strictly stronger than the usual topology of $\mathbb{R}^2$ and yet strictly weaker than the discrete topology of $\mathbb{R}^2$ since, for instance, singletons are not open in this topology. (Later we shall show how to construct *weak* topologies on $\mathbb{R}^n, n \geq 2$ in which some singletons are open and others are not.)

Construction 3.2 *Consider $\mathbb{R}^2$ but now with the horizontal factor space $\mathbb{R}_1$ endowed with the usual topology and the vertical factor space $\mathbb{R}_2$ endowed with the discrete topology. Then the horizontal line open topology on $\mathbb{R}^2$ results.*

As with the open vertical line topology of $\mathbb{R}^2$, the open horizontal line topology is generated by the projection maps. Also it is easy to see that the open horizontal line weak topology is strictly stronger than the standard Euclidean topology of $\mathbb{R}^2$ and strictly weaker than the discrete topology of $\mathbb{R}^2$. We also observe that these two weak topologies (i.e. the open vertical and the open horizontal line topologies of $\mathbb{R}^2$) are not comparable; that is, neither is stronger or weaker than the other. The intersection of these two topologies on $\mathbb{R}^2$ is finer than the usual topology of $\mathbb{R}^2$. This is proved next.

Proposition 3.1 *Let τ_v, τ_h and τ_u denote respectively the open vertical line, the open horizontal line and the usual topologies on $\mathbb{R}^2$. Then*

1. $\tau_u \leq \tau_v$;
2. $\tau_u \leq \tau_h$; *hence*
3. $\tau_u \leq \tau_v \bigcap \tau_h$.

Proof:
1 and 2 are obvious from preceding discussions. For 3, let $G \in \tau_u$. Then from 1, $G \in \tau_v$; and from 2, $G \in \tau_h$ also. Therefore $G \in \tau_v$ and $G \in \tau_h$, for all $G \in \tau_u$, as $G \in \tau_u$ is arbitrary. That is, $G \in \tau_v \bigcap \tau_h, \forall G \in \tau_u$. $\Rightarrow \tau_u \leq \tau_v \bigcap \tau_h$.

Construction 3.3 *Let $n \geq 3$ and let $X = \mathbb{R}^n$ be the product of n copies of $\mathbb{R}$. Let the projection maps $p_i : X \longrightarrow \mathbb{R}_i$, for $1 \leq i \leq n$, be defined in the usual way by $p_i(\bar{x}) = x_i$, where $\bar{x} = (x_1, x_2, \cdots, x_n)$, for all $\bar{x} \in X$. Let m factor subspaces $(1 \leq m < n)$ be endowed with discrete topology and let the remaining $n - m$ factor subspaces retain the usual topology of $\mathbb{R}$. Then the* hyperplane-open topology *of $X(= \mathbb{R}^n)$ is the coarsest topology on $\mathbb{R}^n$ relative to which the projection maps are continuous.*

In $\mathbb{R}^2$, $p_i^{-1}(\{x_{i_0}\})$ is a straight, infinite line perpendicular to the ith axis, $1 \leq i \leq 2$, a 1-dimensional hyperplane perpendicular to the ith axis; for any fixed point x_{i_0} in the ith factor space. In R^3, $p_i^{-1}(\{x_{i_0}\})$ is a straight, infinite plane (a 2-dimensional hyperplane) perpendicular to the ith axis, $1 \leq i \leq 3$; for any fixed point x_{i_0} in the ith factor space. In $\mathbb{R}^n (n \geq 4)$, $p_i^{-1}(\{x_{i_0}\})$ is a hyperplane (of dimension $n - 1$), $1 \leq i \leq n$; for any fixed point x_{i_0} in the ith factor space. However, if $n - 1$ factor spaces are endowed with the discrete topology, and the nth factor space with the usual topology, then the basic open set

$$p_1^{-1}(\{x_0^1\}) \bigcap p_2^{-1}(\{x_0^2\}) \bigcap \cdots \bigcap p_{n-1}^{-1}(\{x_0^{(n-1)}\}) = \bigcap_{i=1}^{n-1} p_i^{-1}(\{x_0^i\})$$

results in a one-dimensional hyperplane; a straight line (parallel to the nth factor space which has the usual topology). So, in the product $X = \mathbb{R}^n$, lines are open in the hyperplane open topology if $n - 1$ factor spaces are endowed with the discrete topology (and the nth factor space has the usual or possibly any other topology on $\mathbb{R}$).

For example in $\mathbb{R}^3$, exactly 2 factor spaces (only) have to be endowed with the discrete topology for lines to emerge really as open sets. If we give all three factor spaces of $\mathbb{R}^3$ the discrete topology, then the resulting open line topology would coincide with the discrete topology of $\mathbb{R}^3$; and this agrees with one of the theorems existing before. If only 1 factor space of $\mathbb{R}^3$ is given the discrete topology, then the resulting weak topology will have no lines as open sets.

REMARK 3.1

1. We observe that m has to be strictly less than n in the last construction since otherwise we would get the discrete topology of $\mathbb{R}^n$.

2. What actually happens is that if we endow 2 factor spaces of $\mathbb{R}^3$ with the discrete topology and the remaining 1 factor space with the usual topology of $\mathbb{R}$, then the line-open (weak) topology results. If we endow

1 factor space of $\mathbb{R}^2$ with discrete topology, then these open lines will *all* be parallel to one axis of $\mathbb{R}^2$; parallel to the vertical axis if the horizontal factor space is endowed with the discrete topology, and vice versa. In $\mathbb{R}^3$, with 2 factor spaces given the discrete topology, all the open lines will be parallel to the only 1 factor space retaining the usual topology, and perpendicular to the plane of the two other factor spaces.

3. May be it is necessary at this point to prove that the line-open topologies constructed in this section are indeed all weak topologies. We only prove that no other topology weaker than the assumed weak topology, say τ, makes each of the projection maps continuous and we do this in the general case of $\mathbb{R}^n$.

Theorem 3.1 *Let τ be the topology on $\mathbb{R}^n$ determined by the projection maps on $\mathbb{R}^n$ when the factor spaces of $\mathbb{R}^n$ are endowed variously with discrete and usual topologies of $\mathbb{R}$. (That is, some factor spaces of $\mathbb{R}^n$ are endowed with discrete topology while the others retain the usual topology of $\mathbb{R}$.) Then τ is a weak topology;* the *weak topology of $\mathbb{R}^n$ with respect to these projection maps.*

Proof:
Let γ be another topology, on $\mathbb{R}^n$, with respect to which each projection map p_i, $1 \leq i \leq n$, on $\mathbb{R}^n$, is continuous; with the factor spaces of $\mathbb{R}^n$ given different topologies between the discrete and the usual topologies of R. Let T_i, $1 \leq i \leq n$ be the topology given $\mathbb{R}_i$ in this arrangement. Then for each open set $G_i \in T_i$, $p_i^{-1}(G_i) \in \gamma$, $1 \leq i \leq n$. But $p_i^{-1}(G_i)$ are the sub-basic sets of τ on $\mathbb{R}^n$. It follows that, as γ is closed under finite intersections and arbitrary unions, τ is weaker than γ.

Chapter 4

MAIN RESULTS AND DISCUSSIONS

4.1 Some Constructions and Their Implications

4.1.1 Point Open Weak Topologies on $\mathbb{R}^n$

POINT-OPEN WEAK TOPOLOGIES OF $\mathbb{R}^n$

We promised (just before Construction 3.2) to show how to construct weak topologies on $\mathbb{R}^n$, $n \geq 2$, in which some singletons are open and others are not. We fulfill that promise immediately.

Construction 4.1 *Let $X = \{x_1, x_2, x_3, \cdots, x_n\}$ be any finite set of real numbers, and let 2^X be the power set of X. Then $\{\mathbb{R}, 2^X\}$ is a topology on $\mathbb{R}$, called (and introduced in this work as) the X-topology on $\mathbb{R}$. The point-open weak topology on $\mathbb{R}^n$ is the weak topology, on $\mathbb{R}^n$, generated by the projection maps when the factor spaces are each endowed with the X-topology.*

REMARK 4.1

We observe that actually some points of $\mathbb{R}^n$ (as singletons) are open in this weak topology while the others are not. This is why we call this *the* point-open *weak* topology of $\mathbb{R}^n$. For example, let $X = \{x_1, x_2\}$; then $2^X = \{\emptyset, X, \{x_1\}, \{x_2\}\}$ and the X-topology on $\mathbb{R}$ is $\{\emptyset, X, \{x_1\}, \{x_2\}, \mathbb{R}\}$. Let the factor subspaces $\mathbb{R}_1$ and $\mathbb{R}_2$ (horizontal and vertical respectively) of $\mathbb{R}^2$ be, each, endowed with this X-topology. Then the only singletons open in the weak topology of $\mathbb{R}^2$, generated by the projection maps this time, are

$$p_1^{-1}(\{x_1\}) \bigcap p_2^{-1}(\{x_1\}) = \{(x_1, x_1)\},$$

$$p_1^{-1}(\{x_1\}) \bigcap p_2^{-1}(\{x_2\}) = \{(x_1, x_2)\},$$
$$p_1^{-1}(\{x_2\}) \bigcap p_2^{-1}(\{x_1\}) = \{(x_2, x_1)\},$$
$$p_1^{-1}(\{x_2\}) \bigcap p_2^{-1}(\{x_2\}) = \{(x_2, x_2)\}.$$

If, say, all three factor spaces of $\mathbb{R}^3$ are given this particular X-topology, then the only open singletons of $\mathbb{R}^3$ in the resulting weak topology would be $\{\{(x_1, x_1, x_1)\}, \{(x_1, x_1, x_2)\}, \{(x_1, x_2, x_1)\},$ $\{(x_2, x_2, x_2)\}, \{(x_2, x_2, x_1)\}, \{(x_2, x_1, x_2)\}, \{(x_2, x_1, x_1)\}, \{(x_1, x_2, x_2)\}\}$.

We also note that some matrices of coordinate points (grid points) in the Cartesian plane $\mathbb{R}^2$ are open sets in the X-topology induced weak topology. For example, we observe that

$$p_1^{-1}(X) = p_1^{-1}(\{x_1, x_2\}) = \{(x, y) \in \mathbb{R}^2 : x = x_1\} \bigcup \{(x, y) \in \mathbb{R}^2 : x = x_2\}$$
$$= \{(x_1, y) \in \mathbb{R}^2\} \bigcup \{(x_2, y) \in \mathbb{R}^2\} = \text{two vertical infinite lines and}$$
$$p_2^{-1}(X) = p_2^{-1}(\{x_1, x_2\}) = \{(x, y) \in \mathbb{R}^2 : y = x_1\} \bigcup \{(x, y) \in \mathbb{R}^2 : y = x_2\}$$
$$= \{(x, x_1) \in \mathbb{R}^2\} \bigcup \{(x, x_2) \in \mathbb{R}^2\} = \text{two horizontal infinite lines.}$$

Therefore

$$p_1^{-1}(X) \bigcap p_2^{-1}(X) = p_1^{-1}(\{x_1, x_2\}) \bigcap p_2^{-1}(\{x_1, x_2\})$$
$$= \{x_1, x_2\} \times \{x_1, x_2\} = \{(x_1, x_1), (x_1, x_2), (x_2, x_1), (x_2, x_2)\};$$

a 2×2 matrix of four coordinate points. The matrix M is shown below.

$$M = \begin{bmatrix} (x_1, x_2) & (x_2, x_2) \\ (x_1, x_1) & (x_2, x_1) \end{bmatrix}$$

This matrix-open weak topology on the Cartesian plane may be compared or contrasted with the cofinite topology induced weak topology (ahead, in Section 4.2) in which matrices are actually closed sets.
REMARK 4.2

The type of topology called X-topology on $\mathbb{R}$ here can be shown to be generally available between any subset and its superset, and we show this immediately in the proposition 4.1 below.

4.1.2 Subset-induced Topologies

Proposition 4.1 *If $X \subset E$, then any topology, say τ_X, on X induces a topology, say τ_{XE}, on E, given by $\tau_{XE} = \tau_X \bigcup \{E\}$.*

Proof:

1. $\emptyset \in \tau_{XE}$, since $\emptyset \in \tau_X$.

2. $E \in \tau_{XE}$, by definition.

3. Let $\{G_i\}_{1\le i\le n} \subset \tau_{XE}$. We show that $\bigcap_{i=1}^{n} G_i \in \tau_{XE}$. Clearly $\bigcap_{i=1}^{n} G_i \in \tau_{XE}$ if any of the G_is, $1 \le i \le n$, comes from τ_X. If all the G_is are equal (each) to E, then $E = \bigcap_{i=1}^{n} G_i \in \tau_{XE}$. So, in either case, $\bigcap_{i=1}^{n} G_i \in \tau_{XE}$ for all $\{G_i\}_{1\le i\le n} \subset \tau_{XE}$. Hence τ_{XE} is closed under finite intersections.

4. Let $\{G_\alpha\}_{\alpha\in\Delta} \subset \tau_{XE}$ be any number of sets in τ_{XE}. If one of the sets is equal to E, say $G_{\alpha_o} = E$, then $\bigcup_{\alpha\in\Delta} G_\alpha = E \in \tau_{XE}$, implying that $\bigcup_{\alpha\in\Delta} G_\alpha \in \tau_{XE}$. If $G_\alpha \ne E, \forall\alpha \in \Delta$, then $\bigcup_{\alpha\in\Delta} G_\alpha \in \tau_X \subset \tau_{XE}$, again implying that $\bigcup_{\alpha\in\Delta} G_\alpha \in \tau_{XE}$. Hence in any case τ_{XE} is closed under arbitrary unions.

Definition 4.1 *The topology τ_{XE}, on E, is called an X-topology on E; or a topology induced on E by the topology τ_X on X.*

Observe that one subset can induce several topologies on its superset.

Proposition 4.2 *Let (E, τ) be a topological space, and let $X \in \tau$ be a τ-open subset of E. Let $\tau_X = \{G \in \tau : G \subset X\}$. Then τ_X is a topology on X.*

Proof:

1. $\emptyset \in \tau_X$, since $\emptyset \in \tau$ and $\emptyset \subset X$.

2. $X \in \tau_X$, since $X \in \tau$ and $X \subset X$.

3. Let $\{G_i\}_{1\le i\le n} \subset \tau_X$ be any finite number of sets of τ_X; and let $N = \bigcap_{i=1}^{n} G_i$ be the intersection of these sets. Then clearly $N \in \tau$, as the intersection of a finite number of sets of τ. Also it is clear that $N \subset X$, since it is the intersection of some subsets of X. Hence $N \in \tau_X$.

4. Let $\{G_\alpha\}_{\alpha\in\Delta} \subset \tau_X$ be any family of sets of τ_X. Then $\bigcup\limits_{\alpha\in\Delta} G_\alpha = U \in \tau$, since τ is closed under arbitrary unions. Also $U \subset X$, as a union of subsets of X. Hence $U \in \tau_X$, implying that τ_X is closed under arbitrary unions and, hence, a topology on X.

Definition 4.2 *With X, E and τ_X as given in proposition 4.2, let $\tau_{XE} = \tau_X \bigcup \{E\}$ be an X-topology on E. Then τ_{XE} is an open subset induced topology on E.*

Note We observe that the open subset induced topology τ_{XE} on E is weaker than the topology τ on E.
It can be seen that Definition 4.2 is a particular case of Proposition 4.1.

EXAMPLES AND APPLICATIONS

Suppose that $f : (X, \tau_X) \to (Y, \tau_Y)$ is not continuous. It might be of interest to determine the strongest topology on Y weaker than τ_Y with respect to which f is continuous. Or, if $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous, to find weaker topology than τ_Y for which the mapping is still continuous.

EXAMPLE 4.1

Let $X = \mathbb{R} = Y$, and let $\tau_X = U$ (where U is the usual topology of $\mathbb{R}$), $\tau_Y = D$ (the discrete topology of R) and let $f(x) = x^2$. Then $f : (X, \tau_X) \to (Y, \tau_Y)$ is not continuous since $G = \{4\} \in \tau_Y$ but $\{-2, 2\} = f^{-1}(G) \notin \tau_X$ because $f^{-1}(G)$ is not U-open subset of $\mathbb{R}$. Let $Z = (0, \infty) \subset \mathbb{R}$ and let $U_1 = \{Z \bigcap G : G \in U\} \bigcup \{\mathbb{R}\}$. Then U_1 is a topology on Y, strictly weaker than D. Let $\eta_Y = U_1$. Then $f : (X, \tau_X) \to (Y, \eta_Y)$ is continuous. We also observe that η_Y is comparable to U and that for instance $(-1, a) \in U$ but $(-1, a) \notin \eta_Y, \forall a \in R, a \neq -1$.

EXAMPLE 4.2

Let $U_2 = \{G \in U : G \subset [0, \infty)\} \bigcup \{\mathbb{R}\}$. Then U_2 is a topology on $Y = \mathbb{R}$

Proof:

1. $\emptyset, \mathbb{R} \in U_2$.

2. Let $G_i \in U_2, i = 1, \cdots, m$. Then $G_i \in U$ and $G_i \subset [0, \infty), \forall i = 1, \cdots, m$; or $G_k = \mathbb{R}$ for some $k \in \{1, \cdots, m\}$. If all the $G_i (i = 1, \cdots, m)$ are not equal to R, then $\bigcap\limits_{i=1}^{m} G_i \subset [0, \infty)$ and $\bigcap\limits_{i=1}^{m} G_i \in U$. So $\bigcap\limits_{i=1}^{m} G_i \in U_2$. And if all the $G_i (i = 1, \cdots, m)$ are each equal to R, then $\bigcap\limits_{i=1}^{m} G_i = \mathbb{R}$ and so $\bigcap\limits_{i=1}^{m} G_i \in U_2$. These imply that U_2 is closed under finite intersections.

3. Let $G_\alpha \in U_2, \alpha \in \Delta$. Then $G_\alpha \in U$ and $G_\alpha \subset [0,\infty), \forall \alpha \in \Delta$ or $G_{\alpha_0} = \mathbb{R}$, for some $\alpha_0 \in \Delta$. Hence $\bigcup_\alpha G_\alpha \subset [0,\infty)$ or $\bigcup_\alpha G_\alpha = R$, either of which implies that $\bigcup_\alpha G_\alpha \in U_2$. The proof is complete.

Clearly $U_2 < U$. Let $\eta_Y = U_2$. Then $f : (X, \tau_X) \to (Y, \eta_Y)$ is continuous. If θ_Y is any topology on Y weaker than η_Y then obviously $f : (X, \tau_X) \to (Y, \theta_Y)$ is continuous since $G \in \theta_Y \Rightarrow G \in \eta_Y$ and so $f^{-1}(G) \in \tau_X \forall G \in \theta_Y$.

EXAMPLE 4.3

Suppose $f : (X, \tau_X) \to (Y, \tau_Y)$ is not continuous. Let $Z = \bigcup\{G \in \tau_Y : f^{-1}(G) \in \tau_X\}$, and let $\eta_Y = \{G \in \tau_Y : f^{-1}(G) \in \tau_X\}$. Then clearly $\eta_Y < \tau_Y$ and

1. $\emptyset, Z \in \eta_Y$; observe that $Z \subset Y$.

2. Let $G_i \in \eta_Y, \forall i = 1, \cdots, m$. Then $G_i \in \tau_Y$. And $f^{-1}(\bigcap_{i=1}^{m} G_i) = \bigcap_{i=1}^{m} f^{-1}(G_i) \in \tau_X$, since τ_X is closed under finite intersections. Hence $\bigcap_{i=1}^{m} G_i \in \eta_Y$.

3. Let $G_\alpha \in \eta_Y, \alpha \in \Delta$. Then $G_\alpha \in \tau_Y$ so $\bigcup_{\alpha\in\Delta} G_\alpha \in \tau_Y$. Also $f^{-1}(\bigcup_{\alpha\in\Delta} G_\alpha) = \bigcup_{\alpha\in\Delta} f^{-1}(G_\alpha) \in \tau_X$. Hence η_Y is a topology on $Z \subset Y$. If $Z = Y$ then η_Y is actually a topology on Y. Otherwise (i.e. if $Z \neq Y$), then $\eta_Y^* = \eta_Y \bigcup \{Y\}$ is a topology on Y and $f : (X, \tau_X) \to (Y, \eta_Y^*)$ is continuous.

In fact, η_Y^* is the strongest topology on Y, weaker than τ_Y, with respect to which f is continuous.

EXAMPLE 4.4

(alternative to Example 4.1)

Let $f : (\mathbb{R}, u) \to [0, +\infty) \subset \mathbb{R}$ be a function defined by $f(x) = x^2$; where $(\mathbb{R}, u)$ is the set of real numbers with its usual topology u. Clearly we intuitively know that f is defined or continuous on all of $\mathbb{R}$. But which topology do we give the range space $\mathbb{R}$ of f in order to illustrate this continuity in terms of open sets of $\mathbb{R}$ as the domain and as the range space? One topology that easily comes to mind is the usual topology itself u of $\mathbb{R}$ since for any u-open subset $G = (a, b)$ of $[0, +\infty)$, $f^{-1}(G) = f^{-1}\{(a, b)\} = (-\sqrt{b}, -\sqrt{a}) \bigcup (+\sqrt{a}, +\sqrt{b})$ is a u-open set.[1] But we know that f^{-1} is not

[1] $x^2 \in (a, b) \Rightarrow a < x^2 < b. \Rightarrow +\sqrt{a} < x < +\sqrt{b}$ or $-\sqrt{a} > x > -\sqrt{b}$. Hence $x \in (-\sqrt{b}, -\sqrt{a}) \bigcup (+\sqrt{a}, +\sqrt{b})$.

defined for some u-open sets (for instance the interval $(-\sqrt{b}, +\sqrt{b})$ does not exist in $\mathbb{R}$ if $b < 0$). Generally f^{-1} is not defined on the open set $(-\infty, 0)$ (and on all u-open subsets of $(-\infty, 0)$). That is, the open set $(-\infty, 0)$ and all u-open subsets of $(-\infty, 0)$ are irrelevant in the range space of f, in discussing u-continuity of f; though we know that f is u-continuous if $\mathbb{R}$ as the range is endowed the usual topology u. So we need a topology u_1 on $\mathbb{R}$, strictly weaker than (and containing all relevant or essential sets of) u in the range space of f so that $f : (\mathbb{R}, u) \to (\mathbb{R}, u_1)$ is still u-continuous. This u_1 may be seen as the 'essential topology of the u-continuity of f'.

Let $X = (0, +\infty)$ and put $u_1 = X$-topology on $\mathbb{R}$ defined as $u_1 = \{G \in u : G \subset X\} \bigcup \{\mathbb{R}\}$. Then we see that

1. u_1 makes f to be u-continuous in that $f^{-1}(G) \in u \forall G \in u_1$.
2. u_1 is strictly weaker than u.
3. All essential or relevant u-open sets for the discussion of continuity of f, from the stand point of the range of f, are collected in u_1 and all the nonessential ones are left out.

The topology u_1 on $\mathbb{R}$ is special in that any topology strictly weaker than u_1 on $\mathbb{R}$ will not contain some *relevant* u-open sets for the analysis of u-continuity of f; and any topology strictly stronger than u_1 will contain some irrelevant u-open sets for this analysis.

EXAMPLE 4.5

Let (X, τ) be a topological space and let $K \subset X$. Let
$\tau_1 = \{G \in \tau : G \subset K\} \cup \{K\}$
$\tau_2 = \{G \subset K : G \in \tau \text{ or } G = K\}$
$\tau_3 = \{G \cap K : G \in \tau\}$.
These are the topologies induced on K by τ. We see that $\tau_i^* = \tau_i \cup \{X\}$, $i = 1, 2, 3$, is a topology on X induced by K.
If $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous, might subset-induced topology still be useful or needed? The answer is 'Yes'. Subset-induced topology might be needed or useful if we wish to find topologies on Y, weaker than τ_Y, with respect to which f is continuous. In general the concept gives us flexibility in the choice of topology we use for the analysis of the continuity of a function—whether the function is continuous or not.

4.1.3 Reducible Topologies

The concepts of *weak* or *strong* reduction of topologies are introduced. Closely related to these, and introduced as well, are the concepts of *weak* and *strong* base reduction of topologies. We also defined *extensible topologies*; and defined *weak* and *strong* base extension of topologies. We proved that there exists a topology γ, weaker than a weak topology τ, on X, which has a chain of strong reductions if one of the range spaces, say (X_α, τ_α) of τ, has a chain of strong reductions. It is proved that the usual topology of the set $\mathbb{R}$ of real numbers can be reduced in the weak sense to chains of infinite families of pairwise comparable topologies; and that the usual topology of $\mathbb{R}$ can neither be reduced in the normal sense nor in the strong sense. We proved that a weak topology has a chain of weaker topologies if one of its range topologies is reducible to a chain of topologies. (References: Agnew and Morse (1938), Lefschetz (1942), Williamson (1956), Kelly and Namioka (1963), Porter (1993), McLemman *etal.,* (year unavailable), Dydak (1997), Karimov and Repovs (2013), and Gothen *etal.,* (2013).)

Throughout, X is a nonempty set.

Definition 4.3 *A topology τ on X is said to be* ***strongly reducible*** *or* ***reducible in the strong sense*** *if $\exists G \in \tau$ such that $\tau_1 = \tau - \{G\}$ is a topology on X. The topology τ_1 is called* ***a strong reduction of*** *τ.*

EXAMPLE 4.6

Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$. Then τ on X is strongly reducible, since there exists $\{a\} \in \tau$ such that $\tau_1 = \tau - \{\{a\}\} \equiv \{\emptyset, X, \{c\}, \{a, c\}\}$ is a topology on X. Conversely, τ_1 is a strong reduction of τ.

Let $X = \{a, b, c\}$ and $\tau = 2^X = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Then $\tau = 2^X$ is not strongly reducible.

Definition 4.4 *A topology τ on X is said to be* ***normally reducible*** *or simply* ***reducible****, or* ***reducible in the normal sense*** *if $\exists G_i \in \tau (i = 1, \cdots, m); m \in \mathbb{N}$ such that $\tau_1 = \tau - \{G_1, \cdots, G_m\}$ is a topology on X. Such a topology τ_1 is called* ***a normal reduction of*** *τ, or simply* ***a reduction of*** *τ.*

EXAMPLE 4.7

Let $X = \{a, b, c\}$ and $\tau = 2^X = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Then $\tau = 2^X$ is normally reducible, to
$\tau_1 = \tau - \{\{c\}, \{b, c\}\} \equiv \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$.

Definition 4.5 *A topology τ on X is said to be* ***weakly reducible*** *or* ***reducible in a weak sense*** *if $\exists\{G_\alpha \in \tau : \alpha \in \Delta\}$ such that $\tau_1 = \tau - \{G_\alpha \in \tau : \alpha \in \Delta\}$ is a topology on X. The topology τ_1 is called* ***a weak reduction of*** *τ.*

EXAMPLE 4.8

Let $(\mathbb{R}, U)$ denote R with its usual topology U. Let $X = (-\infty, 0)$, and $\tau_X = \{G \in U : G \subset X\} \bigcup \{\mathbb{R}\}$. Then τ_X is a weak reduction of U, since $\tau_X = U - \{G \in U : G \text{ is not a subset of } X\}$.

REMARK 4.3

1. Strongly Reducible $\Longrightarrow$ Normally Reducible $\Longrightarrow$ Weakly Reducible. But the converses are not always true.
2. The indiscrete topology of a set cannot be reduced in any sense (strong, normal or weak). In fact it is the weakest reduction of any topology.
3. In the first two examples above we saw that the discrete topology of X is not reducible in the strong sense. This is actually a general fact for the discrete topology of any set X whose cardinality is greater than 2; and we state and prove that below as a theorem.
4. The discrete topology is not the only topology that is irreducible in the strong sense. The usual topology of $\mathbb{R}$ is not reducible in the strong sense. This is stated and proved below as a proposition.

Theorem 4.1 *(a) The discrete topology of X cannot be reduced in the strong sense if the cardinality of X is greater than 2. (b) Every non-indiscrete topology on a set X can be reduced in some sense (strong, normal or weak).*

Proof:

(a) Let the cardinality of X be greater than 2 and let (X, D) be a discrete topological space. Suppose $G \in D$ and $\eta = D - \{G\}$. We need to show that η is not a topology on X.

Without loss of generality, suppose $G \neq \{a\}$. Then there exist at least two proper subsets of G and each is in D (as the discrete topology) and hence separately in η. Since G is the union of all the proper subsets of G, it follows (as $G \notin \eta$) that η is not closed under arbitrary unions and is hence not a topology on X.

Now suppose $G = \{a\}$, a singleton. Then from hypothesis X contains two other mutually distinct elements x_1, x_2, each different from a. The sets $G_1 = \{a, x_1\}$ and $G_2 = \{a, x_2\}$ are in D (as the discrete topology) and hence in η. It is easy to see that $G_1 \cap G_2 = G \notin \eta$; hence η is not a topology on X.

(b) Let τ be a non-indiscrete topology on X. Then the indiscrete topology $\{\emptyset, X\}$ on X is a reduction of τ in some sense. The proof is complete.

Proposition 4.3 *The usual topology U of the set $\mathbb{R}$ of real numbers is not reducible in the strong sense.*

Proof:
Let $(\mathbb{R}, U)$ denote $\mathbb{R}$ with its usual topology. Let $\eta = U - \{(a,b)\}$,for some $(a,b) \in U$. We show that η is not a topology on $\mathbb{R}$. For each $n \in N$ put $G_n = (a + \frac{b-a}{2n}, b - \frac{b-a}{2n})$. Then each G_n is an element of U and an element of η. Clearly $(a,b) = \bigcup_{n=1}^{\infty} G_n$, and since $(a,b) \notin \eta$ it follows that η is not closed under arbitrary unions and is hence not a topology on $\mathbb{R}$.

NOTE

- Not only that the usual topology of $\mathbb{R}$ cannot be reduced in the strong sense; it can also not be reduced in the 'normal' sense.
- There can be found many other topologies which are not reducible in the strong sense. For example the lower limit topology of $\mathbb{R}$ is not strongly reducible and the upper limit topology of $\mathbb{R}$ is not strongly reducible. Yet infinitely many topologies can be reduced in the strong sense—for example, the discrete topology of any set with only two elements has a chain of strong reductions.
- So far it may appear that the only examples of strongly reducible topologies available are finite topologies or topologies on finite sets. Infinite topologies and indeed topologies on infinite sets can be strongly reducible. The next example illustrates this.

EXAMPLE 4.9
Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ denote the set of natural numbers. For each $n \in \mathbb{N}$ let G_n be the set of all real numbers *excluding* the first n natural numbers. Thus for instance

$G_0 = \mathbb{R} - \{\} = \mathbb{R}$;

$G_1 = \mathbb{R} - \{0\}$;

$G_2 = \mathbb{R} - \{0, 1\}$;

$G_3 = \mathbb{R} - \{0, 1, 2\}$;

$\vdots$

$G_n = R - \{0, 1, 2, 3, \cdots, n-1\}$

Let $T_{C\mathbb{N}} = \{\emptyset, G_n\}_{n\in\mathbb{N}}$. Then it is easy to see that

1. The empty set is in $T_{C\mathbb{N}}$, from the way $T_{C\mathbb{N}}$ is defined.
2. The whole set $\mathbb{R}$ of real numbers is in $T_{C\mathbb{N}}$.
3. $T_{C\mathbb{N}}$ is closed under finite intersections.
4. And that $T_{C\mathbb{N}}$ is closed under arbitrary unions.

Hence $T_{C\mathbb{N}}$ is a topology on $\mathbb{R}$. We see that $T_{C\mathbb{N}}$ is strongly reducible since, say $\tau = T_{C\mathbb{N}} - \{G_5\}$ is a topology on $\mathbb{R}$. (The topology $T_{C\mathbb{N}}$ here is one of our interesting constructions in this thesis.)

Definition 4.6 *A topology τ on X, with base B, is said to be* ***strongly base reducible*** *or* ***base reducible in the strong sense*** *if there exists $B_0 \in B$ such that $B_1 = B - \{B_0\}$ is a base for a topology τ_1 on X strictly coarser than τ. Such a topology τ_1 is called* ***a strong base reduction of*** *τ.*

EXAMPLE 4.10

Let $X = \{a, b, c\}$ and τ_1 on X be $\tau_1 = 2^X$ $= \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Let $B_1 = \{\{a\}, \{b\}, \{c\}\}$ be a base for the topology τ_1 on X. Then τ_1 with the base B_1 is not strongly base reducible.

However, if we endow X with the topology $\tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$, with base $B_2 = \{\{a\}, \{b\}, \{a, c\}\}$, then τ_2 would be strongly base reducible, for there exists $\{a\} \in B_2$ such that $B_3 = B_2 - \{\{a\}\} \equiv \{\{b\}, \{a, c\}\}$ is a base for a topology τ_3 on X given by $\tau_3 = \{\emptyset, X, \{b\}, \{a, c\}\}$.

Definition 4.7 *A topology τ on X, with base B, is said to be* ***base reducible*** *if there exists $B_i \in B (i = 1, \cdots, m; m \in N)$ such that $B_1 = B - \{B_i : i = 1, \cdots, m\}$ is a base for a topology τ_1 on X strictly coarser than τ. Such a topology τ_1 is called a* ***base reduction of*** *τ.*

Definition 4.8 *A topology τ on X, with base B, is said to be* ***weakly base reducible*** *or* ***base reducible in the weak sense*** *if $\exists\{B_\alpha \in B : \alpha \in \Delta\}$ such that $B_1 = B - \{B_\alpha : \alpha \in \Delta\}$ is a base for a topology τ_1 on X strictly coarser than τ. Such a topology τ_1 is called a* ***weak base reduction of*** *τ.*

EXAMPLE 4.11

Let $(\mathbb{R}, U)$ denote the usual topological space of $\mathbb{R}$. Then $B = \{(a, b) : a, b \in \mathbb{R}\}$ is a base for U. Let $B_1 = \{B_\alpha \in B : B_\alpha \subset (-\infty, 0)\} \bigcup \{\mathbb{R}\}$. Then

B_1 is a base for a topology on $\mathbb{R}$strictly weaker than U. That is, the topology τ_X is a weak base reduction of $(\mathbb{R}, U)$.

REMARK 4.4

A strongly base reducible topology is base reducible. A base reducible topology is weakly base reducible but converses of these do not hold in general..

Definition 4.9 *A topology τ on X is said to be*

1. ***strongly extensible*** *if $\exists G \subset X$, $G \notin \tau$ such that $\gamma = \tau \bigcup \{G\}$ is a topology on X. The topology γ is then called a strong extension of τ;*

2. ***extensible*** *if $\exists\{G_i \subset X : G_i \notin \tau; i = 1, \cdots, m; m \in \mathbb{N}\}$ such that $\gamma = \tau \bigcup \{G_1, \cdots, G_m\}$ is a topology on X. The topology γ is called an extension of τ;*

3. ***weakly extensible*** *if $\exists\{G_\alpha \subset X : G_\alpha \notin \tau; \alpha \in \Delta\}$ such that $\gamma = \tau \bigcup \{G_\alpha\}_{\alpha\in\Delta}$ is a topology on X. Such a γ is then called a weak extension of τ.*

Definition 4.10 *A topology τ on X with base B is said to be*

1. ***strongly base extensible*** *if $\exists B_0 \subset X$, $B_0 \notin B$ such that $\Omega = B \bigcup \{B_0\}$ is a base for a topology γ on X finer than τ. The topology γ is then called a strong base extension of τ;*

2. ***base extensible*** *if $\exists\{B_i \subset X, B_i \notin B, i = 1, \cdots, m; m \in \mathbb{N}\}$ such that $\Omega = B \bigcup \{B_i; i = 1, \cdots, m\}$ is a base for a topology γ on X finer than τ. The topology γ is called a base extension of τ;*

3. ***weakly base extensible*** *if $\exists\{B_\alpha \subset X: B_\alpha \notin B, \alpha \in \Delta\}$ such that $\Omega = B \bigcup \{B_\alpha : \alpha \in \Delta\}$ is a base for a topology γ on X finer than τ. In this case the topology γ is called a weak base extension of τ.*

The following propositions hold true obviously from the definitions above.

Proposition 4.4 *A topology τ on X is*

1. *strongly extensible if, and only if, τ is a strong reduction of some topology γ on X;*

2. *extensible if, and only if, τ is a reduction of some topology γ on X;*

3. *weakly extensible if, and only if, τ is a weak reduction of some topology γ on X.*

Proposition 4.5 *A topology τ on X with base B is*

1. *strongly base extensible if, and only if, τ is a strong base reduction of some other topology γ on X;*
2. *base extensible if, and only if, τ is a base reduction of some topology γ on X;*
3. *weakly base extensible if, and only if, τ is a weak base reduction of some topology γ on X.*

Definition 4.11 *Let τ be a strongly reducible topology on X. If τ_1 is a strong reduction of τ, τ_2 a strong reduction of τ_1, τ_3 a strong reduction of τ_2, and so on, then the pairwise comparable family*

$$C = \{\tau_n\}_{n \in \mathbb{N}}$$

of topologies on X is called a chain of strong reductions of τ on X.

EXAMPLE 4.12

Let $X = \{a, b, c\}$ and τ on X be $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$. Then $\tau_1 = \{\emptyset, X, \{c\}, \{a, c\}\}$ or $\tau_1 = \{\emptyset, X, \{a\}, \{a, c\}\}$ is a strong reduction of τ. Also $\tau_2 = \{\emptyset, X, \{c\}\}$ or $\tau_2 = \{\emptyset, X, \{a\}\}$ or $\{\emptyset, X, \{a, c\}\}$ is a strong reduction of τ_1. And $\tau_3 = \{\emptyset, X\}$ is a strong reduction of τ_2. Hence the family

$$C_1 = \{\tau_1, \tau_2, \tau_3\}$$

is a chain of strong reductions of τ.

For the topology τ on X given by $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$ a chain of strong reductions can be obtained as follows:

$\tau_1 = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$; $\tau_2 = \{\emptyset, X, \{a\}, \{a, c\}\}$

$\tau_3 = \{\emptyset, X, \{a\}\}$; and $\tau_4 = \{\emptyset, X\}$.

We see that

$$\tau_4 < \tau_3 < \tau_2 < \tau_1 < \tau;$$

and that

$$C_2 = \{\tau_1, \tau_2, \tau_3, \tau_4\}$$

is a chain of strong reductions of τ.

REMARK 4.5

We notice first that a strongly reducible topology can be reduced to a chain of pair-wise comparable topologies. Secondly, there is often more than one way of getting a chain of strong reductions of a strongly reducible topology.

The chains C_1 and C_2 in the last example are simple enough, in that they are (each) finite. Hence one may wonder if the only examples of chain of strong reductions (of a topology) that could be found are those that are finite. Actually examples of denumerable chains of reductions exist. For example, the topology $T_{C\mathbb{N}}$ on $\mathbb{R}$ that we constructed above, just before definition 4.6, has a countably infinite chain of strong reductions. To see this, we observe that

$$T_{C\mathbb{N}} = \bigcup_{n=0}^{\infty} \{\tau_n\}$$

where $\tau_0 = \{\emptyset, \mathbb{R}\}$, $\tau_1 = \tau_0 \cup \{G_1\}$, $\tau_2 = \tau_1 \cup \{G_2\}$, and so on. Then

$$C = \{\tau_0, \tau_1, \tau_2, \cdots\}$$

is a countably infinite family of strong reductions of $T_{C\mathbb{N}}$.

Definition 4.12 *Let τ be a (strongly or weakly) reducible topology on X. If C_1 and C_2 are two chains of (weak or strong) reductions of τ such that for each $\tau_{1i} \in C_1$, there exists $\tau_{2j} \in C_2$ such that τ_{1i} is weaker than τ_{2j}, then we say that the chain C_1 is weaker than the chain C_2.*

Definition 4.13 *Let τ be a (strongly or weakly) reducible topology on X. If C_1 and C_2 are two chains of (weak or strong) reductions of τ such that for each $\tau_{1i} \in C_1$, there exists a $\tau_{2j} \in C_2$ such that τ_{1i} is strictly weaker than τ_{2j}, then we say that the chain C_1 is strictly weaker than the chain C_2.*

Definition 4.14 *If C_1 and C_2 are two chains of reductions of τ on X such that C_1 is weaker than C_2 and C_2 is weaker than C_1, then we say that C_1 is equivalent to C_2.*

Definition 4.15 *If C_1 is not weaker than C_2 and C_2 is not weaker than C_1, then we say that C_1 and C_2 are not comparable.*

EXAMPLE 4.13

Let $X = \{a, b, c\}$ and τ on X be $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$. Let $C_1 = \{\tau_{11}, \tau_{12}, \tau_{13}\}$ where $\tau_{11} = \{\emptyset, X, \{c\}, \{a, c\}\}$, $\tau_{12} = \{\emptyset, X, \{c\}\}$, and $\tau_{13} =$

$\{\emptyset, X\}$. Then C_1 is a chain of strong reductions of τ.
Let $C_2 = \{\tau_{21}, \tau_{22}, \tau_{23}\}$ where $\tau_{21} = \{\emptyset, X, \{a\}, \{a, c\}\}$, $\tau_{22} = \{\emptyset, X, \{a, c\}\}$, and $\tau_{23} = \{\emptyset, X\}$. Then C_2 is another chain of strong reductions of τ.

We see that C_1 and C_2 are not comparable because the topology τ_{12} in C_1 is not comparable to any topology in C_2; and τ_{21} in C_2 is not comparable to any topology in C_1.

EXAMPLE 4.14

Let C_1 remain as in the example above and let $C_3 = \{\tau_{31}, \tau_{32}, \tau_{33}\}$ where $\tau_{31} = \{\emptyset, X, \{c\}, \{a, c\}\}$, $\tau_{32} = \{\emptyset, X, \{a, c\}\}$, and $\tau_{33} = \{\emptyset, X\}$. Then C_3 is another chain of strong reductions of τ and we see that C_1 is weaker (but not strictly) than C_3, since every topology in C_1 is weaker than τ_{31}. And if we also observe that every topology in C_3 is weaker than τ_{11}, then we know that C_1 and C_3 are equivalent.

EXAMPLE 4.15

Let $(\mathbb{R}, u)$ denote the set of real numbers with its usual topology. Let $\mathbb{Z}$ denote the set of integers. For each $z \in \mathbb{Z}$, let X_z be the u-open interval $X_z = (-\infty, z)$. Then clearly

$$\{G \in u : G \subset X_z\} = \{G \in u : G \subset (-\infty, z)\}$$

is a topology on X_z. Let $\tau_z = X_z$-topology on $\mathbb{R}$; in that $\tau_z = \{G \in u : G \subset X_z\} \bigcup \{\mathbb{R}\} = \{G \in u : G \subset (-\infty, z)\} \bigcup \{\mathbb{R}\}$.
Then clearly if $z_1 < z_2$, $X_{z_1} \subset X_{z_2}$ and τ_{z_1} is weaker than τ_{z_2}. Hence the family

$$C_Z = \{\tau_z\}_{z \in \mathbb{Z}}$$

is a chain of weak reductions of the usual topology on R, in that

$$\cdots < \tau_{z_{-2}} < \tau_{z_{-1}} < \tau_{z_0} < \tau_{z_1} < \tau_{z_2} < \cdots < u,$$

where u is the usual topology on $\mathbb{R}$.

For each $n \in \mathbb{N}$, let $X_n = (-n, n)$ and let $\tau_n = \{G \in u : G \subset X_n\} \bigcup \{\mathbb{R}\}$ be an X_n-topology of $\mathbb{R}$ obtained from the usual topology on $\mathbb{R}$. For instance, $X_1 = (-1, 1)$ and $\tau_1 = \{G \in u : G \subset X_1\} \bigcup \{\mathbb{R}\}$ is an X_1-topology on $\mathbb{R}$ strictly weaker than the usual topology on $\mathbb{R}$. Also $X_2 = (-2, 2)$ and $\tau_2 = \{G \in u : G \subset X_2\} \bigcup \{\mathbb{R}\}$ is an X_2-topology of $\mathbb{R}$ obtained from the usual topology on $\mathbb{R}$. And so on. Then

$$C_{\mathbb{N}} = \{\tau_n\}_{n \in \mathbb{N}}$$

is a chain of weak reductions of u. Since, for each $n \in \mathbb{N}$, $(-n, n)$ is a proper subset of $(-\infty, n)$, we see that the chain $C_{\mathbb{N}} = \{\tau_n\}_{n \in \mathbb{N}}$ is strictly weaker than $C_{\mathbb{Z}} = \{\tau_z\}_{z \in \mathbb{Z}}$.

EXAMPLE 4.16
If we replace $\mathbb{Z}$ in example 4.15 with Q (the set of rational numbers) we obtain another chain of weak reductions

$$C_Q = \{\tau_q\}_{q\in Q}$$

of the usual topology of $\mathbb{R}$. And we see that $C_{\mathbb{Q}}$ and C_Z are equivalent.
EXAMPLE 4.17
If we replace $\mathbb{Z}$ in example 4.15 with $\mathbb{Q}^c$ (the irrational numbers) we obtain yet another chain of weak reductions

$$C_{\mathbb{Q}^c} = \{\tau_{q^c}\}_{q^c\in\mathbb{Q}^c}$$

of the usual topology of $\mathbb{R}$. We see that $C_{\mathbb{Q}^c}$ is equivalent to both $C_{\mathbb{Q}}$ and $C_{\mathbb{Z}}$. Also $C_{\mathbb{Q}^c}$ is an uncountable chain of (weak) reductions while $C_{\mathbb{Q}}$ and $C_{\mathbb{Z}}$ are countable.
EXAMPLE 4.18
We may replace $\mathbb{Z}$, in example 4.15, with $\mathbb{R}$ itself and get another chain of reductions

$$C_{\mathbb{R}} = \{\tau_r\}_{r\in\mathbb{R}}$$

of the usual topology u of $\mathbb{R}$.
EXAMPLE 4.19
Another chain of reductions of the usual topology of $\mathbb{R}$ may be obtained in a different way. Let

$X_0 = \mathbb{R}$;

$X_1 = (-\infty, 1)\bigcup(1, \infty)$;

$X_2 = (-\infty, \frac{1}{2})\bigcup(1, \infty)$;

$X_3 = (-\infty, \frac{1}{3})\bigcup(1, \infty)$;

$\vdots$

$X_n = (-\infty, \frac{1}{n})\bigcup(1, \infty)$.

Let $\tau_0 = X_0$-topology on $\mathbb{R}$ (i.e. the usual topology on $\mathbb{R}$), $\tau_1 = X_1$-topology on $\mathbb{R}$, $\tau_2 = X_2$-topology on $\mathbb{R}$, $\tau_3 = X_3$-topology on $\mathbb{R}, \cdots, \tau_n = X_n$-topology on $\mathbb{R}$.
Then the pair-wise comparable chain

$$\{\tau_n\}_{n\in\mathbb{N}}$$

of topologies is a chain of reductions of the usual topology of $\mathbb{R}$.

EXAMPLE 4.20

Let

$X_1 = (-\infty, -1) \bigcup (1, \infty);$

$X_2 = (-\infty, -\frac{1}{2}) \bigcup (\frac{1}{2}, \infty);$

$X_3 = (-\infty, -\frac{1}{3}) \bigcup (\frac{1}{3}, \infty);$

$\vdots$

$X_n = (-\infty, -\frac{1}{n}) \bigcup (\frac{1}{n}, \infty).$

Let $\tau_1 = X_1$-topology on $\mathbb{R}$, $\tau_2 = X_2$-topology on $\mathbb{R}$, $\tau_3 = X_3$-topology on $R, \cdots, \tau_n = X_n$-topology on $\mathbb{R}$. Then $\tau_n \leq \tau_{n+1}$ (since $X_n \subset X_{n+1}$); and, hence, the pair-wise comparable chain

$$\{\tau_n\}_{n\geq 1}$$

of topologies is a chain of reductions of (or, conversely, extensions under) the usual topology of $\mathbb{R}$.

EXAMPLE 4.21

Let $r > 0$ be a positive real number. Then the interval $X_r = (-r, r)$ is a u-open subset of $\mathbb{R}$. The family

$$\tau_r = \{G \in u : G \subset X_r\} \bigcup \{\mathbb{R}\}$$

is an X_r-topology on $\mathbb{R}$. It is easy to see that τ_r is weaker than τ_s if $r < s$; and that τ_r tends to the usual topology of $\mathbb{R}$ as $r \to \infty$. Hence the uncountable chain

$$H_R = \{\tau_r\}_{r>0}$$

of pair-wise comparable topologies is another chain of weak reductions of the usual topology of $\mathbb{R}$.

What happens on a weak topology in terms of reducibility? We show below that if τ is a weak topology on a set X, and one of the range spaces of (X, τ) is reducible in the strong sense, then there exists a chain of weak topologies, each weaker than τ, on X (generated by the fixed family of functions), which are a chain of reductions of τ (not necessarily in the strong sense) if the function associated with the strongly reducible range space has requisite properties. We prove this next in a theorem.
The following lemma will be useful in the theorem that follows after.

Lemma 4.1 *If τ is a topology on X and $\tau_1 = \tau \bigcup \{G\}$ is a topology on X (where $G \notin \tau$), then τ_1 is only one set, G, strictly finer than τ.*

Proof:
τ_1 is a strong extension of τ and is, hence, only one set strictly finer than τ.
NOTE
What Lemma 4.1 says is that the introduction of just one set G into a topology τ to produce another topology τ_1 does not make τ_1 to have more than one open set (either from finite intersections or arbitrary unions) than τ—and that the extra open set is precisely G.

Theorem 4.2 *Let (X, τ) be a weak topological space generated by the family $\{(X_\alpha, \tau_\alpha)\}$ of topological spaces, together with the family $\{f_\alpha\}$ of functions. There exists a chain of weak topologies, each weaker than τ, on X (generated by this fixed family of functions), which are a chain of reductions of τ if (a) one of the range spaces, say τ_α, has a chain of strong reductions, (b) f_α is one-to-one, and (c) f_α maps into all the elements of each topology in the chain of strong reductions of τ_α.*

Proof:
Let (X_α, τ_α) be the range space meeting the hypotheses, for some $\alpha \in \Delta$, and let

$$C_\Gamma = \{\tau_r\}_{r \in \Gamma}$$

be a chain of strong reductions of τ_α. Let τ_{r_1} and τ_{r_2} be any two topologies in C_Γ such that, say τ_{r_1} is strictly weaker than τ_{r_2} by one set. That is, τ_{r_1} is a strong reduction of τ_{r_2}. Let

$$\tau_1 = \{f_\alpha^{-1}(G_{1i}) : G_{1i} \in \tau_{r_1}\}$$

and

$$\tau_2 = \{f_\alpha^{-1}(G_{2i}) : G_{2i} \in \tau_{r_2}\}.$$

Then clearly

$$\bigcap_{i=1}^{n} f_\alpha^{-1}(G_{1i}) = f_\alpha^{-1}\left[\bigcap_{i=1}^{n}(G_{1i})\right] \in \tau_1,$$

as τ_{r_1} is closed under finite intersections. That is, τ_1 is closed under finite intersections. Also

$$\bigcup f_\alpha^{-1}(G_{1i}) = f_\alpha^{-1}(\bigcup G_{1i}) \in \tau_1,$$

implying that τ_1 is closed under arbitrary unions. It is easy to see that $\emptyset, X \in \tau_1$ as $\emptyset, X_\alpha \in \tau_{r_1}$. Hence τ_1 is a topology on X, corresponding to τ_{r_1}. Similarly τ_2 is a topology on X corresponding to τ_{r_2}. It is easy to see that both τ_1 and τ_2 are weaker than τ.

It is obvious that τ_1 is weaker than τ_2 and (by Lemma 4.1) that τ_1 is only one set less than τ_2. That is, τ_1 is a strong reduction of τ_2.

As τ_{r_1} and τ_{r_2} in C_Γ are arbitrary it follows that there corresponds to C_Γ a chain C of topologies on X of pair-wise comparable topologies which can be arranged in such a way that each one is strictly weaker than the next by only one set. If we let the elements of C to represent the (hypothetical) range space (X_α, τ_α)—one after the other—in the collection of sub-base for weak topologies on X while leaving the other range spaces unchanged, the required chain of weaker weak topologies on X will emerge. The proof is complete.

Theorem 4.2 indicates that a fixed family of functions can generate a family of pairwise comparable weak topologies. Further research may now embark on finding more considerations for this result. This is part of the developments in the sections ahead.

NOTE

So far, all the chains of strong reduction of topologies given in this section are countable. The question then arises as to whether there can be an uncountable chain of strong reductions of some topology. For example, can an uncountable chain of strong reductions be obtained for the ususal topology of R? Further, if a range topology for a weak topology has an uncountable chain of strong reductions, what is the implication of this on the weak topology? That is, does the weak topology in this case inherit this property? Can we characterize the weak topologies for which there exist families of other weak topologies which are chains of strong reductions of the given weak topologies? Answers to these questions are as yet unknown.

4.1.4 Reducible Topologies—Lattices

We recall the following definitions.

Definition 4.16 *A relation R on a set X is called a partial order on X if*

1. *R is reflexive; in that xRx, for all $x \in X$,*
2. *R is transitive; in that xRy and yRz implies xRz,*

3. R is anti-symmetric; in that xRy and yRx implies $x = y$.

Definition 4.17 *A set X on which a partial order is defined is called a partially ordered set; in brief, a poset.*

Definition 4.18 *If X is a poset, with partial order R, and xRy, then we say that x precedes y, written $x \prec y$. We then analogously also say that y dominates x. If x precedes y and $x \neq y$, we say that x properly precedes y or y properly dominates x.*

Definition 4.19 *Let X be a poset with R. Then x is called a lower bound of y if $x \prec y$; and then y is called an upper bound of x.*

Definition 4.20 *Let X be a poset with R. An element x_0 of X is called the first or the least element of X if x_0 precedes every other element of X. The last or greatest element of X is that which dominates every other element of X.*

Definition 4.21 *Let X be a poset. An element x_0 of X is called a minimal element if no element of X properly precedes x_0.*

NOTE
If x_0 is a minimal element of a poset X and $x \prec x_0$, then $x = x_0$. Also, every first element is a minimal element but a minimal element may not be a first element.

Definition 4.22 *Let X be a poset. An element y_0 of X is called a maximal element if no element of X properly dominates y_0.*

Definition 4.23 *Let X be a poset. Let T be a subset of X. A lower bound of T is an element of X which precedes every element of T. The greatest lower bound (g.l.b.) of T is the lower bound which dominates every other lower bound of T. The g.l.b. of T is also called the infimum of T, and denoted $inf(T)$.*

Definition 4.24 *Let X be a poset and let T be a subset of X. An upper bound of T is an element of X which dominates every element of T. The least upper bound (l.u.b.) of T is the upper bound which precedes every other upper bound of T. The l.u.b. of T is also called the supremum of T, and denoted $sup(T)$.*

Definition 4.25 *Two elements x, y of a poset X are said to be comparable if either $x \prec y$ or $y \prec x$.*

Definition 4.26 *A lattice is a poset in which every two elements have a g.l.b and an l.u.b.*

DEVELOPMENTS

Let $C = \{\tau_\alpha : \alpha \in \Delta\}$ be a chain of (weak or strong) reductions of a topology τ on a set X. Then C, with the relation of set inclusion $\subseteq$ is a poset. We also see that C is totally ordered (in that any two elements of C are comparable). If τ_{α_1} and τ_{α_2} are two topologies in C such that, say, τ_{α_1} is weaker than τ_{α_2}, then the g.l.b. of the sub-family $T = \{\tau_{\alpha_1}, \tau_{\alpha_2}\}$ of C, that is, $inf(T)$, is τ_{α_1}. Also $sup(T) = \tau_{\alpha_2}$. Hence C is a lattice of topologies by set inclusion.

Let R be another relation on the chain C, where $\tau_\alpha \mathrm{R} \tau_r$ if $\tau_\alpha \leq \tau_r$. That is, the relation R($\leq$) on C, now, is that of comparison of topologies as topologies. With this relation on C, we see again that C is a lattice of topologies.

Corollary 4.1 *Every chain C of reductions of a topology on a set X is a lattice in at least two ways.*

OBSERVATIONS

Every set (on which a partial order is defined) is not a lattice. In particular, every family of topologies is not a lattice. For example, if the topologies in a family F are not comparable, then the family F would not be a lattice in either of the ways; but F would still be a poset in the two ways (of set inclusion and comparison of topologies).

If a family of subsets of a set X is pairwise comparable by set inclusion (i.e. totally ordered by set inclusion), then it generates a topology (on X) which has a chain of reductions. This indeed is a theorem which marks the end and climax of this section.

Theorem 4.3 *Any (set inclusion) pairwise comparable family F of subsets of a set X generates a reducible topology τ on X. And the chain C of reductions of τ can be constructed in such a way that $card(F) = card(C)$.*

Proof:
Let $F = \{A_\alpha : A_\alpha \subset X\}_{\alpha \in \Delta}$ be a family of (set inclusion) pairwise comparable subsets of X. Let A_{α_1} and A_{α_2} be two elements of F such that, say, $A_{\alpha_1} \subset A_{\alpha_2}$. Let $\gamma_1 = A_{\alpha_1}$-induced topology on X and $\gamma_2 = A_{\alpha_2}$-induced topology on X. If γ_1 and γ_2 are not comparable, let $\tau_1 = \gamma_1$ and $\tau_2 = \gamma_1 \bigtriangledown \gamma_2$, the join of γ_1 and γ_2 (defined as the weakest topology, on X, finer than both γ_1 and γ_2). Then τ_1 and τ_2 are two comparable topologies on X. Precisely, τ_1 is strictly weaker than τ_2.

Since F is pairwise comparable, the sets in F can be arranged such that

$$A_\alpha \subset A_r \subset \cdots.$$

It follows from the construction above that these sets in F have, corresponding to them, a family $C = \{\tau_\alpha\}_{\alpha \in \Delta}$ of topologies on X, which is pairwise comparable in that

$$\tau_\alpha \leq \tau_r \leq \cdots.$$

It is easy to see that C is equivalent to F; that is, $card(C) = card(F)$.

NOTE
It is easier to see the existence of the chain C, constructed in the proof of the theorem, if we remember that the construction can actually be done through inducement by the discrete topologies of A_{α_1} and A_{α_2}; or, by what is similar, first getting a topology on A_{α_2} and then using this to induce a topology on A_{α_1}; and then finally using these two topologies to construct subset-induced topologies on X.

4.1.5 Generalization of Hyperplane-Open Topologies of $\mathbb{R}^n$; The X-topology Approach

Construction 4.2 *Let $\mathbb{R}^n$ be a Cartesian product of n copies of $\mathbb{R}$. Let $X \subset \mathbb{R}$ be any proper subset of $\mathbb{R}$. And let $\tau_X = 2^X \bigcup \{\mathbb{R}\}$ be the X-topology on $\mathbb{R}$. Let $m (m < n)$ factor spaces of $\mathbb{R}^n$ be endowed with this X-topology on $\mathbb{R}$; and the remaining $n - m$ factor spaces with the usual topology of $\mathbb{R}$. Then from the factor spaces having the X-topology, $p_i^{-1}(\{x_{i_0}\})$ is a hyperplane of dimension $n-1$, for each $1 \leq i \leq m$. Their intersections*

$$\bigcap_{i=1}^{m} p_i^{-1}(\{x_{i_0}\}) \quad . \; . \; . \; . \; . \; . \; . \; . \quad \textit{(2.1)}$$

are hyperplanes of dimension $n - m$. If $n = m + 1$, so that $n - 1 = m$, then the intersection (2.1) would be a 1-dimensional hyperplane (a straight line) in $\mathbb{R}^n$. If $n = m + 2$, then (2.1) would be a 2-dimensional hyperplane. And so on.

If $X = \mathbb{R}$ then the X-topology, τ_X, on $\mathbb{R}$ will simply coincide with the discrete topology of $\mathbb{R}$ itself; and then the whole analysis would boil down to the statements already made in section 1.2.

Since X is a proper subset of $\mathbb{R}$, 2^X is not equal to the discrete topology of $\mathbb{R}$. Hence the X-topology, τ_X, on $\mathbb{R}$ is strictly weaker than the discrete topology of $\mathbb{R}$. Therefore, even if all the factor spaces of $\mathbb{R}^n$ are given this X-topology of $\mathbb{R}$, the resulting weak topology on $\mathbb{R}^n$ generated by the projection maps, would be strictly weaker than the discrete topology of $\mathbb{R}^n$. Still, some singletons (possibly infinitely many) of $\mathbb{R}^n$ are open in this hyperplane-open weak topology on $\mathbb{R}^n$. These singletons are sets of the form

$$\bigcap_{i=1}^{n} p_i^{-1}(\{x_{i_0}\})$$

where $x_{i_0} \in X \subset \mathbb{R}$.

4.1.6 Sierpinski Weak Topology and the Cartesian Plane

In this subsection, we construct a weak topology generated by the projection maps on the product of a Sierpinski topological space with itself.

Definition 4.27 *Let $X = \{0, 1\}$. Then the Sierpinski topology on X is the collection $\tau = \{\emptyset, X, \{0\}\}$. The Cartesian product $X \times X$ of X with itself is a set of four coordinate points (as depicted in figure 5 of appended pages of figures).*

Define projection maps

$$p_i : X \times X \longrightarrow X$$

by

$$p_i(x, y) = x, \text{ if } i = 1$$
$$p_i(x, y) = y, \text{ if } i = 2.$$

Then, recalling that the topology of each of the range spaces X_1 and X_2 is $\{\emptyset, X, \{0\}\}$, we take inverse images of the open sets under each projection map as follows:

$p_1^{-1}(\emptyset) = \emptyset$;

$p_1^{-1}(X) = \{(0,0),(0,1),(1,0),(1,1)\}$; and

$p_1^{-1}(\{0\}) = \{(0,0),(0,1)\}$

Also

$p_2^{-1}(\emptyset) = \emptyset$;

$p_2^{-1}(X) = \{(0,0),(0,1),(1,0),(1,1)\}$; and

$p_2^{-1}(\{0\}) = \{(0,0),(1,0)\}$.

The collection S of these inverse images will be

$$S = \{\emptyset, \{(0,0),(0,1),(1,0),(1,1)\}, \{(0,0),(0,1)\}, \{(0,0),(1,0)\}\}.$$

The finite intersections B of sets of this collection will be

$$B = \{\emptyset, \{(0,1),(1,1),(0,0),(1,0)\}, \{(0,0),(0,1)\}, \{(0,0),(1,0)\}, \{(0,0)\}\} .$$

The arbitrary unions τ of sets of B is

$$\tau = \{\emptyset, \{(0,1),(1,1),(0,0),(1,0)\}, \{(0,0),(0,1)\}, \{(0,0),(1,0)\}, \{(0,0)\}, \\ \{(0,0),(0,1),(1,0)\}\}$$

The family τ is the (weak) topology on the Cartesian product of the Sierpinski space with itself, generated by the projection maps. We remark that the two-point *connected* space can be modified in several ways to produce similar topological spaces. For example, any family $\gamma = \{\emptyset, X, \{a\}\}$ will be a connected space if X is a set containing two or more elements which include a.

We note the following:

1. This weak topology has only six open sets, as listed out in τ above.

2. This weak topology has (in addition to the empty set and the whole set $X \times X$) four other closed sets, namely the family

   $$\{\{(1,1)\}, \{(0,1),(1,1)\}, \{(1,0),(1,1)\}, \{(0,1),(1,0),(1,1)\}.$$

3. This weak topological space has six sets which are neither closed nor open, namely the family $\{\{(0,1)\}, \{(1,0)\}, \{(0,0),(1,1)\}, \{(0,1),(1,0)\}, \{(0,0),(0,1),(1,1)\}, \{(1,0),(0,0),(1,1)\}\}$. See Figure 5, appended pages of figures, for a view of the landscape of this topology.

4. This weak topology is not Hausdorff, contrary to an existing result which simply states that *The weak topology is Hausdorff.* To see this, observe that no disjoint τ-open sets contain the distinct pair of points (0,0) and (0,1); or the points (0,0) and (1,0) in $X \times X$.

Let us now look at the weak topology on $\mathbb{R}^2$ when each factor space is given the X-topology, where $X = \{0, 1\}$. That is, the topology on $\mathbb{R}_1$, the horizontal factor space, is $\{2^X, \mathbb{R}\} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}, \mathbb{R}\}$. Also the same topology $\{2^X, \mathbb{R}\} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}, \mathbb{R}\}$ is endowed on $\mathbb{R}_2$, the vertical factor space. Then among the sub-basic sets of the resulting weak topology (induced by the projection maps) on $\mathbb{R}^2$ are

$p_1^{-1}(\{0\}) = \{(0, y) : y \in \mathbb{R}\}$, an infinite vertical line in the plane passing through the origin (better seen as the second factor space $\mathbb{R}_2$ itself);

$p_1^{-1}(\{1\}) = \{(1, y) : y \in \mathbb{R}\}$, another vertical infinite line in the plane (better seen as a horizontal translation of the second factor space $\mathbb{R}_2$ to the point (1,0) in the plane);

$p_1^{-1}(\{0, 1\}) = p_1^{-1}(X) = \{(0, y) : y \in \mathbb{R}\} \bigcup \{(1, y) : y \in \mathbb{R}\} = \mathbb{R}_2 \bigcup \{(1, y) : y \in \mathbb{R}\} =$ 2 vertical infinite lines in the plane, one $\mathbb{R}_2$ or the second factor space and the other may be seen as a translation of $\mathbb{R}_2$ to the point (1,0) or simply as a vertical infinite line through (1,0).

In a similar way, the second projection map p_2 will generate (among others) two horizontal infinite lines in the plane; one to be seen as $\mathbb{R}_1$, the horizontal factor space itself, and the other may be seen as a shift upwards of $\mathbb{R}_1$ to the point (0,1) in the plane. In this weak topology on $\mathbb{R}^2$, only these four infinite lines (two horizontal and two vertical) are open lines, and only four points (0,0), (0,1), (1,0), (1,1) in the Cartesian plane $\mathbb{R}^2$ are open singletons.

We can use the X-topology approach, or in general the idea of subset-induced topologies to obtain on $\mathbb{R}^n$ any weak topology in which a desired number of singletons are open sets.

4.1.7 Lower (and Upper) Limit Weak Topologies of $\mathbb{R}^n$

We recall that the lower limit (or Sorgenfrey) topology τ_L on the set $\mathbb{R}$ of real numbers is the topology generated (as a subbase) by sets of the form $\{x \in \mathbb{R} \colon a \leq x < b, a < b, a, b \in \mathbb{R}\} = [a, b)$. That is, τ_L on $\mathbb{R}$ is the topology in which such subintervals $[a, b)$ of $\mathbb{R}$ are open sets.

Construction 4.3 *If we endow* $\mathbb{R}$ *with this topology and define the usual projection maps on* $\mathbb{R} \times \mathbb{R}$ *into* $\mathbb{R}$*, it becomes immediately necessary to find out exactly what topology will result on* $\mathbb{R}^2$ *when we follow the usual procedures (as typified by theorem 1.1). That is, we want to find the weak topology on* $\mathbb{R}^2$ *generated by the projection maps when the factor spaces are given the lower limit topology. The subbase of this weak topology will be made up of sets of the form*

$$p_i^{-1}\{[a,b)\}, \text{ where } 1 \leq i \leq 2$$

The basic sets are of the form

$$p_i^{-1}\{[a,b)\} \bigcap p_j^{-1}\{[c,d)\} = \{[a,b) \times [c,d) : a,b,c,d \in \mathbb{R} \text{ and } a < b, c < d\}$$

where $1 \leq i, j \leq 2$*.*

The topological implication of this is that half-closed, half-open rectangles (as shown in figure 6) are open sets of this topology on $\mathbb{R}^2$. These rectangles will be closed on their left side as well as on their bottom (lower) side; and open on the remaining two sides. This topology on $\mathbb{R}^2$ is what we, here, call the lower limit weak topology of $\mathbb{R}^2$.

Construction 4.4 *If we now endow* $\mathbb{R}$ *with the upper limit topology* τ_U*, generated (as a subbase) by subintervals of the form* $\{(a,b] : a,b \in \mathbb{R}, a < b\}$*, then the resulting weak topology on* $\mathbb{R}^2$*, generated by the projection maps, is what we call the upper limit weak topology of* $\mathbb{R}^2$*; or simply the upper limit topology of* $\mathbb{R}^2$*.*

REMARK 4.6

We see that none of the lower and the upper limit weak topologies of $\mathbb{R}^2$, as constructed here, is weaker or stronger than the open line weak topologies on $\mathbb{R}^2$ constructed earlier.

However, if we endow only one factor space of $\mathbb{R}^2$ with the lower or the upper limit topology (and the other factor space with the usual topology of $\mathbb{R}$), then the weak topology on $\mathbb{R}^2$ that will result would be strictly weaker than one of the open line (weak) topologies: strictly weaker than the open vertical line weak topology if the horizontal axis is endowed the lower limit topology; and strictly weaker than the open horizontal line weak topology on $\mathbb{R}^2$ if the vertical axis is given the lower limit topology. This analysis also holds for the upper limit weak topologies. Also it should be observed that by endowing various (but not all) factor spaces of $\mathbb{R}^n$ with the lower and/or upper limit topologies, leaving the remaining factor spaces with the usual topology of $\mathbb{R}$ and defining n-projection maps on $\mathbb{R}^n$, we can always have such topologies on $\mathbb{R}^n$ where *polygonal solids* with some edges open and some edges closed are seen as open sets. By a polygonal solid we mean a solid with countable number of sides each of which is a polygon. (For instance, a cuboid is a polygonal solid of six sides each of which is a quadrilateral (or precisely a rectangle). A smooth sphere, even though it is a solid, is not a polygonal solid since it has an uncountable number of sides.) In fact this is the subject of the next definition.

Construction 4.5 *Let $X = \mathbb{R}^n$ be the product of n copies of the set $\mathbb{R}$ of real numbers. Define the projection maps $p_i : X \longrightarrow \mathbb{R}_i$, $1 \leq i \leq n$, as usual by $p_i(\bar{x}) = x_i$, where $\bar{x} = (x_1, x_2, \cdots, x_n), \forall \bar{x} \in X$. Let m factor subspaces of $X(= \mathbb{R}^n)$ be endowed with the lower (or upper) limit topologies $(m < n)$ while the remaining $n - m$ factor subspaces retain the usual topology of $\mathbb{R}$. Then the weak topology on $\mathbb{R}^n$ generated by the projection maps of $\mathbb{R}^n$, under this arrangement, is called—for clarification—lower (respectively upper) limit topology of $\mathbb{R}^n$.*

REMARK 4.7

1. For example in $\mathbb{R}^3$, the open sets of this kind of topology will consist of cuboids with some edges open and some edges closed.[2] Conversely every rectangular side of such a cuboid will have some edges open and the others closed.
2. This topology on $\mathbb{R}^n$ is strictly weaker than the discrete topology of $\mathbb{R}^n$.
3. The number of closed or open edges of each open polygonal solid in this topology will depend on the number m of factor subspaces endowed with lower (or upper) bound topology.
4. By way of comparing and contrasting, we have to state here that it is already known (before now) that if the factor spaces of $\mathbb{R}^n$ all retain the usual topology of $\mathbb{R}$, then the resulting weak topology from the projection maps will coincide with the usual Euclidean topology of $\mathbb{R}^n$; and that if the factor spaces of $\mathbb{R}^n$ all retain the discrete topology of $\mathbb{R}$, then the resulting weak topology from the projection maps will coincide with the discrete topology of $\mathbb{R}^n$.

4.1.8 Comparison Theorems For Weak Topologies

Weak topology on a nonempty set X is defined as the smallest or weakest topology on X with respect to which a given (i.e. fixed) family of functions

[2] See Figure 9.

on X is continuous. (See e.g. Lipschutz (1965), page 167; Sims (1976), page 29; Taylor and Lay (1980), page 156; Edwards (1995), page 88; Chidume (1996), page 124; Davis (2005), page 175; Munkres (2007), pages 86 and 114; Royden and Fitzpatrick (2012), page 231; and Morris (2016), page 193.)

Let τ_w be a weak topology generated on a nonempty set X by a family $\{f_\alpha, \alpha \in \Delta\}$ of functions, together with a corresponding family $\{(X_\alpha, \tau_\alpha), \alpha \in \Delta\}$ of topological spaces. If for some $\alpha_0 \in \Delta$, τ_{α_0} on X_{α_0} is not the indiscrete topology and f_{α_0} meets certain requirements, then there exists another topology τ_{w_1} on X such that τ_{w_1} is strictly weaker than τ_w and f_α is τ_{w_1}-continuous, for all $\alpha \in \Delta$. It is observed that

1. The new topology τ_{w_1} on X deserves to be called a weak topology (with respect to the fixed family of functions) in its own right. Hence we call τ_{w_1} a strictly weaker weak topology on X, than τ_w.

2. Every weak topology τ_w does not have a strictly weaker weak topology τ_{w_1}; yet a reasonably good number of weak topologies, including the usual weak topologies of interest, have strictly weaker weak topology τ_{w_1}.

3. Hence a further research agenda to exhaustively establish the relationship between a weak topology τ_w and (when it exists) its strictly weaker weak topology τ_{w_1}, in terms of exchange of topological properties, is set.

4. Also, a research agenda is set to find out why and/or when we must prefer to employ τ_w in analysis to τ_{w_1}; and vice versa.

This subsection basically sets out to establish all the necessary and sufficient conditions for the existence of τ_{w_1} in relation to τ_w. Ample examples are given to illustrate (at appropriate places) the various issues discussed.

Definition 4.28 *If τ_w is the weak topology on X generated by the family $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}$ of topological spaces, together with the family $\{f_\alpha\}_{\alpha\in\Delta}$ of functions, we shall call the triple $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ a weak topological system.*

Definition 4.29 *A product topological system is a triple $[(\bar{X}, \tau_p), \{(X_\alpha, \tau_\alpha)\}, \{p_\alpha\}]_{\alpha\in\Delta}$ of a topological product space $(\bar{X}, \tau_p)$, a family of topological spaces $\{(X_\alpha, \tau_\alpha)\}$ which, together with the family $\{p_\alpha\}$ of projection maps, induce the product topology τ_p on $\bar{X}$.*

We observe that every product topological system is a weak topological system, but not conversely.

Definition 4.30 *Two topologies τ_1 and τ_2 on a nonempty set X are said to be strictly comparable if one of the topologies is strictly weaker than the other.*

Definition 4.31 *Let $[(X,\tau_w),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. The weak topology τ_w is called an* indiscrete weak topology *(or a* minimal weak topology*)* [3] *if the family of functions in this system cannot generate a strictly weaker weak topology than τ_w, on X. Conversely, τ_w is an indiscrete weak topology if τ_w is not reducible to a strictly weaker weak topology within the system.*

Definition 4.32 *Let $[(X,\tau_w),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. The weak topology τ_w is called a* discrete weak topology *(or a* maximal weak topology*)* [4] *if the family of functions in this system cannot generate a strictly stronger weak topology than τ_w, on X. Conversely, τ_w is a discrete weak topology if τ_w is not extensible to a strictly stronger weak topology within the system.*

4.1.9 Some Preliminary Developments

Lemma 4.2 *Let Ψ and Φ be two nonempty subsets of the power set 2^X of a nonempty set X such that (say) Ψ is a proper subfamily of Φ. If f is a 1-1 function (on any set) mapping into all the elements of Φ, then $S_1=\{f^{-1}(G):G\in\Psi\}$ is a proper subfamily of $S_2=\{f^{-1}(G):G\in\Phi\}$.*

Proof:
Ψ is a proper subfamily of Φ. So, there exists $G_0\in\Phi\ni G_0\notin\Psi$, and $G\in\Phi\forall G\in\Psi$. Then from the hypothesis $S_1=\{f^{-1}(G):G\in\Psi\}$ is a subfamily of $S_2=\{f^{-1}(G):G\in\Phi\}$ and—since $G_0\notin\Psi$ and f is 1-1 and maps into each element of Φ—in particular the set $f^{-1}(G_0)\notin S_1$ (for otherwise we will have a contradiction). This means that S_1 is a proper subfamily of S_2.

REMARK 4.8
If f is not 1-1, S_1 may equal S_2 even though Ψ is a proper subfamily of Φ. See examples 4.22 and 4.23 below. And if f is 1-1 and there is no element of the domain of f mapped into an element of Φ not in Ψ, then S_1 may equal S_2; this is illustrated in example 4.25.

[3] As we shall see later, an indiscrete weak topology on X may not equal what may now be called the ordinary indiscrete topology $\{X,\emptyset\}$ of X.

[4] As we shall see later, a discrete weak topology on X may be strictly weaker than what may now be called the ordinary discrete topology 2^X of X.

EXAMPLE 4.22
Let $X = \{a, b, c\}$, $\Psi = \{\emptyset, X, \{a\}\}$ and $\Phi = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let $f : E \to X$ be a map such that $f(E) = \{a\}$, where E is any nonempty set with cardinality greater than 1. Then $S_1 = \{f^{-1}(G) : G \in \Psi\} = \{\emptyset, E\}$ and $S_2 = \{f^{-1}(G) : G \in \Phi\} = \{\emptyset, E\}$. That is, $S_1 = S_2$.
EXAMPLE 4.23
Let $E = \{1, 2, 3, 4, 5, 6\}$, $X = \{a, b, c, d\}$ and let $g : E \to X$ be a map such that $g(1) = a = g(4); g(3) = b = g(5)$. Let $\Psi = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\Phi = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{c\}, \{a, b, c\}\}$. Then Ψ is a proper subfamily of Φ but $S_1 = \{g^{-1}(G) : G \in \Psi\} = \{\emptyset, \{1, 3, 4, 5\}, \{1, 4\}, \{3, 5\}\}$ and $S_2 = \{g^{-1}(G) : G \in \Phi\} = \{\emptyset, \{1, 3, 4, 5\}, \{1, 4\}, \{3, 5\}\}$. So $S_1 = S_2$. We see that $g^{-1}(X) = g^{-1}(\{a, b\}) = g^{-1}(\{a, b, c\}) = \{1, 3, 4, 5\}$.
EXAMPLE 4.24
Let $E = \{1, 3, 4, 5\}$, $X = \{a, b, c, d\}$ and let $h : E \to X$ be a map such that $h(1) = a = h(4); h(3) = b = h(5)$. Let $\Psi = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\Phi = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then Ψ is a proper subfamily of Φ but $S_1 = \{h^{-1}(G) : G \in \Psi\} = \{\emptyset, \{1, 3, 4, 5\}, \{1, 4\}, \{3, 5\}\}$ and $S_2 = \{h^{-1}(G) : G \in \Phi\} = \{\emptyset, \{1, 3, 4, 5\}, \{1, 4\}, \{3, 5\}\}$. So $S_1 = S_2$. We see that (although Ψ and Φ constitute different topologies on X) the weak topology $\tau_w = \{\emptyset, E, \{1, 4\}, \{3, 5\}\}$ generated on E by the function h remains unchanged if the topology of X, as the range space of h, is changed between Ψ and Φ. So it is not obvious that changing the topology of a range space in a weak topological system will result in a change of the weak topology—as someone might think.
EXAMPLE 4.25
Let $X = \{a, b, c\}$, $\Psi = \{\emptyset, X, \{a\}\}$ and $\Phi = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let $f : E \to X$ be a map such that $f(E) = \{a\}$, where E is a singleton. Then f is 1-1 but $S_1 = \{f^{-1}(G) : G \in \Psi\} = \{\emptyset, E\}$ and $S_2 = \{f^{-1}(G) : G \in \Phi\} = \{\emptyset, E\}$. That is, $S_1 = S_2$. This example also shows that change of a range topology may not result in a change of the weak topology.
EXAMPLE 4.26
Let E, X, Ψ and Φ all be as defined in example 4.23 and let $g : E \to X$ be a map defined by $g(1) = a, g(3) = b, g(4) = c, g(5) = d$. We now have $g^{-1}(\emptyset) = \emptyset, g^{-1}(X) = \{1, 3, 4, 5\}, g^{-1}(\{a\}) = \{1\}, g^{-1}(\{b\}) = \{3\}$ and $g^{-1}(\{a, b\}) = \{1, 3\}$. Therefore $S_1 = \{g^{-1}(G) : G \in \Psi\} = \{\emptyset, \{1, 3, 4, 5\}, \{1\}, \{3\}, \{1, 3\}\}$. Now $g^{-1}(\{c\}) = 4$ and $g^{-1}(\{a, b, c\}) = \{1, 3, 4\}$. Hence $S_2 = \{g^{-1}(G) : G \in \Phi\} = \{\emptyset, \{1, 3, 4, 5\}, \{1\}, \{3\}, \{1, 3\}, \{4\}, \{1, 3, 4\}\}$. We now see that S_1 is a proper subfamily of S_2.

It is good to observe that if $E = \{1, 3, 4, 5\}$ in this example, and the two extra sets $\{a, c\}$ and $\{b, c\}$ are included in Φ (to make it a topology), then Φ and Ψ as different topologies on X would generate different weak topologies

on E.
EXAMPLE 4.27
Let $E = \{1, 2, 3, 4, 5\}, X = \{a, b, c, d, e\}$ and let $h : E \to X$ be a function defined by $h(1) = a, h(2) = e, h(3) = b, h(4) = c$, and $h(5) = d$. Let $\Psi = \{\emptyset, X, \{a\}, \{b\}, \{e\}, \{a, b\}, \{a, e\}, \{b, e\}, \{a, b, e\}\}$ and $\Phi =$ $\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{e\}, \{a, b\}, \{a, e\}, \{b, e\}, \{c, e\}, \{c, a\}, \{c, b\},$ $\{a, b, e\}, \{a, b, c\}, \{b, e, c\}, \{a, e, c\}, \{a, b, c, e\}\}$. Then $S_1 = \{h^{-1}(G) : G \in \Psi\} = \{\emptyset, E, \{1\}, \{3\}, \{2\}, \{1, 3\}, \{1, 2\}, \{3, 2\}, \{1, 3, 2\}\}$ and $S_2 = \{h^{-1}(G) : G \in \Phi\} = \{\emptyset, E, \{1\}, \{3\}, \{2\}, \{4\}, \{1, 3\}, \{1, 2\}, \{3, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\},$ $\{1, 3, 2\}, \{1, 2, 4\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$.
It is easy to see that both Ψ and Φ are (strictly comparable) topologies on X. The weak topology on E generated by h when X is endowed the topology $\tau = \Psi$ is τ_{w_1} and if we denote by τ_w the weak topology generated on E by the same function h when X is given the topology $\tau = \Phi$, then we see immediately that τ_{w_1} and τ_w are strictly comparable. This contrasts sharply with the finding in example 4.24 and shows (a) that a change of topologies in a range space of a weak topological system can result in a change of the weak topology and (b) that a sequence of pairwise strictly comparable topologies in a range space of a weak topological system can lead to a sequence of correspondingly strictly comparable weak topologies.

Henceforth whenever we mention 1-1 function in a weak topological system we shall assume that it meets the conditions of lemma 4.2; except otherwise stated.

Proposition 4.6 *Let $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. For some $\alpha_0 \in \Delta$, arbitrary but fixed, let τ_0 be a topology on X_{α_0} such that τ_0 is strictly weaker than τ_{α_0}. If (for this fixed $\alpha_0 \in \Delta$) f_{α_0} is 1-1, then $\exists\tau_{w_1}$, a topology on X, such that (i) $\tau_{w_1} < \tau_w$ and (ii) f_α is continuous with respect to τ_{w_1}, for all $\alpha \in \Delta$.*

Proof:
Let

$$S_1 = \{f_\alpha^{-1}(G_{\alpha_i}) : G_{\alpha_i} \in \tau_\alpha, \alpha \in \Delta, \alpha \neq \alpha_0\} \bigcup \{f_{\alpha_0}^{-1}(G_{\alpha_0}) : G_{\alpha_0} \in \tau_0\}$$

and let

$$S_2 = \{f_\alpha^{-1}(G_{\alpha_i}) : G_{\alpha_i} \in \tau_\alpha, \alpha \in \Delta\}.$$

Then by lemma 4.2 S_1 is a proper subfamily of S_2 since τ_0 is strictly weaker than τ_{α_0} and f_{α_0} is 1-1. We know that S_2 is a sub-base for τ_w; and similarly, since τ_0 is a topology on X_{α_0}, S_1 is a sub-base for another topology τ_{w_1} on X. As S_1 is a proper subfamily of S_2, there exists at least one set, say G, in S_2 such that $G \notin S_1$. It follows that finite intersections of sets in S_2 (that is, base for τ_w) contains at least one set G more than the finite intersections of the sets in S_1 (which is a base for τ_{w_1}). Hence the topology τ_{w_1} is weaker than τ_w by at least one set G. That is, τ_{w_1} is strictly weaker than τ_w. We also observe that f_α is τ_{w_1}-continuous, for each $\alpha \in \Delta$.

Observations:

1. The proposition above and the lemma 4.2 that facilitated its proof relied heavily on the existence of *just one* 1-1 function in a weak topological system, not on the existence of τ_0; since every non-indiscrete topology has a strictly weaker topology (by the reducibility results, in particular theorem 4.2 (b)).

2. Two weak topologies almost always the only ones of interest (so-called *the* weak and *the* weak star topologies) to many authors are about linear maps on linear spaces. The questions now vis-a-vis the proposition 4.6 here are

   - Is every linear map a 1-1 function? The answer is 'No'. Projection maps are linear but not 1-1.
   - Does there exist linear maps which are 1-1? Answer: 'Yes'. The identity maps are linear and 1-1.
   - Is every 1-1 map linear? Answer: 'No'. The function $f(x) = x^3$ is 1-1 but not linear.

3. Since there exist linear maps which are 1-1 and since the usual weak and weak star topologies are general statements about linear maps, proposition 4.6 implies that these topologies have strictly weaker weak or weak star topologies. This is a very important statement; and we therefore state it immediately below as a corollary.

Corollary 4.2 *The usual weak and weak star topologies have chains of pairwise strictly comparable weaker weak or weak star topologies.*

Proof:
Since these topologies are weak topologies generated on sets by all the linear maps on such sets, since some linear maps (namely, the identity maps) are

1-1 functions, proposition 4.6 ensures this result.

It may appear by now that it is only when a function f is 1-1 that S_1 would be a proper subfamily of S_2 given that Ψ is a proper subfamily of Φ. This is not so. In fact, f being 1-1 is only a sufficient condition for S_1 to be a proper subfamily of S_2 (given that Ψ is a proper subfamily of Φ) but it is not a necessary condition. The following example illustrates this.

EXAMPLE 4.28

Let $E = \{1,2,3,4,5\}, X = \{a,b,c,d\}, \Psi = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$ and $\Phi = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{c\}, \{a,b,c\}, \{a,c\}, \{b,c\}\}$. Let $h : E \to X$ be a map defined by $h(1) = a, h(2) = c, h(3) = b, h(4) = a$ and $h(5) = b$. Then we see that

$S_1 = \{h^{-1}(G) : G \in \Psi\} = \{h^{-1}(\emptyset), h^{-1}(X), h^{-1}(\{a\}), h^{-1}(\{b\}), h^{-1}(\{a,b\})\}$
$= \{\emptyset, E, \{1,4\}, \{3,5\}, \{1,3,4,5\}\}$. Also $S_2 = \{h^{-1}(G) : G \in \Phi\}$
$= \{h^{-1}(\emptyset), h^{-1}(X), h^{-1}(\{a\}), h^{-1}(\{b\}), h^{-1}(\{a,b\}), h^{-1}(\{c\}), h^{-1}(\{a,b,c\})\}$
$= \{\emptyset, E, \{1,4\}, \{3,5\}, \{1,3,4,5\}, \{2\}, \{1,4,2\}, \{3,5,2\}\}$.

We observe that S_1 and S_2 are strictly comparable weak topologies on E, just as Ψ and Φ are strictly comparable topologies on X.

A more general form of lemma 4.2 can therefore be stated as follows.

Lemma 4.3 *Let Ψ and Φ be two nonempty subsets of the power set 2^X of a nonempty set X such that Ψ is a proper subfamily of Φ. If f is a function mapping into each element of Φ, and there exists $G_0 \in \Phi - \Psi$ such that $f^{-1}(G_0) \neq f^{-1}(G), \forall G \in \Psi$, then $S_1 = \{f^{-1}(G) : G \in \Psi\}$ is a proper subfamily of $S_2 = \{f^{-1}(G) : G \in \Phi\}$.*

Proof:

Since $\exists G_0 \in \Phi, \ni f^{-1}(G_0) \neq f^{-1}(G), \forall G \in \Psi$ and since $\Psi \subset \Phi$ it follows that the collection $S_1 = \{f^{-1}(G) : G \in \Psi\}$ is a proper subfamily of $S_2 = \{f^{-1}(G) : G \in \Phi\}$.

We can now also obtain a more general form of proposition 4.6.

Proposition 4.7 *Let $[(X,\tau_w), \{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. For some $\alpha_0 \in \Delta$, arbitrary but fixed, let τ_0 be a topology on X_{α_0} such that τ_0 is strictly weaker than τ_{α_0}. If $\exists G_0 \in \tau_{\alpha_0}$ such that $f_{\alpha_0}^{-1}(G_0) \neq f_{\alpha_0}^{-1}(G), \forall G \in \tau_0$, then $\exists \tau_{w_1}$, a topology on X, such that (i) $\tau_{w_1} < \tau_w$ and (ii) f_α is continuous with respect to τ_{w_1}, for all $\alpha \in \Delta$.*

Proof:
Since $\exists G_0 \in \tau_{\alpha_0}$ such that $f_{\alpha_0}^{-1}(G_0) \neq f_{\alpha_0}^{-1}(G), \forall G \in \tau_0$, it follows that $G_0 \in \tau_{\alpha_0} - \tau_0$ and (by lemma 4.3) in particular $S_1 = \{f_{\alpha_0}^{-1}(G) : G \in \tau_0\}$ is a proper subfamily of $S_2 = \{f_{\alpha_0}^{-1}(G) : G \in \tau_{\alpha_0}\}$. Clearly elements of S_2 are among the sub-basic sets of τ_w and, since τ_0 is a topology, S_1 is also a subset of a sub-base for another topology τ_{w_1} on X, strictly weaker than τ_w. Since $[(X, \tau_{w_1}), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ is a weak topological system, f_α is τ_{w_1}-continuous, $\forall \alpha \in \Delta$.

REMARK 4.9
Proposition 4.7 implies that even a product topology can have a strictly weaker product topology. The further implication of proposition 4.7 is that the comparison result of proposition 4.6 can be extended to wider class of weak topological systems in which a 1-1 function may not exist.
EXAMPLE 4.29
Let $X_1 = \{a, b\} = X_2$ be two sets and let $\bar{X} = X_1 \times X_2 = \{(a, a), (a, b), (b, a), (b, b)\}$. Let the projection maps be defined on $\bar{X}$ in the usual way $p_i : \bar{X} \to X_i, 1 \leq i \leq 2$, by $p_i\{(x, y)\} = x$, if $i = 1$ and $p_i\{(x, y)\} = y$, if $i = 2$. Let both factor spaces of $\bar{X}$ be endowed with the topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then the product topology τ_p on $\bar{X}$ is $\tau_p = 2^{\bar{X}}$, the power set of $\bar{X}$; a family of 16 subsets of $\bar{X}$.

If we now let a factor space of $\bar{X}$, say X_1, be endowed with a topology τ_0 strictly weaker than τ such that $\exists G_0 \in \tau$ and such that $p_1^{-1}(G_0) \neq p_1^{-1}(G), \forall G \in \tau_0$ we shall get a strictly weaker product topology τ_{p_1}, on $\bar{X}$, than τ_p. To see this, let τ_0 on X_1 be $\tau_0 = \{\emptyset, X_1, \{a\}\}$. Then (with the topology of X_2 still being τ) the product topology now on $\bar{X}$ would be $\tau_{p_1} = \{\emptyset, \bar{X}, \{(a, a), (a, b)\}, \{(a, a), (b, a)\}, \{(a, b), (b, b)\}, \{(a, a)\}, \{(a, b)\}, \{(a, a), (a, b), (b, a)\}, \{(a, a), (a, b), (b, b)\}\}$; a family of only 9 subsets of $\bar{X}$.

It can also be verified easily that both projection maps p_1 and p_2 are continuous with respect to τ_{p_1} if τ_0 and τ are endowed respectively on X_1 and X_2.
NOTE:
Example 4.29 actually represents a general phenomenon in product topological systems; namely that if $[(\bar{X}, \tau_p), \{(X_\alpha, \tau_\alpha)\}, \{p_\alpha\}]_{\alpha\in\Delta}$ is a product topological system, and that there exists $\alpha_0 \in \Delta$ such that τ_{α_0} has a strictly weaker topology τ_0, on X_{α_0}, then there exists a strictly weaker product topology τ_{p_1} than τ_p on $\bar{X}$ with respect to which all the projection maps are continuous. We shall give a formal proof of this later, but for now, let's have another lemma.

Lemma 4.4 *Let $p_\alpha : \bar{X} \to X_\alpha$ be a projection map of a Cartesian product set onto a factor space. If x_{α_1} and x_{α_2} are two different elements of X_α, then $p_\alpha^{-1}(x_{\alpha_1}) \neq p_\alpha^{-1}(x_{\alpha_2})$.*

Proof:
Since projection maps count coordinates and return them to respective (or corresponding) factor spaces, we have
$p_\alpha^{-1}(x_{\alpha_1}) = \{\bar{x} \in \bar{X} : p_\alpha(\bar{x}) = x_{\alpha_1}\} = \{(x_r)_{r\in\Delta} \in \bar{X} : x_\alpha = x_{\alpha_1}\}$.
Also
$p_\alpha^{-1}(x_{\alpha_2}) = \{\bar{x} \in \bar{X} : p_\alpha(\bar{x}) = x_{\alpha_2}\} = \{(x_r)_{r\in\Delta} \in \bar{X} : x_\alpha = x_{\alpha_2}\}$.
As tuples (or vectors) are equal if and only if their corresponding components are equal, and since $x_{\alpha_1} \neq x_{\alpha_2}$, we must have $p_\alpha^{-1}(x_{\alpha_1}) \cap p_\alpha^{-1}(x_{\alpha_2}) = \emptyset$; that is, $p_\alpha^{-1}(x_{\alpha_1})$ and $p_\alpha^{-1}(x_{\alpha_2})$ have no element in common. As both $p_\alpha^{-1}(x_{\alpha_1})$ and $p_\alpha^{-1}(x_{\alpha_2})$ are nonempty, it follows that $p_\alpha^{-1}(x_{\alpha_1}) \neq p_\alpha^{-1}(x_{\alpha_2})$.

Corollary 4.3 *Let $p_\alpha : \bar{X} \to X_\alpha$ be a projection mapping. If A and B are two nonempty subsets of X_α such that (say) A is a proper subset of B, then $p_\alpha^{-1}(A) \subset p_\alpha^{-1}(B)$ and $p_\alpha^{-1}(A) \neq p_\alpha^{-1}(B)$; that is, $p_\alpha^{-1}(A)$ is a proper subset of $p_\alpha^{-1}(B)$.*

Proof:
Since $A \subset B$ and $A \neq B, \exists b_0 \in B \ni b_0 \notin A$. This implies that $b_0 \neq a, \forall a \in A$. This implies (by lemma 4.4) that $p_\alpha^{-1}(b_0) \neq p_\alpha^{-1}(a), \forall a \in A$. This implies that $p_\alpha^{-1}(b_0) \notin \{p_\alpha^{-1}(a) : a \in A\} = p_\alpha^{-1}(A)$. But $\{p_\alpha^{-1}(a) : a \in A\} \subset \{p_\alpha^{-1}(b) : b \in B\}$, because $A \subset B$. And we also know that $p_\alpha^{-1}(b_0) \in \{p_\alpha^{-1}(b) : b \in B\}$ as $b_0 \in B$. Hence $p_\alpha^{-1}(A)$ is a proper subset of $p_\alpha^{-1}(B)$.

Corollary 4.4 *Let $p_\alpha : \bar{X} \to X_\alpha$ be a projection mapping and let Ψ and Φ be two nonempty subsets of the power set 2^{X_α} of X_α. If Ψ is a proper subfamily of Φ, then $S_1 = \{p_\alpha^{-1}(G) : G \in \Psi\}$ is a proper subfamily of $S_2 = \{p_\alpha^{-1}(G) : G \in \Phi\}$.*

Proof:
Clearly $S_1 = \{p_\alpha^{-1}(G) : G \in \Psi\}$ is a subfamily of $S_2 = \{p_\alpha^{-1}(G) : G \in \Phi\}$, from hypothesis. We only show that $S_1 \neq S_2$. Let $G_0 \in \Phi - \Psi$. Since each set is the union of its own elements, we have

$$p_\alpha^{-1}(G_0) = \bigcup_{g\in G_0} p_\alpha^{-1}(g) \neq \bigcup_{g\in G} p_\alpha^{-1}(g), \forall G \in \Psi.$$

This implies that $p_\alpha^{-1}(G_0) \neq p_\alpha^{-1}(G), \forall G \in \Psi$. This implies that $p_\alpha^{-1}(G_0) \notin S_1$ and since $p_\alpha^{-1}(G_0) \in S_2$, it follows that $S_1 \neq S_2$. That is, S_1 is a proper subfamily of S_2.

Proposition 4.8 *Let* $[(\bar{X}, \tau_p), \{(X_\alpha, \tau_\alpha)\}, \{p_\alpha\}]_{\alpha \in \Delta}$ *be a product topological system. If (for some* $\alpha_0 \in \Delta$*)* τ_{α_0} *has a strictly weaker topology* τ_0*, on* X_{α_0}*, then the product topology* τ_p *on* $\bar{X}$ *has a strictly weaker product topology,* τ_{p_1}*.*

Proof:
From hypothesis τ_0 is a proper subfamily of τ_{α_0}. By corollary 4.4, $S_1 = \{p_{\alpha_0}^{-1}(G) : G \in \tau_0\}$ is a proper subfamily of $S_2 = \{p_{\alpha_0}^{-1}(G) : G \in \tau_{\alpha_0}\}$. Since τ_0 is a topology on X_{α_0}, $S_1 = \{p_{\alpha_0}^{-1}(G) : G \in \tau_0\}$ is part of a sub-base for a product topology τ_{p_1} on $\bar{X}$ (with the topologies of the other factor spaces unchanged). Since S_2 is part of a sub-base for τ_p and since S_1 is a proper subfamily of S_2, τ_{p_1} is strictly weaker than τ_p.

REMARK 4.10

1. It is now clearer that the condition of 1-1-ness in proposition 4.6 is only a sufficient, but not necessary, requirement for a strictly weaker weak topology to be obtained, given that the topology of a range space has a strictly weaker topology.

2. The reasoning in propositions 4.6 and 4.7 implies that if τ_1 is strictly weaker than τ_0, τ_2 strictly weaker than τ_1, and so on, then there exist correspondingly weak topologies τ_{w_2}, τ_{w_3}, etc., on X, such that $\tau_w > \tau_{w_1} > \tau_{w_2} > \tau_{w_3} > \cdots$.

3. If we have a weak topological system $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}, \{f_\alpha\}_{\alpha \in \Delta}]$, one question is whether we can always find another topology τ_{w_1} on X such that $\tau_w > \tau_{w_1}$ and such that each function in the family is continuous? That is, does τ_{w_1} always exist for every weak topology τ_w? Another question (if it be found that τ_{w_1} does not exist for all weak topologies τ_w) is whether we can characterize such weak topologies τ_w for which we can find such τ_{w_1}. And yet another question is: What (if any) topological property can τ_w transmit to, or induce on τ_{w_1}? This last question can be seen as property inheritance question—and it is as important here as it is in human society. These questions and more are what we shall be looking at in the next subsection.

In the following developments, when we discuss a weak topological system we shall assume that there exists in it a range space whose associated function is such that it returns distinct open sets (in the range space) to distinct pre-images in the domain space; and when we give attention to a range space in a weak topological system, we shall assume (without loss of generality) that the function associated with that range space returns distinct open sets to distinct pre-images.

If in a weak topological system there is no range space for which the associated function returns distinct open sets to distinct pre-images, then we shall assume that the weak topology τ_w of the system is an indiscrete weak topology in that it has no strictly weaker weak topology τ_{w_1}.

4.1.10 More General Results and Examples

CASE I—τ_{w_1} Does Not Exist For Every Weak Topology τ_w

EXAMPLE 4.30

Let $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system in which each of the topological range spaces is an indiscrete space and the domain of each of the functions is all of X. Then necessarily (X, τ_w) is an indiscrete weak topological space; hence τ_w has no strictly weaker weak topology τ_{w_1}.

EXAMPLE 4.31

Let $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system in which each of the topological range spaces is an indiscrete space. Let the domain of one of the functions not be all of X. Then again (X, τ_w) is an indiscrete weak topological space in the sense of our definitions—that is, τ_w has no strictly weaker weak topology τ_{w_1}.

It is also good at this juncture to again look at example 4.24 above. In example 4.24, change of range topologies did not result in a change of the weak topology for the single function h. This, however, does not mean that the weak topology $\tau_w = \{\emptyset, E, \{1, 4\}, \{3, 5\}\}$ on E generated by h is indiscrete. For if we endow X with the topology $\tau = \{\emptyset, X, \{a\}\}$, then h will generate a weak topology $\tau_{w_1} = \{\emptyset, E, \{1, 4\}\}$ on E, which is strictly weaker than τ_w. (See also theorem 4.4 ahead.)

CASE II—τ_{w_1} Exists For Many Weak Topologies τ_w

EXAMPLE 4.32

Let $X = \{0, 1\}$. A Sierpinski topology on X is the collection $\tau = \{\emptyset, X, \{0\}\}$. The Cartesian product of X with itself is the set $\bar{X} = X \times X$ $= \{(0, 0), (0, 1), (1, 1), (1, 0)\}$ of 4 coordinate points. We can define the projection maps $p_i : \bar{X} \to X, i = 1, 2$ in the usual way by $p_i\{(x, y)\} = x$ if $i = 1$, and $p_i\{(x, y)\} = y$ if $i = 2$. Let us also endow each factor space X_1 and X_2 of $\bar{X}$ with this Sierpinski topology. Then we have obtained all the conditions for a product topological system $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}, \{p_\alpha\}]$ where the family of functions is made up of only two projection maps; and the product topology is the family

$$\tau_w = \{\emptyset, \bar{X}, \{(0, 0), (0, 1)\}, \{(0, 0), (1, 0)\}, \{(0, 0)\}, \{(0, 0), (1, 0), (0, 1)\}\}$$

of 6 subsets of $\bar{X}$.

Now let us endow only one factor space of $\bar{X}$ with the Sierpinski topology, and the remaining factor space with the indiscrete topology. The product (weak) topology that would now emerge on $\bar{X}$ is seen to be

$$\tau_{w_1} = \{\emptyset, \bar{X}, \{(0,0), (1,0)\}\}$$

a family of only 3 subsets of $\bar{X}$. It is also easily seen that τ_{w_1} is a strictly weaker weak topology than τ_w, on $\bar{X}$. Yet both weak topologies are generated by the same fixed family of functions.

EXAMPLE 4.33

The Euclidean (or usual) topology of the Cartesian plane $\mathbb{R}^2$ is known as the weak topology τ_w of the plane when its factor spaces $\mathbb{R}_1, \mathbb{R}_2$ (respectively the horizontal and the vertical axes) are themselves given their usual (Euclidean) topology, and the projection maps are the family of functions.

If we endow any of the axes of the plane $\mathbb{R}^2$ with a topology strictly weaker than the usual topology of $\mathbb{R}$ the weak topology that would then be generated on the plane by the projection maps would be strictly weaker than (what may now be called) the usual weak topology of the plane. And only a second thought is all we need to see that virtually every topology on an axis of the Cartesian plane $\mathbb{R}^2$ has a strictly weaker topology—hence virtully every weak topology (including of course product topology) on the plane has a strictly weaker weak (or product) topology. This somewhat strong statement will find illustration in further examples and propositions here.

EXAMPLE 4.34

Let $X = (a, b) \in U$ be a fixed open interval in the usual topology U of $\mathbb{R}$. Let $\gamma = \{G \in U : G \subset X\}$. Then it is easy to see that γ is a topology on X. If we now let $\tau = \gamma \cup \{\mathbb{R}\}$, we see that τ is a topology strictly weaker than U on $\mathbb{R}$.[5] If we have the two factor spaces of $\mathbb{R}^2$ endowed with the topology τ and have the projection maps as the family of functions on R^2, the weak (product) topology now on the plane $\mathbb{R}^2$ would be strictly weaker than the usual weak topology of the plane.

EXAMPLE 4.35

Let $n \in \mathbb{N}$ be a natural number, and let $X_n = (-n, n) \in U$, a U-open interval, where U is the usual topology on $\mathbb{R}$. We can let τ_n be the topology induced on $\mathbb{R}$ by its U-open subset X_n following the process of construction in example 4.34 above. Then we observe the following.

1. Each τ_n on $\mathbb{R}$ is strictly weaker than the usual topology U on $\mathbb{R}$ for all $n \in \mathbb{N}$. Hence by endowing each factor space of $\mathbb{R}^2$ with τ_n we can

[5] We can see that the development on subset-induced topologies on supersets is very important here.

obtain a strictly weaker weak topology (than the Euclidean topology) on $\mathbb{R}^2$, generated by the projection maps.

2. If $m > n$ then τ_n is strictly weaker than τ_m on $\mathbb{R}$. Hence corresponding to any pair m, n of natural numbers there exists a pair τ_m and τ_n of strictly comparable and strictly weaker topologies than U on $\mathbb{R}$.

3. Hence corresponding to any pair m, n of natural numbers there exists a pair τ_{w_m} and τ_{w_n} of strictly comparable and strictly weaker weak topologies than the usual weak topology τ_w on $\mathbb{R}^2$. Hence

4. There exists a chain $\{\tau_{w_n}\}_{n\in\mathbb{N}}$ of pairwise strictly comparable and strictly weaker weak topologies than the usual weak topology τ_w on $\mathbb{R}^2$ in that

$$\tau_{w_1} < \tau_{w_2} < \tau_{w_3} < \cdots < \tau_w$$

5. As $n \to \infty$, $\tau_{w_n} \to \tau_w$; and finally

6. Any nonempty subset of the set $\mathbb{R}$ of real numbers can be used as the indexing set here in place of $\mathbb{N}$ and the subset-induced topologies can be constructed in many other ways than what is done here.

REMARK 4.11

- The analysis above, particularly in example 4.34, copiously holds for any weak topology on any nonempty set which has a range topological space that in turn has a strictly weaker topology. And this scenario is a very fortuitous one as it tells us that we can seek and find a strictly weaker weak topology τ_{w_1}, than τ_w, provided τ_w is not an indiscrete weak topology; that we can further seek and find a strictly weaker weak topology τ_{w_2}, than τ_{w_1}, provided τ_{w_1} is not an indiscrete weak topology; and so on.

- All the range (topological) spaces must not be endowed with only one type of topology in order to get a strictly weaker weak topology than a given weak topology.

- The expositions in the examples above can be extended to (particularly) general Euclidean topology of $\mathbb{R}^n$—and in general, to many weak topological systems.

- From the observations above it is clear that every pair of strictly comparable topologies in a range space of a weak topological system equally has a pair of strictly comparable weak topologies generated (if it can be so said) by them. This is a very important result which we state below in lemma 4.5. (In this lemma it is assumed that conditions of lemma 4.3 are met.)

Lemma 4.5 *Let* $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If in a range space, say* (X_r, τ_r) *there exist two strictly comparable topologies* τ_{r_1} *and* τ_{r_2} *where, say* $\tau_{r_1} < \tau_{r_2}$ *(and both are strictly weaker than* τ_r*), then there exist two strictly comparable weaker weak topologies* τ_{w_1} *and* τ_{w_2}*, generated by the fixed family of functions, on* X *in that* $\tau_{w_1} < \tau_{w_2} < \tau_w$.

EXAMPLE 4.36
It is known that a finite product of discrete topological spaces is discrete. We add that if the cardinality of any of the factor spaces of a finite dimensional discrete product space is greater than 1, then such a discrete product topology has a strictly weaker product topology. The last statement was proved in Proposition 4.8 above.

The strictly weaker weak topologies obtained in respect of a given weak topology may not be pairwise strictly comparable; in fact they may not be comparable at all. The next example illustrates this. That is, if we look at the foregoing examples it may appear that all the strictly weaker weak topologies τ_{w_i} (when they exist) of a weak topology τ_w are always pairwise comparable. This is not actually so.

Definition 4.33 *Let* $K = \{\frac{1}{n}\}_{n\in N} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\} \subset [0,1]$*, and let* $(\mathbb{R}, U)$ *denote* $\mathbb{R}$ *with its usual topology* U*. Let* $B = (a,b) - K$*, where* (a,b) *is an open interval of the set of real numbers with its usual topology. (We observe that* $(a,b) - K = (a,b)$*, if* $(a,b) \bigcap K = \emptyset$*) Then the* K*-topology* $\mathbb{R}_k$ *on the set* $\mathbb{R}$ *of real numbers is the topology generated on* $\mathbb{R}$ *by using sets of the forms* (a,b) *and* $(a,b) - K$*, that is, the family* $\{(a,b), (a,b) - K : a < b \in \mathbb{R}\}$*, as subbase.*

EXAMPLE 4.37
As we have seen (in subsection 4.1.7), the lower limit topology $\mathbb{R}_l$ on $\mathbb{R}$ is generated by taking subintervals of the form $[a,b)$ as subbase. It is clear that the lower limit topology $\mathbb{R}_l$ and the K-topology $\mathbb{R}_k$ on $\mathbb{R}$ are not comparable—since for instance the $\mathbb{R}_k$-open set

$$G = (-1,1) - K = (-1,0] \cup \left(\bigcup_{n=1}^{\infty} (\tfrac{1}{n+1}, \tfrac{1}{n}) \right) \ . \ . \ . \ (*)$$

is not $\mathbb{R}_l$-open, and the $\mathbb{R}_l$-open subinterval $[0,1)$ is not $\mathbb{R}_k$-open.
Let $\mathbb{R}_k \bigtriangledown \mathbb{R}_l$ be the join of $\mathbb{R}_k$ and $\mathbb{R}_l$ (defined as the weakest topology stronger than both $\mathbb{R}_k$ and $\mathbb{R}_l$). Let $(\mathbb{R}, u)$ remain the usual topological space of $\mathbb{R}$. Then it is clear that $(\mathbb{R}, u) < (\mathbb{R}, \mathbb{R}_l)$ (that is, u is strictly weaker than $\mathbb{R}_l$) since $(a,b) = \bigcup_{n=n_0}^{\infty} [a + \frac{1}{n}, b)$ for some $n_0 \in \mathbb{N}$ implies that every u-open set is $\mathbb{R}_l$-open, but not conversely, since no subinterval of the form $[a,b)$ is u-open. It is also easy to see that $u < \mathbb{R}_k$ as the u-basic open intervals (a,b) are among the subbasic sets of $\mathbb{R}_k$—and for instance the $\mathbb{R}_k$-open set G above is not u-open. So $u \leq \mathbb{R}_l \cap \mathbb{R}_k$ and since $\mathbb{R}_l < \mathbb{R}_k \bigtriangledown \mathbb{R}_l$ and $\mathbb{R}_k < \mathbb{R}_k \bigtriangledown \mathbb{R}_l$ it follows that $u < \mathbb{R}_k \bigtriangledown \mathbb{R}_l$.
If we endow all the factor spaces of $\mathbb{R}^2$ with the topology $\mathbb{R}_k \bigtriangledown \mathbb{R}_l$ and have the projection maps defined in the usual way on $\mathbb{R}^2$ we shall have a weak topology τ_w on $\mathbb{R}^2$. Let τ_{w_l}, τ_{w_k} and τ_{w_u} denote the weak topology generated on $\mathbb{R}^2$ by the projection maps when all the factor spaces are endowed with respectively the lower limit topology $\mathbb{R}_l$, the K-topology $\mathbb{R}_k$ and the usual topology u. Then it is easy to establish the following:

- $\tau_{w_u} < \tau_w$;
- $\tau_{w_l} < \tau_w$;
- $\tau_{w_k} < \tau_w$;
- $\tau_{w_u} < \tau_{w_K}$;
- $\tau_{w_u} < \tau_{w_l}$;
- τ_{w_l} and τ_{w_k} are not comparable. This is because $\mathbb{R}_l$ and $\mathbb{R}_k$ are not comparable. (See also proposition 4.11(2) ahead.)

We remark that the topologies we endow the two factor spaces of $\mathbb{R}^2$ can be mixtures of the four topologies ($\mathbb{R}_l, \mathbb{R}_k, u$, and $\mathbb{R}_k \bigtriangledown \mathbb{R}_l$) here. (That is, we do not have to endow all the factor spaces of $\mathbb{R}^2$ with only one of these topologies.) If we do this, the comparison considerations will be different. We also remark that the analysis here can be extended to $\mathbb{R}^n, n > 2$ using these four topologies.

Construction 4.6 (K-topology-induced weak topology) *Suppose the Cartesian plane $\mathbb{R}^2$ has the projection maps defined on it, as usual, and that the factor spaces $\mathbb{R}_1$ and $\mathbb{R}_2$ (respectively horizontal and vertical) are* *each endowed with the K-topology. The K-topology-induced weak topology of $\mathbb{R}^2$ is the weak topology generated on $\mathbb{R}^2$ by the projection maps under this arrangement; i.e. where the factor spaces are given the K-topology.*

We may want to know one or two things about the landscape of this topology. Suppose $G = [(a, b) - K] \in K_{\mathbb{R}}$ is an arbitrary open set in the K-topology of $\mathbb{R}$. Then two cases arise: namely, that either $(a, b) \bigcap K = \emptyset$ or $(a, b) \bigcap K \neq \emptyset$.
Suppose, first, that $(a, b) \bigcap K \neq \emptyset$. Then $p_1^{-1}(G) = \{\bar{x} \in \mathbb{R}^2 : p_1(\bar{x}) \in G\} = \{\bar{x} \in \mathbb{R}^2 : p_1(\bar{x}) \in [(a, b) - K]\} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [(a, b) - K]\} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in (a, b) \text{ and } x_1 \notin K\}$. This is an open vertical infinite strip with deleted infinite vertical lines through the common points of (a, b) and K.
If $(a, b) \bigcap K = \emptyset$, then $p_1^{-1}(G) = \{\bar{x} \in \mathbb{R}^2 : p_1(\bar{x}) \in (a, b)\} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in (a, b)\}$. This is the usual open vertical infinite strip, with no demarcations in it.
In the same way, $p_2^{-1}(G)$ will either be an open horizontal infinite strip with deleted horizontal infinite lines or the usual horizontal infinite strips, without demarcations.
NOTE:

1. Let $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}, \{f_\alpha\}_{\alpha \in \Delta}]$ be a weak topological system. If, for some $r \in \Delta$, τ_r has two distinct strictly weaker topologies τ_{r_1} and τ_{r_2} then it is clear from the foregoing that we can get a strictly weaker weak topology τ_{w_1}, than τ_w, on X in at least two ways.

2. Now we can confidently say that any non-indiscrete weak topology has a strictly weaker weak topology. The last assertion is clearly an important statement which needs to be proved. The proof of this will be given below in theorem 4.4.

3. In terms of topological properties (like the separation axioms, compactness, etc.) there is now a challenge to identify or characterize the weak topologies whose strictly weaker weak topologies inherit their property; and it will equally be important and interesting to find those topological properties that are preserved under the operation of getting strictly weaker weak topologies.

Lemma 4.6 *Let τ and η be two topologies on a set X and let S_τ and S_η denote the subbases for τ and η respectively. Then $S_\tau \subset S_\eta \Rightarrow \tau \leq \eta$.*

Proof:
Let $B_\tau = \left\{ \bigcap_{i=1}^{n} G_i : G_i \in S_\tau \right\}$ be the base for τ and let $B_\eta = \left\{ \bigcap_{i=1}^{n} G_i : G_i \in S_\eta \right\}$ be the base for η. If $S_\tau \subset S_\eta$ then clearly $B_\tau \subset B_\eta$, and hence that $\tau = \left\{ \bigcup_{\alpha\in\Delta} B_\alpha : B_\alpha \in B_\tau \right\}$ is a subfamily of $\eta = \left\{ \bigcup_{\alpha\in\Delta} B_\alpha : B_\alpha \in B_\eta \right\}$. That is, $\tau \leq \eta$.

Clearly the following result has many fundamental and far-reaching implications.

Theorem 4.4 *Let* $[(X,\tau_w), \{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If* $\exists (X_r, \tau_r)$*, for some* $r \in \Delta$, $\ni Card(\tau_r) > 2$ *then* τ_w *has a strictly weaker weak topology. Hence any non-indiscrete weak topology has a strictly weaker weak topology in its system.*

Proof:
$Card(\tau_r) > 2$ implies that τ_r contains at least 3 subsets of X_r. So, let $\tau_r = \{\emptyset, X_r, G\}$, where G is a nonempty proper subset of X_r. Then $\tau_{r_1} = \tau_r - \{G\}$ is a topology on X_r strictly weaker than τ_r—or, as has been proved in theorem 4.1, τ_r can be reduced in some sense. Let τ_{w_1} be the weak topology generated on X by the fixed family of functions when X_r has the topology τ_{r_1} and the remaining range spaces have their topologies unchanged. Then τ_{w_1} is strictly weaker than τ_w since in particular $f_r^{-1}(G) \notin \tau_{w_1}$. The proof is complete.

The meaning of theorem 4.4 is that a weak topology τ_w generated on a set X by a given family F of functions has a strictly weaker weak topology τ_{w_1} on X generated by the same family F of functions provided one of its range spaces is not an indiscrete topological space. (This is why for instance, under example 4.31, as we looked more closely at example 4.24, we saw that the function h generated a weaker weak topology.) And an alternative way to say this is that a weak topology is non-indiscrete if and only if it has at least one non-indiscrete range space.

If all the range spaces are indiscrete topological spaces in the usual sense of having topologies of cardinality 2, it does not follow or mean that the weak topology—being then an indiscrete weak topology—would have cardinality equal to 2.

EXAMPLE 4.38
Let $X = \{a, b, c\}$, $X_1 = \{x, y\}$ and $X_2 = \{p, q, r, s, t\}$. Let $f_1 : X \to X_1$ be

a function defined by $f_1(a) = x$, $f_1(b) = x$ and $f_1(c) = y$. Let $f_2 : X \to X_2$ be a function defined by $f_2(b) = q$ and $f_2(c) = p$. Let $\tau_1 = \{X_1, \emptyset\}$ be the topology on X_1 and let $\tau_2 = \{X_2, \emptyset\}$ be the topology on X_2. Then (X_1, τ_1) and (X_2, τ_2) are indiscrete topological spaces and the cardinality of each of the range topologies is 2. It can easily be verified that the weak topology τ_w on X generated by the family $F = \{f_1, f_2\}$ of these two functions is $\tau_w = \{\emptyset, X, \{b, c\}\}$; a family of 3 subsets of X.

EXAMPLE 4.39

Let $X = \{a, b, c\}$, $X_1 = \{x, y\}$ and $X_2 = \{p, q, r, s, t\}$. Let $f_1 : X \to X_1$ be a function defined by $f_1(a) = x$ and $f_1(b) = x$. Let $f_2 : X \to X_2$ be a function defined by $f_2(b) = q$ and $f_2(c) = p$. Let $\tau_1 = \{X_1, \emptyset\}$ be the topology on X_1 and let $\tau_2 = \{X_2, \emptyset\}$ be the topology on X_2. Then (X_1, τ_1) and (X_2, τ_2) are indiscrete topological spaces and the cardinality of each of the range topologies is 2. Now the weak topology τ_w on X generated by the family $G = \{f_1, f_2\}$ of two functions is $\tau_w = \{\emptyset, \{a, b, c\}, \{a, b\}, \{b, c\}, \{b\}\}$; a family of 5 subsets of X.

It is important to observe that the family F of functions in example 4.38 is different from the family G of functions in example 4.39. This observation will help us not to think that a fixed family of functions can generate two indiscrete weak topologies on the same set—as really a fixed family of functions cannot generate more than one indiscrete weak topology on a set. And the indiscrete weak topology of a family of functions must emerge only when all the range topologies are themselves indiscrete.

An indiscrete weak topology may also emerge in the usual form (with cardinality 2) in which we have known indiscrete topologies.

EXAMPLE 4.40

Let X, X_1 and X_2 all be as given in example 4.39 above and let X_1 and X_2 retain their indiscrete topologies. If the domain of f_1 is all of X and the domain of f_2 is all of X, then the weak topology τ_w on X generated by these two functions will be $\tau_w = \{\emptyset, X\}$; with cardinality 2. So, when we say *an indiscrete weak topology* we only know or mean that it is one which has no strictly weaker weak topology; the matter of the determination of its cardinality is something else.

Proposition 4.9 *An indiscrete weak topology can have cardinality greater than 2; however, it cannot have a strictly weaker weak topology in its own system.*

Since we have seen (from examples 4.38 and 4.39 above) that an indiscrete weak topology can have cardinality greater than 2, since such an indiscrete weak topology is also a topology in the ordinary sense and hence by theorem

4.1 can further be reduced in some sense (though not as a weak topology), we have yet another very important exposition.

Theorem 4.5 *Let* $[(X,\tau_w),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. The following statements are equivalent.*
(a) The weak topology τ_w *is not reducible to a strictly weaker weak topology in any sense.*
(b) All the range topologies of τ_w*, including any which may itself be a weak topology, have cardinality 2.*
(c) τ_w *is an indiscrete weak topology.*

Proof:
(a) If the weak topology τ_w is not reducible as a weak topology in any sense, then all the range topologies have cardinality 2; for if a range topology has cardinality greater than 2, theorem 4.4 would imply that τ_w has a strictly weaker weak topology. That is, (a) implies (b).
(b) Clearly τ_w is an indiscrete weak topology if all the range topologies of τ_w have cardinality 2.
(c) implies (a) by definition.

Theorem 4.4 again has this very important implication which we state below as a corollary.

Corollary 4.5 *Every non-indiscrete weak topology on a nonempty set* X *is at the peak of a chain of pairwise strictly comparable* ***weaker weak topologies****.*

Note
The cardinality of such a chain will depend on (a) the cardinality of X and (b) the creative way we choose to develop the chain. If X is a finite set, then the chain will necessarily be finite; and if X is infinite the chain can be made to be finite or infinite. The usual Euclidean topologies of $\mathbb{R}^n (n \geq 2)$, as weak topologies, can have finite chain, denumerable chain, or uncountable chain of pairwise strictly comparable weaker weak topologies.

Proposition 4.10 *Let* $[(X,\tau_w),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. Let* τ_{w_i} *denote the weak topology on* X *when* X_α *has the topology* τ_{α_i}*. Then*

1. $\tau_{\alpha_i} \leq \tau_{\alpha_j} \Longrightarrow \tau_{w_i} \leq \tau_{w_j}$;

2. τ_{α_i} *and* τ_{α_j} *not comparable, implies* τ_{w_i} *and* τ_{w_j} *not comparable;*

3. $\tau_{\alpha_i} < \tau_{\alpha_j}$ *and* $\tau_{r_i} > \tau_{r_j}$ *implies* τ_{w_i} *and* τ_{w_j} *not comparable; and*

4. $\{\tau_{\alpha_r}\}$*, a chain, implies that* $\{\tau_{w_r}\}$ *is a chain.*

Proof:

1. Lemma 4.6 makes this easy to see.

2. If τ_{α_i} and τ_{α_j} are not comparable, then the subbases of τ_{w_i} and τ_{w_j} (and hence the topologies τ_{w_i} and τ_{w_j}) are not comparable.

3. If $\tau_{\alpha_i} < \tau_{\alpha_j}$ then from the foregoing, $\tau_{w_i} < \tau_{w_j}$. But then $\tau_{r_i} > \tau_{r_j}$ implies that $\tau_{w_i} > \tau_{w_j}$. That is, $\tau_{w_i} < \tau_{w_j}$ and $\tau_{w_i} > \tau_{w_j}$. This is a contradiction; implying that τ_{w_i} and τ_{w_j} are not comparable.

4. $C = \{\tau_{\alpha_r}\}$ being a chain implies that the topologies in C are pairwise comparable. Lemma 4.5 then implies that the family $\{\tau_{w_r}\}$ of weak topologies on X is also in chain.

4.1.11 Strictly Stronger Weak Topologies?

So far we have only been looking at the possibility of getting strictly weaker weak topologies when and if requisite conditions are met. Let us now look at the possibility of obtaining strictly stronger weak topologies.

This particular idea has been explored before by other researchers; however, the development here is an extension of the approach adopted before in getting strictly stronger weak topologies. For instance, only four weak topologies have been constructed and compared before by others. Secondly, these four weak topologies constructed and compared before were achieved by using polars of subsets of normed linear spaces. (In that sense it was more or less a development on only *normed linear spaces.*)

Here we show the link between constructing weak topologies by the use of polars and constructing them by the general method which we have since adopted, and then we proved that the general method is *indeed general enough* as it encompasses (what may now be called) the polar method. Then thirdly we showed that between the four already compared weak topologies there exist many other weak topologies—constructible even by the use of polars. Finally, we proved that if two weak topologies (generated by one family of functions) on a set are strictly comparable, then there exist in a range space two strictly comparable topologies which induce the weak topologies.

This last exposition is a converse way of proving the earlier assertion that the polar method is part of the general method of construction of weak topologies.

In making this inquiry, we follow our established tradition and do not assume that there is a linear structure or a norm on X. We also do not assume that it would be useful (in application) to look for strictly stronger weak topologies instead when a goal is to find the smallest topology with respect to which a given family of functions is continuous.

This flipside exposition is necessitated by the fact (from the title of this section) that we wish to explore and establish results on how weak topologies can, and do actually compare with one another; and to give that inquiry a reasonable treatment we have to, we believe, look at other situations in which there may exist strictly stronger weak topologies (when a weak topology τ_w is not the discrete topology).

It is a well known proposition that if B is a family of subsets of a given set X, then B is a base for *some* topology on X if (a) $\bigcup_{G \in B} G = X$; and (b) if $G_1, G_2 \in B$, then $G_1 \cap G_2 \in B$. This fact will be crucially used in what follows.

Proposition 4.11 *Let* $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}, \{f_\alpha\}_{\alpha \in \Delta}]$ *be a weak topological system. If* τ_w *is not the discrete topology on* X*, and there exists* $\alpha_0 \in \Delta$ *such that* τ_{α_0} *is not the discrete topology on* X_{α_0} *then there exists a topology* τ_{w+1} *on* X *such that (i)* $\tau_w < \tau_{w+1}$*; and (ii)* f_α *is* τ_{w+1}*-continuous, for all* $\alpha \in \Delta$.

Proof:
Since $\tau_{\alpha_0} < 2^{X_{\alpha_0}}$ there exists $G_0 \subset X_{\alpha_0}$ such that $G_0 \notin \tau_{\alpha_0}$. Let $S_0 = \tau_{\alpha_0} \cup \{G_0\}$ and let $B_0 = \{\text{finite intersections of elements of } S_0\}$. Then we see that (a) $\bigcup_{G \in B_0} G = X_{\alpha_0}$; and (b) if $G_1, G_2 \in B_0$, then $G_1 \cap G_2 \in B_0$. Hence B_0 is a base for a topology τ_0 on X_{α_0}. It is clear that $\tau_{\alpha_0} < \tau_0$, and if we replace τ_{α_0} with τ_0 in the weak topological system we shall get a weak topology τ_{w+1}, on X generated by this fixed family of functions. Then it is easy to verify that (i) $\tau_w < \tau_{w+1}$; and that (ii) each f_α is continuous with respect to τ_{w+1}.

If all the range spaces (X_α, τ_α) are power set discrete topological spaces, then the weak topology τ_w, being then a discrete weak topology—though it may not be equal to the power set of X—will not be extensible in any sense (strong, normal or weak) to a strictly stronger weak topology.

EXAMPLE 4.41
Let $X = \{a, b, c\}, X_1 = \{x, y\}, X_2 = \{p, q, r, s, t\}$ and let $f_1 : X \to X_1$ on

X into X_1 be defined by $f_1(a) = x$, $f_1(b) = x$; and let $f_2 : X \to X_2$ be defined by $f_2(b) = q$, $f_2(c) = p$. Let $\tau_1 = 2^{X_1}$ be the discrete topology of X_1 and let $\tau_2 = 2^{X_2}$ be the discrete topology of X_2. Then (X_1, τ_1) and (X_2, τ_2) are discrete topological spaces. The weak topology τ_w on X generated by f_1 and f_2 is $\tau_w = \{\emptyset, \{a, b, c\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$ and it is easy to see that τ_w is not extensible within this system to a strictly stronger weak topology. That is, it is impossible to obtain a weak topology generated by $\{f_1, f_2\}$ on X which is strictly stronger than τ_w.

EXAMPLE 4.42

Let X, X_1, X_2, f_1 and f_2 all be as given in example 4.41 above and let X_1 retain its discrete topology while X_2 is given its indiscrete topology. Then this time the weak topology generated on X by f_1 and f_2 is
$\tau_w = \{\emptyset, \{a, b\}, \{b\}, \{b, c\}, \{a, b, c\}\}$. Clearly τ_w now is extensible (in the strong sense) to a strictly stronger weak topology within this system. From example 4.41, we see that a discrete weak topology τ_w on X may actually be strictly weaker than the normal discrete topology 2^X of X; similar to what we have seen before that an indiscrete weak topology may actually have cardinality greater than 2.

Since it has been known before that if all the factor spaces are discrete, then the product topology (in finite dimensions) would be discrete, the development here is an important revelation which we now state below as a proposition. So, when we say 'a discrete weak topology' we only know or mean that it has no strictly stronger weak topology in its own system; the matter of determining its size in comparison to the power set of X is something else.

Corollary 4.6 *If all the range spaces of any weak topology are discrete topological spaces, then the weak topology is a discrete weak topology in that it has no strictly stronger weak topology.*

Corollary 4.6 is important in one aspect: It generalizes and extends the existing discreteness result on product topologies (in finite dimensions) to arbitrary weak topological systems. Then Proposition 4.12 (below) explains that discreteness inherited by a weak topology from its range spaces may not be powerset-discreteness.

Proposition 4.12 *A discrete weak topology on X (i.e. one for which there exists no strictly stronger weak topology) can be strictly weaker than the normal discrete topology 2^X of X.*

REMARK 4.12

While it is our conjecture that a discrete product topology in infinite dimensions will not coincide with the power set of the product set,[6] it is also our conjecture that an important research question is now set to the effect that: *Since it is now clear that every discrete weak topology on a set X does not coincide with the power set of X, (1) can we characterize the discrete weak topologies that coincide with the power set, (2) can the discrete weak topology of a weak topological system coincide with the indiscrete weak topology of the system since we now also know that an indiscrete weak topology may have cardinality greater than 2, in other words, can we find a weak topology that is neither reducible nor extensible?* Our answer to the two questions is that any weak topology on a singleton set meets the requirement of the questions. So, the research will focus on sets with cardinality at least 2.

Theorem 4.6 *Let $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. The following statements are equivalent.*
(a) The weak topology τ_w is not extensible to a strictly stronger weak topology in any sense.
(b) All the range topologies of τ_w are discrete topologies and if any is itself a weak topology, then it is a discrete weak topology coinciding with the power set.
(c) τ_w is a discrete weak topology.

Proof:
From (c) if τ_w is a discrete weak topology, then it means from definition that (a) it is not extensible in any sense to a strictly stronger weak topology. And if τ_w is not extensible in any sense then (b) it follows that all the range topologies are power set discrete topologies. Finally, if all the range topologies are power set discrete topologies, then certainly τ_w is a discrete weak topology.

Theorem 4.7 *Let $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. (a) If all the range topologies are discrete topologies and none is a weak topology, or (b) all the range topologies are discrete topologies equal to the power set of their range spaces, then τ_w is a discrete weak topology*

The following corollary statement is certainly a very important derivation from the last two theorems.

[6] Later we shall see that what we would call the *supra* of the discrete product topology in infinite dimensions would coincide with the power set discrete topology of the infinite dimensional product set if all the factor spaces are power set discrete topological spaces.

Corollary 4.7 *If all the factor spaces of a product topology in infinite dimensions are power set discrete topologies, then the product topology is a discrete product (weak) topology in the sense of our existing definitions.*

Theorem 4.8 *Let* $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If* τ_w *is a non-trivial weak topology on* X*, then* τ_w *is in the middle (midst) of weak topologies on* X*, some strictly weaker and some strictly stronger than* τ_w*, in that* $\{\emptyset, X\} \leq \cdots < \tau_{w_1} < \tau_w < \tau_{w+1} < \cdots \leq 2^X$.

Proof:
This is the conclusive meaning of all the foregoing results.

4.1.12 Relationship With Existing Results

As we have pointed out, researchers have in the past constructed weak topologies on normed linear spaces by using polars of sets. Let E be a normed linear space, E^* its algebraic dual, and E' its topological dual. What was called *the* weak topology $\sigma(E, E^*)$, on E (or $\sigma(E^*, E)$, on E^*) is the topology on E (or on E^*) made up of polars of finite subsets of E^* (or of E). And what was called *the weakened topology* $\sigma(E, E')$, on E (or $\sigma(E', E)$, on E') is the topology on E (or on E') formed by taking polars of finite subsets of E' (or of E). (See Edwards (1995), pages 88 and 89.)

A particular issue was then called the Mackey problem and it is as follows: If we have a dual system (E, E'), how may we characterize those locally convex topologies on E compatible with the duality (E, E')—in the sense that every element of E' is not only continuous but can in addition be represented by an element of E? It was remarked (in Edwards (1995), page 504) that such topologies do exist and that the weakest of them is $\sigma(E, E')$. It was also "shown" that there is a strongest such topology, and that the others are lying between these two. The strongest such topology is denoted by $\tau(E, E')$ and is called the Mackey topology. The Mackey topology $\tau(E, E')$ on E is then constructed as the topology made up of polars of weakly compact and convex subsets of E'. And the conclusion is that a locally convex topology T on E is compatible with the duality (E, E') if and only if

$$\sigma(E, E') \leq T \leq \tau(E, E').$$

If E is a LCTVS (locally convex topological vector space) and E' its topological dual, the relation above will hold with T the initial topology of E. Usually the ordering is strict, it is observed, but the equality $T = \tau(E, E')$

holds for certain important types of LCTVS, which are then accordingly called *relatively strong*.

Now if E is a TVS (topological vector space), E' its topological dual, Arens introduced the topology on E' having polars A^o of compact, convex and balanced subsets A of E as a base of neighborhoods at 0. This is called the Arens topology on E' and is denoted by $k(E', E)$; and it is locally convex and weaker than the Mackey topology $\tau(E', E)$, as every compact subset of E is weakly compact for the weakened topology $\sigma(E, E')$. (If we interchange the roles played by E and E', we get the Arens topology $k(E, E')$ on E.) Since it is obviously stronger than $\sigma(E', E)$, $k(E', E)$ is compatible with the duality between E and E'; that is, $k(E', E)$ makes all maps of the form $x' \mapsto \langle x, x' \rangle$ continuous, for any fixed element x of E.

The so-called strong topology, $\eta(E, E')$ on E is made up of polars of the (norm-) bounded subsets of E'. Again, interchanging the roles of E and E' gets us the strong topology $\eta(E', E)$ on E'. (Edwards (1995), pages 507 and 508) It is noted that in general the topology $\eta(E', E)$ is *not* compatible with the duality between E and E'. That is, it is not generally the case that each linear form on E', continuous with rescpect to $\eta(E', E)$, is generated by an element of E.

QUESTIONS

1. What is the place of our comparison results on the four weak topologies (namely *the* weakened, the Arens, the Mackey and the strong topologies) already known to be comparable, viz: $\sigma(E, E') \leq k(E, E') \leq \tau(E, E') \leq \eta(E, E')$?

2. The conclusion of earlier researches is that $\sigma(E, E')$ is the weakest and that $\tau(E, E')$ is the strongest of all those locally convex topologies on E, compatible with the duality existing between E and E', which make elements of E' continuous.

    - Except for the Arens topology, existing research finding did not tell whether the intermediate locally convex topologies compatible with the duality (E, E') are weak topologies—that is, constructible by any known process of forming a weak topology, such as by using polars. In short, no systematic process of looking for such topologies (be they 'weak' or not) is given.
    - Between the Mackey (weak) topology $\tau(E, E')$ and the strong (but 'weak') topology $\eta(E, E')$, is there no intermediate weak topology (akin to, say the Arens topology)?

- If there exists an intermediate weak topology between the Mackey and the strong topology, what role can such a topology play or not play in analysis? And if there are no such intermediate weak topologies, why?

3. Can we find (or not find) a weak topology stronger than the so-called strong (weak) topology; and if we cannot find such a weak topology, why?

FURTHER DEVELOPMENTS

In order to know the full impact of our comparison theorems on the existing results, we need to clearly establish the connection between constructing weak topologies in the way we have done *and* constructing them by the use of polars. And to do this, we will now recast the meaning of a 'polar' (by way of definition) and then look deeper into it, to see its place in the collection of open sets of a weak topology.

Definition 4.34 *Let (A, B) be a dual system over a scalar field $K(= \mathbb{R}$ or $\mathbb{C})$. (We recall that the meaning of this is that, first, A and B are linear spaces over the same scalar field K; and secondly there exist linear maps $\phi_b : A \to K$, on A into K defined by $\phi_b(a) = \langle a, b \rangle$, for each element b of B and linear maps $\phi_a : B \to K$, on B into K defined by $\phi_a(b) = \langle a, b \rangle$, for each element a of A.) Let $G \subset B$ be any subset of B. Then the polar G^o of G is a subset of A given by $G^o = \{a \in A : |\langle a, b \rangle| \leq 1,\ b \in G\}$.*

REMARK 4.13

- Some use the strict inequality $<$ in the definition of polar above; and for obvious reasons we may have to resort to that use in the sequel.
- Any $\varepsilon > 0$ can be used in the definition above, in place of 1.

EXAMPLE 4.43

Let E be a linear space over K and let E^* be its algebraic dual. Then (E, E^*) is a dual system, for the maps $\phi_f : E \to K$ defined by $\phi_f(x) = \langle x, f \rangle = f(x)$ are linear, for all $f \in E^*$; and the maps $\phi_x : E^* \to K$ defined by $\phi_x(f) = \langle x, f \rangle = f(x)$ are also linear.

Let A be a finite subset of E^*. Then the polar A^o of A is

$$A^o = \{x \in E : |\langle x, f \rangle| \leq 1, f \in A\}.$$

This is typically the set we need to understand clearly in our present discussion of weak topologies. Taking a look again:
$A^o = \{x \in E : |\langle x, f\rangle| \leq 1, f \in A\} = \{x \in E : |f(x)| \leq 1, f \in A\}$
$= \{x \in E : -1 \leq f(x) \leq 1, f \in A\} = \{x \in E : f(x) \in [-1, 1], f \in A\}$
$= \{x \in E : x \in f^{-1}([-1, 1]), f \in A\} = \bigcap_{f \in A} f^{-1}([-1, 1]) = \bigcap_{i=1}^{n} f_i^{-1}([-1, 1]),$
as A is finite.

OBSERVATIONS:

1 If the strict inequality $<$ is used, then the polar A^o is the intersection of a finite number of inverse image of open sets of the scalar field K with its usual topology, under an equally finite number of linear maps.

2 If A is infinite, the polar A^o of A is the intersection of inverse image of an infinite number of sets under an infinite number of linear maps.

3 The collection $\{A^o : A \subset E^*\}$ of polars is always a collection of intersections of inverse images of (open or not open) sets—finite or infinite intersections according to whether the subsets of E^* considered are finite or infinite.

4 These intersections (the polars) are always a base for a weak topology and if subsets of E^* are used to generate the polars, the weak topology would be that generated by the elements of E^*.

5 If E^* is replaced by E', the topological dual of E, we would have a weak topology (on E) with respect to elements of E'.

6 For the four weak topologies $\sigma(E, E'), k(E, E'), \tau(E, E')$, and $\eta(E, E')$, all elements of E' are continuous. The difference is that while $k(E, E')$ and $\tau(E, E')$ may contain some infinite intersections, $\sigma(E, E')$ will not have such intersections. Also $\eta(E, E')$ will contain more exotic intersections than both $k(E, E')$ and $\tau(E, E')$ (but not necessarily all arbitrary intersections).

7 Hence the differences among these four weak topologies lie on the kind of intersections they contain, and this in turn lies on the kind of sets whose polars are used as base for the topologies. And it is indeed just *that polars are used as bases for the topologies*; the collection of polars are not directly (necessarily) the topologies in question. So, when we say that a weak topology is made up of polars of (some) subsets, what we really mean is that such a weak topology is built up (or constructed) from polars of such subsets as a base.

EXPOSITIONS

Let $P_0 = \{A^o : A \subset E', A \text{ is finite }\}$ be the collection of polars of finite subsets of E', a base for the weak topology $\sigma(E, E')$ on E; and let $G \subset E'$ be a relatively compact, infinite and convex subset of E', and let G^o be the polar of G. Let $B = P_0 \cup \{G^o\}$. Then it is easy to see that

- $\bigcup_{p \in B} P = E$; and
- If $P_1, P_2 \in B$ then $P_1 \cap P_2 \in B$, as a subset of a polar is a polar.

Hence B is a base for *some* topology τ_w, on E—a weak topology generated by elements of E'. We now observe that

1. The (polar) base for $\sigma(E, E')$ does not contain the polar of any infinite, relatively compact and convex subset of E'. Hence $\sigma(E, E')$ is strictly weaker than τ_w.
2. The polar base for τ_w does not contain the polars of all infinite, relatively compact and convex subsets of E'. Hence τ_w is strictly weaker than $\tau(E, E')$. That is, $\sigma(E, E') < \tau_w < \tau(E, E')$.
3. τ_w has a base of neighborhoods at zero; hence it is locally convex. In short (by 8.3.1 on page 505, of Edwards) a locally convex topology T on E is compatible with the duality between E and E' if and only if $\sigma(E, E') \leq T \leq \tau(E, E')$. Hence τ_w is a locally convex topology on E compatible with the duality (E, E').

What we have proved is that τ_w is a locally convex, (E, E')-compatible weak topology on E, generated by elements of E', which is strictly stronger than $\sigma(E, E')$ and strictly weaker than $\tau(E, E')$.

Since we can find[7] other relatively compact, infinite and convex subsets $G_1, G_2, G_3, \cdots$ of E', each different from one another (and different from G_0), we can by analogy get a sequence $\{\tau_{w_1}, \tau_{w_2}, \tau_{w_3}, \cdots\}$ of locally convex, (E, E')-compatible, pairwise strictly comparable weak topologies lying between $\sigma(E, E')$ and $\tau(E, E')$, in that

$$\sigma(E, E') < \tau_w < \tau_{w_1} < \tau_{w_2} < \tau_{w_3} < \cdots < \tau(E, E').$$

Let $P_0 = \{A^o : A \subset E', A \text{ is weakly compact and convex }\}$, base for the weak topology $\tau(E, E')$ on E; let $G \subset E'$ be a subset of E' which is not weakly compact but a bounded subset of E', and let G^o be the polar of G. Let $B = P_0 \cup \{G^o\}$. Then B is a base for *some* topology T on E—a weak topology generated by elements of E'. And we can again see that

1. $\tau(E, E')$ is strictly weaker than T;
2. T is strictly weaker than $\eta(E, E')$;

[7]Non-existence of these subsets would imply non-existence of the Mackey topology as a different topology from the weakened topology.

3. T is locally convex (since it is for instance strictly stronger than the Mackey topology);

4. If E' has other bounded subsets which are not weakly compact and convex, there exists a family $\{T_n\}$ of T-like topologies on E, lying between $\tau(E, E')$ and $\eta(E, E')$, such that $\tau(E, E') < T < T_1 < \cdots < \eta(E, E')$; and if E' has no bounded subsets which are NOT weakly compact and convex (i.e. all bounded subsets of E' are weakly compact and convex) then necessarily $\tau(E, E')$ would coincide with $\eta(E, E')$;

5. These weak topologies lying between $\tau(E, E')$ and $\eta(E, E')$ are, like $\eta(E, E')$, in general not guaranteed to be compatible with the duality (E, E') (because the Mackey topology is the strongest locally convex weak topology on E which is compatible with the duality (E, E')).

Now let G be an unbounded subset of E' which is also not weakly compact, and let G^o be its polar. Let P_o be the polar base of $\eta(E, E')$ and let $B = P_o \cup \{G^o\}$. Then again B is easily seen to be a base for some weak topology T on E, generated by the elements of E'. And we notice that $\eta(E, E')$—the strong topology—is strictly weaker than this weak topology T.

We have shown that the bases (coming as collections of polars) for these weak topologies are actually nothing but intersections (finite or infinite) of pre-images (under the linear maps) of some subsets of the scalar field underlying the dual system. Finally, the next question to consider is whether the four weak topologies ($\sigma(E, E')$, $k(E, E')$, $\tau(E, E')$ and $\eta(E, E')$) which have traditionally been compared are actually induced or generated on E by some correspondingly compared topologies in a range space or two. This will show that Proposition 4.10 (1) and/or Lemma 4.5 apply also to these four weak topologies.

Proposition 4.13 *Let B_1 be a base for a topology τ_1 on X and let B_2 be a base for another topology τ_2 on X. If B_1 is a proper subfamily of B_2 (or conversely that τ_1 is strictly weaker than τ_2), then the subbase S_1 for τ_1 is a proper subfamily of the subbase S_2 for τ_2. Hence for a weak topological system $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$, if B_1 is a polar base for another weak topology τ_{w_1} on X generated by the same family of functions, and that B_1 is a proper subfamily of a polar base B_2 of τ_w (or conversely that $\tau_{w_1} < \tau_w$), then there exist two topologies τ_1 and τ_2 on a range space X_{α_0}, for some $\alpha_0 \in \Delta$, such that τ_1 is strictly weaker than τ_2 and τ_{w_1} is the weak topology on X when X_{α_0} has the topology τ_1 and τ_w is the weak topology on X when X_{α_0} has the topology τ_2.*

Proof:
From the hypothesis and since $B_1 = \{\text{finite intersections of sets in } S_1\}$, and since B_2 is analogously defined for S_2, B_1 is a proper subfamily of B_2 if and only if S_1 is a proper subfamily of S_2.

Let $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}, \{f_\alpha\}_{\alpha \in \Delta}]$ be a weak topological system and let τ_{w_1} be another weak topology, on X, generated by the fixed family of functions. Let B_1 and B_2 be the respective polar bases for τ_{w_1} and τ_w. Then $\tau_{w_1} < \tau_w$ if and only if B_1 is a proper subfamily of B_2; and B_1 is a proper subfamily of B_2 if and only if the subbase S_1 for τ_{w_1} (in relation to B_1) is a proper subfamily of S_2, the subbase for τ_w (in relation to B_2). Clearly S_1 is of the form

$$\{f_\alpha^{-1}(G_\alpha) : G_\alpha \in \tau_\alpha, \alpha \in \Delta\}$$

and S_2 is also of the (same) form

$$\{f_\alpha^{-1}(G_\alpha) : G_\alpha \in \tau_\alpha, \alpha \in \Delta\}.$$

Since $S_1 \subset S_2$ and $S_1 \neq S_2$, there must be a range space $(X_{\alpha_0}, \tau_{\alpha_0})$, $\alpha_0 \in \Delta$, in the weak topological system such that X_{α_0} has another topology τ_0, strictly weaker than τ_{α_0}, and such that

$$S_1 = \{f_\alpha^{-1}(G_\alpha) : G_\alpha \in \tau_\alpha, \alpha \in \Delta, \alpha \neq \alpha_0\} \bigcup \{f_{\alpha_0}^{-1}(G_{\alpha_0}) : G_{\alpha_0} \in \tau_0\}$$

and

$$S_2 = \{f_\alpha^{-1}(G_\alpha) : G_\alpha \in \tau_\alpha, \alpha \in \Delta\}.$$

Let $\tau_0 = \tau_1$ (of the proposition) and $\tau_{\alpha_0} = \tau_2$, and the proof is complete.

4.1.13 Cursory Look at an Existing Result

Let us again point out some of the benefits of taking a constructive approach to the study of weak topology: (1) The constructive approach enables us to create or obtain weak topologies of virtually all kinds of topological properties; (2) It can help us to check the correctness or otherwise of our intended general result; for example there is a theorem in literature which simply states that *The weak topology is Hausdorff.* (See proposition 6.9, page 124 of Chidume (1996) for this.) The questions relating to this proposition are: (1) Are all weak topologies Hausdorff? and (2) Is it to be accepted that only one weak topology—that which is Hausdorf—is in existence, even though both present and older works have shown the existence of several weak topologies? Our answer to these two questions is that one topological property of a weak topology may not be shared by other weak topologies, and, in particular, all weak topologies are NOT Hausdorff. We take a few illustrative examples.

EXAMPLE 4.44

Let $X = \{a, b, c\}, Y = \{1, 2, 3\}$ and $Z = \{p, q, r, s, t\}$ be three sets. Let $f : X \to Y$ be a function defined by $f(a) = 2$ and $f(b) = 1$; and let $g : X \to Z$ be a function defined by $g(b) = p$ and $g(c) = p$. Let Y be endowed with its indiscrete topology and let Z be given any topology. Then the weak topology τ_w on X generated by these two functions f, g is $\tau_w = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$. And it is easy to see that this weak topology on X is not Hausdorff.

EXAMPLE 4.45

Let X, Y, Z, f, g all be as given in example 4.44. Let Z be endowed with any topology but Y now with its discrete topology. Then the weak topology τ_w on X generated by these two functions f, g now is
$\tau_w = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. And it is easy to see that this weak topology on X is not Hausdorff, as $b \neq c$ and there are no disjoint τ_w-open sets containing b and c.

EXAMPLE 4.46

Let X, Y, Z, f all be as given in example 4.45. Let Z be endowed with the topology $\{\emptyset, Z, \{p\}, \{t\}, \{p, t\}\}$ and Y with its discrete topology. And let $g : X \to Z$ be defined by $g(b) = p$ and $g(c) = t$. Then the weak topology τ_w on X generated by these two functions f, g now is $\tau_w =$
$\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. And we now see that this weak topology on X is Hausdorff.

EXAMPLE 4.47

The Sierpinski weak topology constructed earlier in subsection 4.1.6 is not Hausdorff.

We can therefore make the following proposition without need of other proof.

Proposition 4.14 *Not every weak topology is Hausdorff.*

4.1.14 On Seminorm Topologies

Topologies have been generated on linear spaces by norms and families of *seminorms* without recourse to the process of Theorem 1.1 which is a typical process of constructing weak topologies. We are here set to investigate the link (if any) between topologies generated by families of seminorms in such traditionally special way (hereafter simply referred to as 'seminorm topologies' and denoted by T) and weak topologies generated by families of seminorms (henceforth referred simply to as 'seminorm weak topologies' and as usual here, denoted by τ_w).

SOME EXISTING CONCEPTS AND RESULTS

All the concepts, ideas and results of this subsection are in existing literature. (See e.g. Angus Taylor and David Lay, pages 96 to 107.)

Theorem 4.9 *A topological linear space X has a base Ψ at 0 with the following properties:*
(a) Each member of Ψ is balanced and absorbing.
(b) If $U \in \Psi$, there exists $V \in \Psi$ such that $V + V \subset U$.
(c) If U_1 and U_2 are in Ψ, there exists $U_3 \in \Psi$ such that $U_3 \subset U_1 \cap U_2$.
Conversely, any nonempty collection Ψ of nonempty subsets of a linear space X satisfying (a) to (c) is a base at 0 for a unique linear topology on X.

Definition 4.35 *A topological linear space X is said to be locally convex if every neighborhood of 0 has a convex neighborhood of 0.*

REMARK 4.14

If X is a locally convex space and Ψ is the family of all absolutely convex neghborhoods of 0, then Ψ is a base at 0. The description or construction of base at 0 for locally convex spaces is often given in terms of seminorms.

Definition 4.36 *Let X be a (real or complex) linear space. A seminorm on X is a real-valued function p defined on X such that*
(1) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$.
(2) $p(\lambda x) = |\lambda| p(x)$, for all $x \in X$ and scalar λ.

REMARK 4.15

If p has the further property that $p(x) \neq 0$ if $x \neq 0$, then p is a norm.

Lemma 4.7 *Let p be a seminorm on a linear space X. Then the sets*

$$V_1 = \{x \in X : p(x) < 1\} \text{ and } V_2 = \{x \in X : p(x) \leq 1\}$$

are absorbing and absolutely convex.

Theorem 4.10 *Let P be a nonempty family of seminorms on a linear space X. For each $p \in P$ let $V(p)$ be the set $\{x \in X : p(x) < 1\}$. Let Ψ be the family of all finite intersections*

$$r_1V(p_1) \cap r_2V(p_2) \cap \cdots \cap r_nV(p_n),\ r_k > 0, p_k \in P.$$

Then Ψ is a base at 0 for a topology T that makes X a locally convex space. Furthermore, this topology T is the smallest linear topology for X with respect to which all the seminorms in P are continuous.

4.1.15 Seminorm Weak Topologies Versus Seminorm Topologies

REMARK 4.16

1. The seminorm topology T described in the last theorem is actually constructed without special attention to what may happen if the topology of the set $\mathbb{R}$ of real numbers, as the range space of all the seminorms in P, is changed or varied. Hence conceptually, seminorm topologies are different from seminorm weak topologies. Our expositions here show that seminorm topologies T exist strictly in the midst of weak topologies generated by the given family of seminorms.

2. We prove that every seminorm topology T (which we know is locally convex) is at the top of a chain of pairwise strictly comparable non-locally-convex weak topologies; and yet the same family of seminorms would generate a weak topology strictly stronger than the seminorm topology. And an associated question here is: Can the seminorm topology itself be shown to be also a weak topology generated by the given family of seminorms? Interestingly our exposition here results in an affirmative answer to this question. We showed that the seminorm topology T is actually a weak topology τ_w generated by the (same) family of seminorms.

3. The cardinality $n(P)$ of P can range from 1 to infinity, it is to be observed. If $n(P) = 1$, then the seminorm topology T would be one generated by a single seminorm, just as one norm on X can induce a topology on X.

4. If P_1 and P_2 are two different families of seminorms on a linear space X, the locally convex topologies T_1 and T_2 generated on X by P_1 and P_2 may not be comparable, and may be comparable but not coincident.

EXAMPLE 4.48
Let $X = \mathbb{R}^2$ and let $p_1 : X \longrightarrow \mathbb{R}$ be a seminorm defined on X by $p_1\{(x, y)\} = |x|$. The family $P = \{p_1\}$ of "seminorms" is now a singleton. Let $V(p_1) = \{(x, y) \in X : p_1\{(x, y)\} < 1\}$. Then the family P of this single seminorm nevertheless generates a seminorm topology T_1 on X. The base for T_1 is the collection

$$\Psi = \{\bigcap_{k=1}^{n} r_k V(p_1) : r_k > 0, n \in \mathbb{N}\}$$

of absorbing and absolutely convex subsets of X. The set $V(p_1)$ is the vertically infinite open strip whose center is the vertical line $x = 0$ or the y axis of the Cartesian plane—the "p_1-seminorm ball" centered on the vertical line $(0, y)$ with radius $r = 1$. (See figure 10.) Its width stretches along the interval from -1 to 1 on the horizontal axis. Hence the sets $rV(p_1)$ $(r > 0)$ are as well vertically infinite open strips with widths running from $-r$ to r along the horizontal. (We should observe that $V(p_1) = P_1^{-1}\{(-1, 1)\}$. [8]) This single-seminorm topology is easily seen to be locally convex but strictly weaker than the usual topology u of $X(= \mathbb{R}^2)$. To see that T_1 is strictly weaker than the usual topology u of X, observe that every T_1-open set is u-open but, for instance, no u-open rectangle is T_1-open.

EXAMPLE 4.49

Let $X = \mathbb{R}^2$ and let $p_2 : X \longrightarrow \mathbb{R}$ be a seminorm defined on X by $p_2\{(x, y)\} = |y|$. The family $P = \{p_2\}$ of "seminorms" is a singleton. Let $V(p_2) = \{(x, y) \in X : p_2\{(x, y)\} < 1\}$. (And again we see that $V(p_2) = P_2^{-1}\{(-1, 1)\}$.) Then the family P of this single seminorm generates a seminorm topology T_2 on X. The base for T_2 is the collection

$$\Psi = \{\bigcap_{k=1}^{n} r_k V(p_2) : r_k > 0, n \in \mathbb{N}\}$$

of absorbing and absolutely convex subsets of X. The set $V(p_2)$ is the horizontally infinite open strip whose center is the horizontal line $y = 0$ or the x axis of the Cartesian plane—the "p_2-seminorm ball" centered on the line $(x, 0)$. (See figure 11.) Its width stretches along the interval from -1 to 1 on the vertical axis. Hence the sets $rV(p_2)$ $(r > 0)$ are horizontally infinite open strips with widths running from $-r$ to r along the vertical. This single-seminorm topology is also locally convex and strictly weaker than the usual topology u of $X(= \mathbb{R}^2)$. And to see this, we only need to observe that every T_2-open set is u-open but, for instance, no u-open rectangle is T_2-open. We also see that T_1 and T_2 are not comparable.

EXAMPLE 4.50

If we now have a family of seminorms $P = \{p_1, p_2\}$ on $X = \mathbb{R}^2$, where elements of P are as defined in the two examples above, we shall get a third seminorm topology T_3 on X. We shall continue to look at these and other seminorm topologies in the sequel; but for now we need some background results.

Lemma 4.8 (Topology of Balanced Sets) *Let X be a linear space and let $\Psi = \{B \subset X : B$ is balanced, convex and absorbent$\}$. If $\tau = \Psi \cup \{\emptyset\}$ and Ψ is pairwise comparable, then τ is a topology on X.*

[8] This observation will help us soon to see this single-seminorm topology T_1 as a weak topology τ_{w_1} generated by this family of only one seminorm.

Proof:
We first observe that $X \in \tau$ since X (as the linear space) is balanced, convex and absorbent; so we only need to prove that τ is closed under finite intersections and arbitrary unions.

1. Let $B_i, i = 1, \cdots, n$ be a finite number of sets in τ, and let $N = \bigcap_{i=1}^{n} B_i$ be their intersection. We show that N is balanced, convex and absorbent. Let $x \in N, -1 \leq t \leq 1$. Then $x \in B_i, i = 1, \cdots, n$; so $tx \in B_i, i = 1, \cdots, n$ since each B_i is balanced. This implies that $tx \in N$, and hence that N is balanced.
Let $x, y \in N, 0 \leq t \leq 1$. Then $x, y \in B_i, i = 1, \cdots, n$ and $tx+(1-t)y \in B_i, i = 1, \cdots, n$ as each B_i is convex. Hence $tx+(1-t)y \in N$, implying that N is convex.
Let $x \in X$ be arbitrary. Then since each B_i is absorbent, there exists for each i, $t_i \in \mathbb{R}$ such that $t_i x \in B_i, i = 1, \cdots, n$. Let $t = min\{t_i, i = 1, \cdots, n\}$. Then $tx \in B_i, i = 1, \cdots, n$; and so $tx \in N$. (Or alternatively since elements of Ψ or indeed of τ are pairwise comparable, $N = B_{i_0}$, for some $i_0 \in \{1, \cdots, n\}$. Since B_{i_0} is absorbing, N is absorbing.) That is, N is balanced, convex and absorbing; so $N \in \Psi$ and hence is in τ.

2. Let $\{B_\alpha\} \subset \tau$ be an arbitrary collection of sets in τ. We need to show that $U = \bigcup_\alpha B_\alpha \in \tau$ by showing that it is balanced, convex and absorbing.
Let $x \in U, -1 \leq t \leq 1$. Then $x \in B_{\alpha_0}$, for some α_0. Since B_{α_0} is balanced, $tx \in B_{\alpha_0}$. Hence $tx \in U$, implying that U is balanced.
Let $x, y \in U, 0 \leq t \leq 1$. Then x and y are in one set B_{α_0} since Ψ is pairwise comparable. As B_{α_0} is convex, $tx + (1-t)y \in B_{\alpha_0}$; and since $B_{\alpha_0} \subset U$ it follows that $tx + (1-t)y \in U$ and hence that U is convex. Clearly it is easy to see that U is absorbent since the sets that make up U are absorbent. So $U \in \Psi$ and hence $U \in \tau$. The proof is complete.

NOTE
The family Ψ in Lemma 4.8 can be constructed by restricting the elements (of Ψ) to subsets of a fixed subset of X. That is, if $E \subset X$, we can define Ψ to be $\Psi = \{B \subset E : B$ is balanced, convex and absorbent$\}$, and then $\tau = \Psi \cup \{\emptyset, X\}$ would be a topology (on X) of the balanced subsets of E.
For example, for any fixed real number $r > 0$, the set $X_r = (-r, r)$ is a balanced, convex and absorbing subset of $\mathbb{R}$. If we let $\Psi = \{B \subset X_r : B$ is balanced, convex and absorbent$\}$, then $\tau = \Psi \cup \{\emptyset, \mathbb{R}\}$ would be a topology (on $\mathbb{R}$) of the balanced subsets of X_r. We shall in the sequel make use of this type of topology of balanced subsets *of subsets* of $\mathbb{R}$ to construct useful topologies.

Proposition 4.15 *If P_1 and P_2 are two different families of seminorms on a linear space X, the respective locally convex topologies T_1 and T_2 generated on X by P_1 and P_2 may not be comparable, and may be comparable but not coincident.*[9]

Now we have to explore the relationship of the three seminorm topologies above with *weak topologies* that may be generated on X by each of the families of seminorms.

EXAMPLE 4.51

Let $X = \mathbb{R}^2$ and let $p_1 : X \longrightarrow \mathbb{R}$ be a seminorm defined on X by $p_1\{(x, y)\} = |x|$. The family $P = \{p_1\}$ of "seminorms" is only a singleton. To construct a weak topology on X using this singleton as "a family of functions" we have to consider the topology endowed $\mathbb{R}$ as the range space since each weak topology is built in consideration of the range topologies of the family of functions.

Let $(\mathbb{R}, u)$ denote the usual topological space of $\mathbb{R}$. Then it is easy to see that the weak topology τ_w on X generated by this singleton of seminorm is strictly stronger than the p_1-seminorm topology T_1 of example 4.48 above. That is, $T_1 < \tau_w$; and to see this we observe that $G \in T_1 \Longrightarrow G = rV(P_1) = p_1^{-1}\{(-r, r)\} \in \tau_w$, where $(-r, r)$ is a u-open subinterval of $\mathbb{R}$, but for instance $p_1^{-1}\{(2, 4)\} \in \tau_w$ but $p_1^{-1}\{(2, 4)\} \notin T_1$ since $p_1^{-1}\{(2, 4)\} = \{\bar{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1 \in (-4, -2) \cup (2, 4)\}$.

Let us now change the topology of $\mathbb{R}$ as the range space of p_1. Let $X_1 = (-1, 1)$ be a u-open interval and let $\tau_1 = X_1$-topology on $\mathbb{R}$ given as $\tau_1 = \{(-r, r) \in u : 0 < r \leq 1\} \cup \{\emptyset, \mathbb{R}\}$, generated (in line with Lemma 4.8) by all the balanced u-open subsets of X_1. Then the weak topology now τ_{w_1} on $\mathbb{R}^2$ generated by this seminorm is strictly weaker than the seminorm topology T_1 on $\mathbb{R}^2$ since every τ_{w_1}-open set (which actually is of the form $rV(p_1)$) is T_1-open but $rV(p_1)$ is not τ_{w_1}-open if $r > 1$.

If (similarly) we let $X_2 = (-2, 2)$ be a u-open interval and let $\tau_2 = X_2$-topology on $\mathbb{R}$ given as $\tau_2 = \{(-r, r) \in u : 0 < r \leq 2\} \cup \{\emptyset, \mathbb{R}\}$, generated (according to Lemma 4.8) by the balanced u-open subsets of X_2. Then the weak topology now τ_{w_2} on $\mathbb{R}^2$ generated by this seminorm will again be strictly weaker than the seminorm topology T_1 on $\mathbb{R}^2$ since every τ_{w_2}-open set (which also is of the form $rV(p_1)$) is T_1-open but $rV(p_1)$ is not τ_{w_2}-open if $r > 2$. And we observe that $\tau_{w_1} < \tau_{w_2}$.

If we continue this process of using the lemma on topology of balanced sets to construct weak topologies on $X = \mathbb{R}^2$ we shall get a chain

$$C = \{\tau_{w_n}\}, n \in \mathbb{N}$$

[9] And a research question exists here: Can two different families P_1 and P_2 of seminorms on a linear space X generate the same locally convex topology T on X?

of pairwise strictly comparable (**vertical-wise**, but **horizontally expanding**) weak topologies (on $\mathbb{R}^2$) at the peak of which sits the seminorm topology T_1 which itself in turn is strictly weaker than the weak topology τ_w obtained when the usual topology of $\mathbb{R}$ is assumed on the range space of p_1. That is

$$\tau_{w_n} < \tau_{w_{n+1}} < T_1 < \tau_w \text{ for all } n \in \mathbb{N}.$$

We then observe that τ_{w_1} is not locally convex since (for instance) there is no convex neighborhood of the coordinate point (2,0) in the topology τ_{w_1}, except of course $\mathbb{R}^2$ itself. By similar analysis we also see that τ_{w_2} is not locally convex.

If we change the process of constructing the seminorm weak topologies, the resulting pairwise comparable chain of weak topologies may not be comparable to the seminorm topology T_1. For example, let $X_0 = (0, \infty)$ be a u-open subset of $\mathbb{R}$. And let $\tau_0 = \{G \in u : G \subset X_0\} \cup \{\mathbb{R}\}$ be an X_0-topology on $\mathbb{R}$. Then for the τ_0-open set $G = (0, 1)$, we have

$$p_1^{-1}\{(0,1)\} = \{(x, y) \in \mathbb{R}^2 : x \in (-1, 0) \cup (0, 1)\}.$$

This is the vertically infinite open strip centered on the y axis, with radius 1, but with the y axis itself deleted. This is shown in figure 12. In general, for any $G = (a, b)$ in τ_0,

$$p_1^{-1}(G) = \{(x, y) \in \mathbb{R}^2 : x \in (-b, -a) \cup (a, b)\}.$$

Then we see that (now) the weak topology τ_{w_1} on X is not comparable to the 'p_1 seminorm topology', of example 4.48, since no nonempty τ_{w_1}-open proper subset (of $X = \mathbb{R}^2$) is T_1-open and no nonempty T_1-open proper subset of X is τ_{w_1}-open.
To see this, we observe that all the T_1-open sets are balanced, convex and absorbing while no nontrivial τ_{w_1}-open set is balanced, convex or absorbent. For example, the set

$$G = p_1^{-1}\{(0,1)\} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in (-1, 0) \cup (0, 1)\}$$

is τ_{w_1}-open and it is not balanced, since for the coordinate point $\bar{x} = (-\frac{1}{2}, -\frac{1}{5})$ in G and the scalar $\lambda = 0 \in [-1, 1]$, $\lambda\bar{x} = (0, 0) \notin G$. Also G is not convex since for the coordinate points $\bar{x} = (-\frac{1}{2}, -\frac{1}{5})$ and $\bar{y} = (\frac{1}{2}, \frac{3}{4})$ in G and the scalar $\lambda = \frac{1}{2}$ we have $\lambda\bar{x} + (1 - \lambda)\bar{y} = (0, \frac{11}{40}) \notin G$. And G is not absorbent since, for instance, the coordinate point $(0, \frac{11}{40})$ cannot be absorbed into G; that is, no scalar multiple of this point is an element of G. This is because $\lambda.(0, \frac{11}{40}) = (0, \frac{11\lambda}{40}), \forall \lambda \in \mathbb{R}$. Let $X_1 = (1, \infty)$ be another u-open subset of $\mathbb{R}$ and let $\tau_1 = \{G \in u : G \subset X_1\} \cup \{\mathbb{R}\}$ be an X_1-topology on $\mathbb{R}$. Then for the τ_1-open set $G = (1, 2)$, we have

$$p_1^{-1}\{(1,2)\} = \{(x,y) \in \mathbb{R}^2 : x \in (-2,-1) \cup (1,2)\}.$$

This is a pair of vertically infinite open strips, one centered on the vertical line $(\frac{3}{2}, y)$ with radius $\frac{1}{2}$, and the other is centered on the vertical line $(-\frac{3}{2}, y)$ with radius of $\frac{1}{2}$. This is shown in figure 13. Generally, for all $G = (a,b)$ in τ_1,

$$p_1^{-1}(G) = \{(x,y) \in \mathbb{R}^2 : x \in (-b,-a) \cup (a,b)\},$$

a pair of vertically infinite open strips, one centered on the vertical line $(\frac{a+b}{2}, y)$ with radius $\frac{b-a}{2}$, and the other is centered on the vertical line $(-\frac{a+b}{2}, y)$ with radius of $\frac{b-a}{2}$. Then the weak topology τ_{w_2} now on X generated by $P = \{p_1\}$ is strictly weaker than the earlier weak topology τ_{w_1}. That is

$$\tau_{w_2} < \tau_{w_1}.$$

We can continue this process and the result is that there exists a chain

$$C = \{\tau_{w_n}, n \in \mathbb{N}\}$$

of pairwise strictly comparable non locally convex weak topologies generated by $P = \{p_1\}$ which are not comparable to the seminorm topology T_1 generated by $P = \{p_1\}$. That is

$$\cdots < \tau_{w_3} < \tau_{w_2} < \tau_{w_1}.$$

EXAMPLE 4.52
Let $X = \mathbb{R}^2$ and now let $p_2 : X \longrightarrow \mathbb{R}$ be a seminorm defined on X by $p_2\{(x,y)\} = |y|$. The family $P = \{p_2\}$ of "seminorms" is also a singleton. Let $(\mathbb{R}, u)$ denote the usual topological space of $\mathbb{R}$. The weak topology τ_w on X generated by this singleton of seminorm is strictly stronger than the p_2-seminorm topology T_2 of example 4.49 above. That is, $\tau_w > T_2$, where T_2 is as given in example 4.49 above. And to see this, it is enough to observe that $V(p_2) = p_2^{-1}\{(-1,1)\}$; so every T_2-open set is τ_w-open but not conversely as, for instance $p_2^{-1}\{(2,4)\}$ is not T_2-open but τ_w-open.
Let us now change the topology of $\mathbb{R}$ as the range space of p_2.
Let $X_1 = (-1,1)$ be a u-open interval and let $\tau_1 = X_1$-topology on $\mathbb{R}$ given as $\tau_1 = \{(-r,r) \in u : 0 < r \leq 1\} \cup \{\emptyset, \mathbb{R}\}$, generated (in line with Lemma 4.8) by all the balanced u-open subsets of X_1. Then the weak topology now τ_{w_1} on $\mathbb{R}^2$ generated by this seminorm is strictly weaker than the seminorm topology T_2, of $P = \{p_2\}$ in example 4.49 above, on $\mathbb{R}^2$ since every τ_{w_1}-open set (which again is of the form $rV(p_2)$) is T_2-open but $rV(p_2)$ is not τ_{w_1}-open if $r > 1$.

If (similarly) we let $X_2 = (-2, 2)$ be a u-open interval and let $\tau_2 = X_2$-topology on $\mathbb{R}$ given as $\tau_2 = \{(-r, r) \in u : 0 < r \leq 2\} \cup \{\emptyset, \mathbb{R}\}$, generated (according to Lemma 4.8) by the balanced u-open subsets of X_2, then the weak topology now τ_{w_2} on $\mathbb{R}^2$ generated by this seminorm will again be strictly weaker than the seminorm topology T_2 on $\mathbb{R}^2$ since every τ_{w_2}-open set (which also is of the form $rV(p_2)$) is T_2-open but $rV(p_2)$ is not τ_{w_2}-open if $r > 2$. We then observe that $\tau_{w_1} < \tau_{w_2}$.
If we continue this process of using the lemma on topology of balanced sets to construct weak topologies on $X = \mathbb{R}^2$ we shall get a chain

$$C = \{\tau_{w_n}\}, n \in \mathbb{N}$$

of pairwise strictly comparable (**horizontal-wise**, **vertically expanding**) weak topologies (on $\mathbb{R}^2$) at the peak of which sits the seminorm topology T_2 which itself in turn is strictly weaker than the weak topology τ_w obtained when the usual topology of $\mathbb{R}$ is assumed on the range space of p_2. That is

$$\tau_{w_n} < \tau_{w_{n+1}} < T_2 < \tau_w \text{ for all } n \in \mathbb{N}.$$

We then observe that τ_{w_1} is not locally convex since (for instance) there is no convex neighborhood of the coordinate point (4,0) in the topology τ_{w_1}. And similarly we also see that τ_{w_2} is not locally convex.

Now let $X_0 = (0, \infty)$ be a u-open subset of $\mathbb{R}$ and let $\tau_0 = \{G \in u : G \subset X_0\} \cup \{\mathbb{R}\}$ be an X_0-topology on $\mathbb{R}$. Then for the τ_0-open set $G = (0, 1)$, we have

$$p_2^{-1}\{(0, 1)\} = \{(x, y) \in \mathbb{R}^2 : y \in (-1, 0) \cup (0, 1)\}.$$

This is the horizontally infinite open strip centered on the x axis, with radius 1, but with the x axis itself deleted. This is shown in figure 14. And in general, for any $G = (a, b)$ in τ_0,

$$p_2^{-1}(G) = \{(x, y) \in \mathbb{R}^2 : y \in (-b, -a) \cup (a, b)\}.$$

The weak topology τ_{w_1} on X is not weaker than the 'p_2 seminorm topology'.

Let $X_1 = (1, \infty)$ be another u-open subset of $\mathbb{R}$. Let $\tau_1 = \{G \in u : G \subset X_1\} \cup \{\mathbb{R}\}$ be an X_1-topology on $\mathbb{R}$. Then for the τ_1-open set $G = (1, 2)$, we have

$$p_2^{-1}\{(1, 2)\} = \{(x, y) \in \mathbb{R}^2 : y \in (-2, -1) \cup (1, 2)\}.$$

This is a pair of horizontally infinite open strips, one centered on the horizontal line $(x, \frac{3}{2})$ with radius $\frac{1}{2}$, and the other is centered on the horizontal line $(x, -\frac{3}{2})$ with radius of $\frac{1}{2}$. This is shown in figure 15. And generally, for any $G = (a, b)$ in τ_1,

$$p_2^{-1}(G) = \{(x,y) \in \mathbb{R}^2 : y \in (-b,-a) \cup (a,b)\},$$

a pair of horizontally infinite open strips, one centered on the horizontal line $(x, \frac{a+b}{2})$ with radius $\frac{b-a}{2}$, and the other is centered on the horizontal line $(x, -\frac{a+b}{2})$ with radius of $\frac{b-a}{2}$. Then the weak topology τ_{w_2} now on X generated by $P = \{p_2\}$ is strictly weaker than the earlier weak topology τ_{w_1}. That is

$$\tau_{w_2} < \tau_{w_1}.$$

We can continue this process and the result is that there exists a chain

$$C = \{\tau_{w_n}, n \in \mathbb{N}\}$$

of pairwise strictly comparable non locally convex weak topologies generated by $P = \{p_2\}$ not comparable to T_2. That is,

$$\cdots < \tau_{w_3} < \tau_{w_2} < \tau_{w_1}.$$

EXAMPLE 4.53
Now let $X = \mathbb{R}^2$ and let $p_1, p_2 : X \longrightarrow \mathbb{R}$ be the two seminorms defined on X respectively by $p_1\{(x,y)\} = |x|$ and $p_2\{(x,y)\} = |y|$. Then again $P = \{p_1, p_2\}$ is a family of seminorms on X. Let $(\mathbb{R}, u)$ denote the usual topological space of $\mathbb{R}$. The weak topology τ_w on X generated by this family P of seminorms is strictly stronger than the 2-seminorm topology T_3 of example 4.50 above for the reasons already explained using topology of balanced sets in example 4.51 here. The only rectangles open in the seminorm topology T_3 are those of the form $(-r,r) \times (-r,r)$—that is, those centered at the origin. Any rectangle not centered at the origin is not open in T_3. This is one of the reasons why $T_3 < \tau_w$. By contrast if $b > a > 0$, then $p_1^{-1}\{(a,b)\} \cap p_2^{-1}\{(a,b)\} \in \tau_w$ but $p_1^{-1}\{(a,b)\} \cap p_2^{-1}\{(a,b)\} \notin T_3$.
Also $\tau_w < u =$ the usual topology of $\mathbb{R}^2$ because single rectangles not centered at the origin are not τ_w-open. (Only even mumbers of such rectangles are τ_w-open.) So, we have

$$T_3 < \tau_w < u.$$

And as we have seen, T_3 may be seen as a topology of only "origin-centered concentric rectangles", τ_w as a "topology of (in addition to T_3-open sets) pairs of non-concentric rectangles", and the usual topology u as a "topology of (in addition to τ_w-open sets) arbitrarily single rectangles".

Now let the two factor spaces of $\mathbb{R}^2$ be given the topology of balanced subsets of $X_1 = (-1,1)$ (after the rules outlined in Lemma 4.8 on topology of balanced sets, and as already constructed in example 4.51 above). The weak topology τ_{w_1} now on $X = \mathbb{R}^2$ generated by the family $P = \{p_1, p_2\}$ of these two seminorms will (as can easily be seen) be strictly weaker than the seminorm topology T_3: To see this, we observe that the only origin-centered rectangles open in τ_{w_1} are those of the form $(-r, r) \times (-r, r), 0 < r < 1$; that is, origin-centered sub-rectangles of $(-1, 1) \times (-1, 1)$. The complete picture of the landscape of τ_{w_1} is ***cross-like*** after the axes of the plane, with the origin as the center of the cross. So

$$\tau_{w_1} < T_3 < \tau_w < u. \dots\dots\dots\dots\dots (1)$$

If we repeat the construction process here, with $X_2 = (-2, 2)$ replacing $X_1 = (-1, 1)$ we get another cross-like (or **cross-wise**) weak topology τ_{w_2} on $\mathbb{R}^2$ generated by this family P of two seminorms. And it is easy to see that τ_{w_1} is strictly weaker than τ_{w_2} and that both weak topologies are not locally convex. Hence $\tau_{w_1} < \tau_{w_2} < T_3 < \tau_w < u$.
Indefinite continuation of the construction process here gives a sequence $\{\tau_{w_n}\}$ of non-locally-convex, pairwise strictly comparable cross-wise weak topologies on $\mathbb{R}^2$ generated by the family P of these two seminorms; so that

$$\tau_{w_1} < \tau_{w_2} < \cdots < T_3 < \tau_w < u. \dots\dots\dots\dots\dots (2)$$

As n tends to ∞, $X_n = (-n, n)$ tends to $X_\infty = (-\infty, \infty) = \mathbb{R}$ and the process of constructing these crosswise weak topologies leads to τ_{w_∞} which would then coincide with T_3. So we finally have

$$\tau_{w_1} < \tau_{w_2} < \cdots < \tau_{w_\infty} = T_3 < \tau_w < u, \dots\dots\dots\dots (3)$$

so that the seminorm topology T_3 is a weak topology which is the limit of a chain of non-locally-convex weak topologies generated by the same family of seminorms.
By analoguous processes, it is relatively easy to show that for a family $P = \{p_1, p_2, \cdots, p_n\}$ of n similar seminorms on $X(= \mathbb{R}^n)$ we can gradually move from relation of type (1) to relations of type (2) and (3) for the seminorm weak topologies and the seminorm topologies on $X(= \mathbb{R}^n)$. The question now is whether this scenario or trend will be the case if P is another family of seminorms on $X(= \mathbb{R}^n)$. And a more general and hence better question is whether relations of any of the three types above hold for an arbitrary family P of seminorms on an arbitrary nonempty linear space X. These questions amplify the importance of the statement and proof of the following theorem.

Theorem 4.11 *Let X be any nonempty (real or complex) linear space. Let P be any nonempty family of seminorms on X. Let T be the locally convex topology on X generated by the family P of seminorms. There exists a pairwise strictly comparable chain $C = \{\tau_{w_n}, n \in \mathbb{N}\}$ of non locally convex weak* *topologies generated on X by the family P of seminorms at which peak is a locally convex weak topology τ_w generated by P. Hence $T = \tau_w$.*

Proof:
Let u be the usual topology of $\mathbb{R}$, $r > 0$ any positive real number and let $X_r = (-r, \infty)$ be a u-open subinterval of $\mathbb{R}$. Let $\tau_r = \{G \in u : G \subset X_r\} \cup \{\mathbb{R}\}$ be a topology induced on $\mathbb{R}$ by X_r using some u-open sets.
For any $0 < k < r$ and $b > 0$, $(-k, b) \in \tau_r$. And for any $p \in P$, we have

$$\begin{gathered} p^{-1}\{(-k,b)\} = \{x \in X : p(x) \in (-k,b)\} \\ = \{x \in X : -k < p(x) < b\} \\ = \{x \in X : -b < p(x) < b\} \text{ as } p(x) \geq 0 \text{ for all } x \in X \\ = \{x \in X : p(x) < b\} \\ = bV(p) \text{ where } V(p) = \{x \in X : p(x) < 1\}. \end{gathered}$$

Hence for all $p \in P$, $p^{-1}\{(-k,b)\} = bV(p)$ where $V(p) = \{x \in X : p(x) < 1\}$. This equation and theorem 4.10 imply that the weak topology τ_w and the seminorm topology T generated on X by the family P of seminorms are the same. That is, $T = \tau_w$.

Gradual application of Lemma 4.8 following the pattern of developments in example 4.53 will generate a sequence $C = \{\tau_{w_n}, n \in \mathbb{N}\}$ of pairwise strictly comparable non locally convex weak topologies lying below $T = \tau_w$. That is

$$\tau_{w_1} < \tau_{w_2} < \cdots < \tau_{w_\infty} = \tau_w = T.$$

NOTE: Proof of the last theorem is facilitated greatly by the idea that subsets can induce topologies on their supersets. Also the proof is made possible by the careful notice taken of the fact that all seminorms are real-valued functions; and hence we can change the landscape of *seminorm weak topology* by only tweaking the topology of the set $\mathbb{R}$ of real numbers.
Future researches will have to find out how and if two different families of seminorms on a linear space X can generate the same seminorm topology on X—or whether this can never happen.
(References: Wehausen (1938), Arens (1947), Kelly (1958), Liusternik and Sobolev (1974), McMaster (1990), Usher (2009), Ramachandran and Wolfston (2009), Yong *etal.,* (2012), and Karimov and Repovs (2013).)

4.2 Cofinite Topology: Constructions and Considerations

4.2.1

Here we show that every infinite set admits infinitely many *cofinite-like* topologies which we then called *semi-cofinite* topologies. We then showed that the infinity of semi-cofinite topologies can be constructed to form a chain of topologies at which peak stands the cofinite topology of the set. For a finite set, the semi-cofinite topologies forming a chain will only be finite in number and the discrete topology of the finite set will be at the top of the chain of the semi-cofinite topologies, as the cofinite topology. We also proved that each of the semi-cofinite topologies is itself at the top of yet another chain of pairwise comparable (semi-cofinite) topologies. This last exposition resulted into what we finally called the branching theorem—that every cofinite topology is a tree of many semi-cofinite topologies. Finally in the section, we constructed the cofinite topology induced weak topology on $\mathbb{R}^2$. And it is seen that matrices of coordinate points come as closed sets of this topology. This particular development again has an interesting contrast with the point-open weak topology (constructed in subsection 4.1.1) on $\mathbb{R}^2$ in which matrices of coordinate points actually come as open sets. (References: Kelly (1950), Warner (1958), Mehdi (1959), Arsove and Edwards (1980), Uribe (2008), Kapovich and Lustig (2009), Albin and Melrose (2009), and Wang and Huang (2013), and Bilan (2013),.)

Definition 4.37 *Let X be an infinite set and let $C = \{A \subset X : A^c$ is finite $\} \bigcup \{\emptyset\}$. Then C is a topology on X, called* the *cofinite topology on X.*

Proposition 4.16 *Let X be an infinite set and let $C = \{A \subset X : A^c$ is finite $\} \bigcup \{\emptyset\}$. Then C is a topology on X.*

Proof:

1. $\emptyset \in C$, by definition.
2. $X \in C$, because $X^c = \emptyset$ is finite.
3. Let $\{A_i\}_{i=1}^n$ be a finite number of sets of C. Let $N = \bigcap\limits_{i=1}^{n} A_i$ be the intersection of this finite number of sets. Then $N^c = \left(\bigcap\limits_{i=1}^{n} A_i\right)^c = \bigcup\limits_{i=1}^{n} A_i^c$ must be finite, as a finite union of finite sets. Hence, $\bigcap\limits_{i=1}^{n} A_i \in C$.

4. Let $\{A_\alpha, \alpha \in \Delta\}$ be an arbitrary family of sets of C. We consider the union $U = \bigcup_{\alpha \in \Delta} A_\alpha$ of this family. The complement U^c of this union is the intersection $\bigcap_{\alpha \in \Delta} A_\alpha^c$ of finite sets and hence U belongs to C since its complement is finite.

Even though one may loosely say that open sets of a cofinite topology are either finite or have finite complement, it is not quite correct to say that the cofinite topology on an infinite set X is a collection of all subsets of X which are either finite or have finite complement. Reason: Consider the set $\mathbb{R}$ of real numbers. If we define cofinite topology on $\mathbb{R}$ using the loose definition above, then a problem will occur as follows. Suppose C is the family of all subsets of R which are either finite or have finite complement. For each natural number n, let $G_n = \{n\}$, the singleton of n. Then each G_n belongs to C, as a finite set. Take the union $\bigcup_{n=1}^{\infty} G_n$ of all such sets. Then this union does not belong to C since it is infinite and its complement, $\left(\bigcup_{n=1}^{\infty} G_n\right)^c = \bigcap_{n=1}^{\infty} G_n^c$, is infinite because it includes (among others) all the irrational numbers, which are themselves even uncountable.

Some of the well known properties of the cofinite topology C on a set X are as follows:

1. For an infinite set X, the complement of every C-open set (apart from the empty set) is finite—this is the actual *complement finite* or co-finite property.

2. If X is infinite, then C has infinitely many open sets.

3. If X is infinite, then C is not closed under arbitrary intersections.

4. There is one, and only one cofinite topology C on a set X.

5. The cofinite topology C on a set X is always T_1.

The properties of a cofinite topology outlined above will soon be compared and contrasted with those of *semi-cofinite* topologies defined and constructed below in the next subsection.

4.2.2 Peak of a Sequence of Pair-wise Comparable Topologies

Definition 4.38 *If a collection C_s of subsets of an infinite set X is such that C_s contains the empty set and that every (other) set in C_s has finite complement, then C_s is called a semi-cofinite topology on X if it is a strictly weaker topology than the cofinite topology on X.*

The following are the properties against which each semi-cofinite topology, when constructed, will be checked.

1. The complement of every C_s-open set (apart from the empty set) is finite.
2. If X is infinite, then C_s can have infinitely many open sets or only a finite number of open sets—all depends on how we choose to construct C_s.
3. If X is infinite, C_s can be closed under arbitrary intersections, or not closed under arbitrary intersections—depending on how C_s is constructed.
4. A set X can (and do often) have more than one semi-cofinite topology.
5. No semi-cofinite topology C_s on a set X is T_1.

So, one main difference between a cofinite topology C and a semi-cofinite topology C_s is that C is T_1 and C_s is never T_1. And one special relationship between a cofinite topology C and a semi-cofinite topology C_s is that C_s is always strictly weaker than C, on X. Last, but not the least, the cofinite topology C and each semi-cofinite topology C_s have the *co-finite or complement finite* property in common. These differences and similarities necessitated the new definition—for if only one topology has a name, then a very large class of other topologies related to the named topology should have their name in autonomy.

EXAMPLE 4.54

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ denote the set of natural numbers. For each $n \in \mathbb{N}$ let G_n be the set of all real numbers *excluding* the first n natural numbers. Thus for instance

$G_0 = \mathbb{R} - \{\} = \mathbb{R}$;

$G_1 = \mathbb{R} - \{0\}$;

$G_2 = \mathbb{R} - \{0, 1\};$

$G_3 = \mathbb{R} - \{0, 1, 2\};$

$\vdots$

$G_n = \mathbb{R} - \{0, 1, 2, 3, \cdots, n-1\}$

Let $T_{C\mathbb{N}} = \{\emptyset, G_n\}_{n \in \mathbb{N}}$.
Then it is easy to see that

1. The empty set is in $T_{C\mathbb{N}}$, from the way $T_{C\mathbb{N}}$ is defined.
2. The whole set $\mathbb{R}$ of real numbers is in $T_{C\mathbb{N}}$.
3. The complement G_n^c of every set in $T_{C\mathbb{N}}$, apart from the empty set, is finite; precisely G_n^c contains the first n natural numbers.
4. And that $T_{C\mathbb{N}}$ is closed under finite intersections and arbitrary unions.
5. Hence $T_{C\mathbb{N}}$ is a topology on $\mathbb{R}$, satisfying all but one property of the cofinite topology, on $\mathbb{R}$, namely that it is not the family of *all* subsets of $\mathbb{R}$ whose complements are finite, together with the empty set. Hence $T_{C\mathbb{N}}$ is an example of what we, in this thesis, call semi-cofinite topology, on the set $\mathbb{R}$ of real numbers. It is strictly weaker than the cofinite topology on $\mathbb{R}$.

We shall call $T_{C\mathbb{N}}$ the semi-cofinite topology on $\mathbb{R}$ generated by the set $\mathbb{N}$ of natural numbers.

EXAMPLE 4.55

Let $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \cdots\}$ denote the set of all integers, arranged thus. For each $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$, let G_n be $\mathbb{R}$ without the first n integers under the arrangement thus made of $\mathbb{Z}$.
Hence for instance

$G_0 = \mathbb{R};$

$G_1 = \mathbb{R} - \{0\};$

$G_2 = \mathbb{R} - \{0, 1\};$

$G_3 = \mathbb{R} - \{0, 1, -1\};$

$G_4 = \mathbb{R} - \{0, 1, -1, 2\};$

$G_5 = \mathbb{R} - \{0, 1, -1, 2, -2\};$

$$\vdots$$

etc.

Then $T_{C\mathbb{Z}} = \{\emptyset, G_n\}_{n\in\mathbb{N}}$ is a semi-cofinite topology (different from the one above) on $\mathbb{R}$, as can easily be verified.
We observe that in $T_{C\mathbb{N}}, G_2 = \mathbb{R} - \{0,1\}$ and that in $T_{C\mathbb{Z}}, G_2 = \mathbb{R} - \{0,1\}$. However in $T_{C\mathbb{N}}, G_3 = \mathbb{R} - \{0,1,2\}$ and in $T_{C\mathbb{Z}}, G_3 = \mathbb{R} - \{0,1,-1\}$ and we see that though G_2 is common to both $T_{C\mathbb{N}}$ and $T_{C\mathbb{Z}}$, G_3 is not common to the two topologies on $\mathbb{R}$. It may be further verified that G_n is not common to the two topologies if $n \geq 3$. Hence these two topologies are not comparable. But we can obtain two comparable semi-cofinite topologies on $\mathbb{R}$ based on the two subsets $\mathbb{N}$ and $\mathbb{Z}$. Let $\mathbb{Z}$ be written in the alternative (and usual) form $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ and let $G_n = \mathbb{R} - \{\text{n integers }\}$ for each $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$. That is, G_n is $\mathbb{R}$ without a finite number of whole numbers. We can also observe that the complement of each G_n is a finite number of integers. This observation is helpful in proving that the family $T_{\mathbb{Z}} = \{\emptyset, G_n\}_{n\in\mathbb{N}}$ is closed under arbitrary unions; for if $\{G_\alpha\}_{\alpha\in\Delta} \subset T_{\mathbb{Z}}$ is any family of sets in $T_{\mathbb{Z}}$, then $\bigcap\limits_{\alpha\in\Delta} G_\alpha^c$ is a finite number of integers. It follows that

$$\left(\bigcap_{\alpha\in\Delta} G_\alpha^c\right)^c = \bigcup_{\alpha\in\Delta} G_\alpha$$

is $\mathbb{R}$ without n whole numbers. Similarly we see that

$$\bigcap_{i=1}^{n} G_i = \left(\bigcup_{i=1}^{n} G_i^c\right)^c$$

is the complement of a finite number of whole numbers. Hence $T_{\mathbb{Z}}$ is a semi-cofinite topology on $\mathbb{R}$. And it is easy to see that $T_{C\mathbb{N}}$ is strictly weaker than $T_{\mathbb{Z}}$. There are other ways of constructing strictly comparable pairs of semi-cofinite topology on any infinite set. Later developments here will show that.

EXAMPLE 4.56

Let $\mathbb{P} = \{p_1, p_2, p_3, \cdots\}$ be the ordered (ascendingly) set of all prime numbers, and $\mathbb{R}$ and $\mathbb{N}$ as earlier introduced. Let G_n denote $\mathbb{R}$ without the first n prime numbers. Then $T_{C\mathbb{P}} = \{\emptyset, G_n\}_{n\in\mathbb{N}}$ is yet another semi-cofinite topology on $\mathbb{R}$, different from the two introduced earlier.

EXAMPLE 4.57

Let $\mathbb{Q} = \{q_1, q_2, q_3, \cdots\}$ be the set of all rational numbers, and $\mathbb{R}$ and

$\mathbb{N}$ as earlier introduced. Put $G_n = \mathbb{R} - \{n \text{ rational numbers }\}$. Then $T_{C\mathbb{Q}} = \{\emptyset, G_n\}_{n\in\mathbb{N}}$ is another semi-cofinite topology on $\mathbb{R}$.

EXAMPLE 4.58

Let $\mathbb{Q}^c$ be the set of all irrational numbers, and $\mathbb{R}$ and $\mathbb{N}$ remain as introduced before. Put $G_n = \mathbb{R} - \{n \text{ irrational numbers}\}$, and let $T_{C\mathbb{Q}^c} = \{\emptyset, G_n\}_{n\in\mathbb{N}}$. Then $T_{C\mathbb{Q}^c}$ is a semi-cofinite topology on $\mathbb{R}$.

EXAMPLE 4.59

With $\mathbb{R}$ and $\mathbb{N}$ as earlier introduced, let $G_n = \mathbb{R} - \{n \text{ real numbers }\}$. Put $T_{C\mathbb{R}} = \{\emptyset, G_n\}_{n\in\mathbb{N}}$. Then $T_{C\mathbb{R}}$ is *the* cofinite topology on $\mathbb{R}$.

NOTE:

We observe that the topology constructed last above (i.e. example 4.59), $T_{C\mathbb{R}}$, is *the* family of *all* subsets of $\mathbb{R}$ whose complements are finite, together with the empty set—hence only this topology is *the* cofinite topology on $\mathbb{R}$. The general method of constructing cofinite topologies supplied here can be used to construct cofinite topology on any set. For example if we put $\mathbb{N} = \mathbb{R}$ then example 4.54 will coincide with what is found in Lipschutz (1965). If $\mathbb{N} = \mathbb{R}$ in example 4.59 we get the cofinite topology on $\mathbb{N}$.

We have proved that all the constructions 4.54 to 4.58 are semi-cofinite topologies. We have also proved, in Proposition 4.16, that the family defined in definition 4.37 is indeed a topology on X. We provide below an alternative, rigorous and particular proof that the family in example 4.59 is indeed the cofinite topology on the set $\mathbb{R}$ of real numbers.

Proposition 4.17 *The family $T_{C\mathbb{R}}$ as constructed in example 4.59 is* the *cofinite topology on the set $\mathbb{R}$ of real numbers.*

Proof:

We only prove that $T_{C\mathbb{R}}$ is closed under finite intersections and arbitrary unions, since the remaining two properties of a (cofinite) topology are easily seen to be satisfied by $T_{C\mathbb{R}}$; we also note that $T_{C\mathbb{R}}$ is a T_1-space (property of all cofinite topologies) as singletons are $T_{C\mathbb{R}}$-closed subsets of $\mathbb{R}$. We recall that

$G_0 = \mathbb{R}$;

$G_1 = \mathbb{R} - \{r_{11}\}$;

$G_2 = \mathbb{R} - \{r_{21}, r_{22}\}$;

$G_3 = \mathbb{R} - \{r_{31}, r_{32}, r_{33}\}$;

$G_4 = \mathbb{R} - \{r_{41}, r_{42}, r_{43}, r_{44}\}$;

$\vdots$

$G_n = \mathbb{R} - \{r_{n1}, r_{n2}, r_{n3}, \cdots, r_{nn}\}$; where $r_{ni} \in \mathbb{R}$

Now let $N = \bigcap_{k=1}^{n} G_k = \bigcap_{k=1}^{n} (\mathbb{R} - \{r_{k1}, r_{k2}, r_{k3}, \cdots, r_{kk}\})$
$= \mathbb{R} - \bigcup_{k=1}^{n} \{r_{k1}, r_{k2}, r_{k3}, \cdots, r_{kk}\}$. Hence the complement N^c of N is $N^c = \bigcup_{k=1}^{n} \{r_{k1}, r_{k2}, r_{k3}, \cdots, r_{kk}\}$, a finite union of finite sets; and hence must be finite. Alternatively we can consider directly the cardinality of N^c. Card(N^c) is such that Card$(N^c) \leq 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} < \infty$. So N^c is finite, implying that $T_{C\mathbb{R}}$ is closed under finite intersections.
Now let $U = \bigcup_{\alpha \in \Delta} G_\alpha \equiv \bigcup_{m \geq t} G_m = \bigcup_{m \geq t} (\mathbb{R} - \{r_{mi}\}_{i=1}^{m}) = \mathbb{R} - (\bigcap_{m \geq t} \{r_{mi}\}_{i=1}^{m})$. Hence $U^c = \bigcap_{m \geq t} \{r_{mi}\}_{i=1}^{m} \subset \{r_{ti}\}_{i=1}^{t}$. Hence Card$(U^c) \leq t < \infty$, implying that $T_{C\mathbb{R}}$ is also closed under arbitrary unions.

OBSERVATIONS

[**1**] We have said that the topologies $T_{C\mathbb{N}}$ and $T_{C\mathbb{Z}}$ (in examples 4.54 and 4.55) are not comparable, from the observation that G_3 in $T_{C\mathbb{N}}$ is $G_3 = \mathbb{R} - \{0, 1, 2\}$ while in $T_{C\mathbb{Z}}$, $G_3 = \mathbb{R} - \{0, 1, -1\}$. Hence G_3 in $T_{C\mathbb{N}}$ is not the same as G_3 in $T_{C\mathbb{Z}}$. If we denote G_n in $T_{C\mathbb{N}}$ by $T_{C\mathbb{N}}(G_n)$, and G_n in $T_{C\mathbb{Z}}$ by $T_{C\mathbb{Z}}(G_n)$ then it is easy to see that $T_{C\mathbb{N}}(G_n) \notin T_{C\mathbb{Z}}$ if $n \in N$ and $n \geq 3$; and, conversely, $T_{C\mathbb{Z}}(G_n) \notin T_{C\mathbb{N}}$ if $n \in \mathbb{N}$ and $n \geq 3$, though $T_{C\mathbb{N}}(G_n) = T_{C\mathbb{Z}}(G_n)$ if $n = 0, 1, 2$.

[**2**] However, if we define G_n in $T_{C\mathbb{N}}$ by $G_n = \mathbb{R} - \{n$ natural numbers $\}$, and define G_n in $T_{C\mathbb{Z}}$ by $G_n = \mathbb{R} - \{n$ whole numbers $\}$, then it would be seen that, since the set $\mathbb{N}$ of natural numbers is a subset of the set $\mathbb{Z}$ of integers, the collection $T_{C\mathbb{N}}$ of the set of real numbers without some natural numbers is a sub-collection of the collection $T_{C\mathbb{Z}}$ of $\mathbb{R}$ without some integers. Hence the semi-cofinite topology $T_{C\mathbb{N}}$ on $\mathbb{R}$ (this time) is strictly weaker than the semi-cofinite topology $T_{C\mathbb{Z}}$ on $\mathbb{R}$. That is, $T_{C\mathbb{N}} < T_{C\mathbb{Z}}$. Similarly $T_{C\mathbb{P}} < T_{C\mathbb{Z}}$, and $T_{C\mathbb{Z}} < T_{C\mathbb{Q}}$. And all the five semi-cofinite topologies, on $\mathbb{R}$, above can be summarized as follows:

1. $T_{C\mathbb{N}} < T_{C\mathbb{Z}} < T_{C\mathbb{Q}} < T_{C\mathbb{R}}$;
2. $T_{C\mathbb{P}} < T_{C\mathbb{Z}} < T_{C\mathbb{Q}} < T_{C\mathbb{R}}$;
3. $T_{C\mathbb{Q}^c} < T_{C\mathbb{R}}$; and

4. $T_{C\mathbb{P}} < T_{C\mathbb{N}}$.

The procedure for constructing the semi-cofinite topologies, on $\mathbb{R}$, above can be applied to any infinite set and summaries similar to 1 to 4 can be put under a lemma as follows.

Lemma 4.9 (The Cofinite Topology Lemma) *Let A and B be two infinite subsets of an infinite set X such that $A \subset B$ (where A is a proper subset of B). Then there exist semi-cofinite topologies T_{CA} and T_{CB}, on X, induced respectively by A and B such that $T_{CA} < T_{CB}$; that is, T_{CA} is strictly weaker than T_{CB}.*

EXAMPLE 4.60

Let X be an infinite set and let $B = X - \{x_1\}$ and $A = B - \{x_2\} = X - \{x_1, x_2\}$. Then A is an infinite and proper subset of B, and (W.L.O.G) B is an infinite and proper subset of X. Let

$G_0 = X - \{\} = X;$

$G_1 = B = G_0 - \{x_1\} \equiv X - \{x_1\};$

$G_2 = A = G_1 - \{x_2\} = B - \{x_2\} \equiv X - \{x_1, x_2\};$

$G_3 = G_2 - \{x_3\} = A - \{x_3\} = G_1 - \{x_2, x_3\} = B - \{x_2, x_3\} \equiv X - \{x_1, x_2, x_3\};$

$$\begin{aligned} G_4 &= G_3 - \{x_4\} \\ &= G_2 - \{x_3, x_4\} \\ &= G_1 - \{x_2, x_3, x_4\} \\ &= G_0 - \{x_1, x_2, x_3, x_4\} \end{aligned}$$

$$\vdots$$

$$\begin{aligned} G_n &= G_{n-1} - \{x_n\} \\ &= G_{n-2} - \{x_{n-1}, x_n\} \\ &= G_{n-3} - \{x_{n-2}, x_{n-1}, x_n\} \end{aligned}$$

$$\vdots$$

$$= G_0 - \{x_1, x_2, \cdots, x_n\}.$$

Let $T_{CA} = \{\emptyset, G_0, G_2, G_3, \cdots, G_n, \cdots\}$. Then T_{CA} is a semi-cofinite topology on X.
Let $T_{CB} = \{\emptyset, G_n\}_{n\in\mathbb{N}}$. Then T_{CB} is another semi-cofinite topology on X.

And we see that T_{CA} is a strictly weaker topology than T_{CB} on X—since $G_1 \in T_{CB}$ and $G_1 \notin T_{CA}$ and every T_{CA}-open set is T_{CB}-open.

We also see that the open sets of both semi-cofinite topologies satisfy the inclusions

$$\cdots \subset G_3 \subset G_2 \subset G_0 = X, \text{ for } T_{CA};$$

and

$$\cdots \subset G_3 \subset G_2 \subset G_1 \subset G_0 = X, \text{ for } T_{CB}.$$

Hence both semi-cofinite topologies are closed under arbitrary intersections. Finally, we remark that one must not follow the process of construction used here to have two comparable semi-cofinite topologies induced on X by A and B. For example we could have used procedures exactly similar to those used in examples 4.57 and 4.58, and still have T_{CA} to be strictly weaker than T_{CB}—only that this time the semi-cofinite topologies may not be closed under arbitrary intersections.

Let $C_{\mathbb{N}}$ and $C_{\mathbb{Z}}$ be respectively the cofinite topology on $\mathbb{N}$ and $\mathbb{Z}$. Let $T_{\mathbb{N}} = C_{\mathbb{N}} \bigcup \{\mathbb{Z}\} \bigcup \{\mathbb{R}\}$ and $T_{\mathbb{Z}} = C_{\mathbb{Z}} \bigcup \{\mathbb{R}\}$. Then, by subset-induced topologies, both $T_{\mathbb{N}}$ and $T_{\mathbb{Z}}$ are topologies on $\mathbb{R}$ and $T_{\mathbb{N}}$ is strictly weaker than $T_{\mathbb{Z}}$. Finally we observe that these two topologies are semi-cofinite topologies on $\mathbb{R}$.

REMARK 4.17

We know that every infinite set, say $X = \{x_1, x_2, x_3, \cdots\}$, has an infinite proper subset, say $X_1 = \{x_2, x_3, x_4, \cdots\}$. From the cofinite topology lemma, above, we can construct (or there exists) a semi-cofinite topology T_{CX_1} on X, and T_{CX}, such that $T_{CX} > T_{CX_1}$. Since X_1 is itself infinite, it has an infinite proper subset, say X_2. By the cofinite topology lemma again, there exists a semi-cofinite topology T_{CX_2} on X such that $T_{CX} > T_{CX_1} > T_{CX_2}$. The reasoning can continue like that, and what we have proved is the following.

Theorem 4.12 (Cofinite Topology Theorem) *Let X be any infinite set. There exists a sequence $\{\tau_1, \tau_2, \tau_3, \cdots\}$ of (semi-cofinite) topologies on X, forming a chain in that*

$$C = T_{CX} > \tau_1 > \tau_2 > \tau_3 > \cdots$$

where $C = T_{CX}$ is the cofinite topology on X.

4.2.3 The Branching Theorem

Definition 4.39 *A topology is called a* chain-element *topology if it is an element of a family of topologies which form a (decreasing or increasing) chain on a set.*

NOTE

One implication of the cofinite topology theorem is that every infinite set has infinitely many semi-cofinite topologies. The other implication is that an infinity of semi-cofinite topologies on any infinite set can be constructed to form a chain, on the top of which sits the cofinite topology of the set.

We observe that each of the semi-cofinite topologies so far constructed here has an infinite number of open sets. However, we also have to point out that they alone are not the only semi-cofinite topologies: there are some semi-cofinite topologies with only finite numbers of open sets. In fact, each of the chain-element semi-cofinite topologies with infinite number of open sets can be shown to be (themselves) the limit of an increasing sequence of pair-wise comparable semi-cofinite topologies with finite numbers of open sets.

EXAMPLE 4.61

Let us take another look at example 4.54, the semi-cofinite topology $T_{C\mathbb{N}}$ on $\mathbb{R}$ generated by the set of natural numbers. If we serially collect finite numbers of open sets of $T_{C\mathbb{N}}$, we shall get an increasing sequence of semi-cofinite topologies on $\mathbb{R}$ forming a chain at the top of which sits $T_{C\mathbb{N}}$. To see this, we go as usual and let

$G_0 = \mathbb{R} - \{\} = \mathbb{R}$;

$G_1 = \mathbb{R} - \{0\}$;

Then $\tau_1 = \{\emptyset, G_0, G_1\}$ is a semi-cofinite topology, on $\mathbb{R}$, strictly weaker than $T_{C\mathbb{N}}$.

Let $G_2 = \mathbb{R} - \{0, 1\}$. Then, with G_0 and G_1 as earlier defined, $\tau_2 = \{\emptyset, G_0, G_1, G_2\}$ is yet another (semi-cofinite) topology, strictly weaker than $T_{C\mathbb{N}}$, on $\mathbb{R}$.

$$\vdots$$

Continuing like that, with $G_n = \mathbb{R} - \{0, 1, 2, \cdots, n-1\}$ and G_i $(1 \leq i \leq n-1)$ as earlier defined, we see that $\tau_n = \{\emptyset, G_k\}_{k=0}^{n}$ is a (semi-cofinite) topology on $\mathbb{R}$, strictly weaker than $T_{C\mathbb{N}}$.

We finally observe that τ_1 is strictly weaker than τ_2, and τ_2 is strictly weaker than τ_3, and so on. That is

$$\tau_1 < \tau_2 < \tau_3 < \cdots < T_{C\mathbb{N}},$$

where $T_{C\mathbb{N}}$ is as earlier introduced in example 4.54. That is, some semi-cofinite topologies can *branch* out into the limit of another sequence of pair-wise comparable topologies. Since each of the chain element topologies (τ_n on X) in the cofinite topology theorem is induced on X by an infinite set, X_n, each of these semi-cofinite topologies can be made to sit at the top of yet another sequence of pairwise comparable topologies. For example, if τ_n is induced on X by X_n, let X_{n_i} be X_n without i elements (where $i \in \mathbb{N}$). That is, $X_{n_1} = X_n - \{x_1\}$, where $x_1 \in X_n$; $X_{n_2} = X_{n_1} - \{x_2\}$, where $x_2 \in X_{n_1}$; and so on. We see that $X_n \supset X_{n_1} \supset X_{n_2} \cdots$. It follows from the cofinite topology lemma that the semi-cofinite topology $T_{CX_{n_1}}$ on X is strictly weaker than $T_{CX_n} = \tau_n$; $T_{CX_{n_2}}$ is strictly weaker than $T_{CX_{n_1}}$; $T_{CX_{n_3}}$ is strictly weaker than $T_{CX_{n_2}}$; and so on. That is

$$\cdots < T_{CX_{n_3}} < T_{CX_{n_2}} < T_{CX_{n_1}} < T_{CX_n} = \tau_n < T_{CX} = C, \quad . \; . \; . \; (*)$$

where C is the cofinite topology on X.

That is, the topology τ_n on X, induced by the subset X_n of X, sits at the top of a chain of pairwise comparable topologies. Each X_{n_i}, subset of X_n, induces the topology $T_{X_{n_i}}$ on X, strictly weaker than τ_n, as seen in $(*)$. By a process similar to what has been used to generate $(*)$, under τ_n, we can have another chain

$$H = \{T_{X_{n_{i_j}}}\}_{j=1}^{\infty}$$

of topologies on X, pairwise comparable, and such that each $T_{X_{n_{i_j}}}$ is strictly weaker than $T_{X_{n_i}}$. The process can continue for each of the (infinite) subsets $X_{n_i}(i = 1, 2, 3, \cdots)$ of X—and their own subsets—and what we have proved is the following.

Theorem 4.13 (Branching) *Each of the chain element topologies under the cofinite topology theorem is itself at the peak of yet another chain of (semi-cofinite) topologies. If the original set X is infinite, then this branching will be endless; if X is finite, the branching will terminate.*

For example each of the semi-cofinite topologies 4.54 to 4.58 above, on $\mathbb{R}$, is the limit of a sequence of pair-wise comparable monotone increasing semi-cofinite topologies.

EXAMPLE 4.62

We may again let

$G_0 = \mathbb{N} = \{0, 1, 2, \cdots\}$;

$G_1 = \mathbb{N} - \{0\} = \{1, 2, 3, \cdots\}$.

Then $\tau_1 = \{\emptyset, G_0, G_1\}$ is a (semi-cofinite) topology on the set $\mathbb{N}$ of natural numbers. If we also let

$$G_0 = \mathbb{N},\ G_1 = \mathbb{N} - \{0\},\ G_2 = \mathbb{N} - \{0, 1\} = \{2, 3, 4, \cdots\},$$

then $\tau_2 = \{\emptyset, G_0, G_1, G_2\}$ is another topology, strictly stronger than τ_1, on $\mathbb{N}$. If we continue like that, for each $n \in \mathbb{N}$ then

$$\tau_n = \{\emptyset, G_k\}_{k=0}^{n}$$

is a semi-cofinite topology on $\mathbb{N}$, strictly stronger than τ_{n-1}. We then see that the family $H_1 = \{\tau_n\}_{n=1}^{\infty}$ of topologies on $\mathbb{N}$ form an increasing chain of topologies, on $\mathbb{N}$, at which peak lies *the* cofinite topology on $\mathbb{N}$. We note that each chain element of H_1 has only finitely many open sets. This is to be contrasted with $H_2 = \{\tau_n\}_{n=1}^{\infty}$ whose elements $\tau_1 = T_{CX_1}, \tau_2 = T_{CX_2}, \tau_3 = T_{CX_3}, \cdots$, have each infinitely many open sets, where $X_1 = N - \{0\} = \{1, 2, 3, \cdots\}, X_2 = N - \{0, 1\} = \{2, 3, 4, \cdots\}$, etc., and the topologies $\tau_1 = T_{CX_1}, \tau_2 = T_{CX_2}, \tau_3 = T_{CX_3}, \cdots$ are constructed according to the remark in Lemma 4.9.

Each of the chain-element semi-cofinite topologies with only a finite number of open sets (i.e. elements of H_1) is an example of *complement topology* (defined in the next section).

Finite Sets

Since the complement of every subset of a finite set is finite, any search for subsets of a finite set whose complements are finite is not an interesting exercise. Hence we do not often talk about cofinite topologies on finite sets. However if X is finite, then 2^X the power set of X is the cofinite topology on X since it satisfies the definition of cofinite topology. Also if X is finite, say $X = \{x_1, x_2, x_3, \cdots, x_n\}$, then we have some semi-cofinite topologies on X, forming a chain also. These semi-cofinite topologies can be constructed as follows:

Let $G_0 = X$;

$G_1 = X - \{x_1\}$. Then $\tau_1 = \{\emptyset, G_0, G_1\}$ is a semi-cofinite topology on X.

Let $G_2 = X - \{x_1, x_2\}$. Then $\tau_2 = \{\emptyset, G_k\}_{k=0}^{2}$, where G_0, G_1 are as earlier defined, is another semi-cofinite topology on X, strictly stronger than τ_1.

With $G_3 = X - \{x_1, x_2, x_3\}$, we see that $\tau_3 = \{\emptyset, G_k\}_{k=0}^3$ is another semi-cofinite topology, strictly stronger than τ_2.

$\vdots$

Continuing like that, with $G_n = X - \{x_1, x_2, x_3, \cdots, x_n\} = \emptyset$, we see that $\tau_n = \{\emptyset, G_k\}_{k=0}^n$ is a topology, strictly stronger than τ_{n-1}.

That is

$$\tau = \tau_n = \{\emptyset, G_k\}_{k=0}^n$$

is a topology on X, stronger than all the other ones. So, we have a finite sequence $\{\tau_k\}_{k=1}^n$ of topologies on X forming a chain in that

$$\tau_n > \tau_{n-1} > \cdots > \tau_1$$

and the power set 2^X of X, or its cofinite topology, is at the top of this finite sequence of topologies.

4.2.4 Cofinite Topology Induced Weak Topology

Let us reconsider the projection maps $p_i : \mathbb{R}^2 \longrightarrow \mathbb{R}$, where $1 \leq i \leq 2$, such that $p_1(x, y) = x$ and $p_2(x, y) = y$. We wish to find the weak topology, induced by the projection maps, on $\mathbb{R}^2$ when the two factor spaces of $\mathbb{R}^2$ are endowed with the cofinite topology of $\mathbb{R}$, as has been constructed in example 4.59

Let $G_n \in T_{C\mathbb{R}}$ be an open subset of $\mathbb{R}$ when $\mathbb{R}$ is endowed with the cofinite topology. Then the inverse image[10] of G_n under, say p_1, is $p_1^{-1}(G_n) =$

$\{(x_1, x_2) = \bar{x} \in \mathbb{R}^2 : p_1(\bar{x}) \in G_n\}$

$= \{\bar{x} \in \mathbb{R}^2 : p_1(\bar{x}) \in \mathbb{R} \text{ and } p_1(\bar{x}) \notin G_n^c\}$

$= \{\bar{x} \in \mathbb{R}^2 : p_1(\bar{x}) \notin \{r_{n1}, r_{n2}, \cdots, r_{nn}\} \text{ and } p_1(\bar{x}) \in \mathbb{R}\}$

$= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \notin G_n^c, x_1 \in \mathbb{R}\}$

$= \{\mathbb{R}^2$, without a finite number of infinite vertical lines $\}$

$= \{\mathbb{R}^2$, without exactly n vertical infinite lines $\}$.

By the same token we find that

[10] See Figure 7.

$$p_2^{-1}(G_n) = \{\mathbb{R}^2, \text{ without exactly } n \text{ horizontal infinite lines}\}.$$

The above constitute the subbasic sets of the (weak) topology of $\mathbb{R}^2$, under this arrangement. The base for this (weak) topology has some interesting sets. For example, the whole plane $\mathbb{R}^2$, excluding some matrices of coordinate points, are among the basic sets of the cofinite topology induced weak topology on $\mathbb{R}^2$. This means that among the open sets of this topological space are the whole plane itself without some matrices of coordinate points. To see this, we observe that

$$p_1^{-1}(G_n) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \notin \{r_{n1}, r_{n2}, \cdots, r_{nn}\}\}; \text{ and}$$
$$p_2^{-1}(G_n) = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \notin \{r_{n1}, r_{n2}, \cdots, r_{nn}\}\}.$$

Therefore
$p_1^{-1}(G_n) \bigcap p_2^{-1}(G_n) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \notin \{r_{n1}, r_{n2}, \cdots, r_{nn}\}\}$
$= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \notin G_n^c, x_2 \notin G_n^c\}$
$= \mathbb{R}^2 - (G_n^c \times G_n^c)$
$= \mathbb{R}^2 - \{\{r_{n1}, r_{n2}, \cdots, r_{nn}\} \times \{r_{n1}, r_{n2}, \cdots, r_{nn}\}\}$
$= \mathbb{R}^2 - \{(r_{n1}, r_{n1}), (r_{n1}, r_{n2}), \cdots, (r_{nn}, r_{nn})\}$
$= \mathbb{R}^2 - \{(r_{ni}, r_{nj})\}_{i,j=1}^{n}$,
the whole plane without a square $n \times n$ matrix of coordinate points. We also see that
$p_1^{-1}(G_n) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \notin \{r_{n1}, r_{n2}, \cdots, r_{nn}\}\}$; and
$p_2^{-1}(G_m) = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \notin \{r_{m1}, r_{m2}, \cdots, r_{mm}\}\}$.
Hence
$p_1^{-1}(G_n) \bigcap p_2^{-1}(G_m) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \notin \{r_{n1}, r_{n2}, \cdots, r_{nn}\}$
and $x_2 \notin \{r_{m1}, r_{m2}, \cdots, r_{mm}\}\}$
$= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \notin G_n^c, x_2 \notin G_m^c\}$
$= \mathbb{R}^2 - (G_n^c \times G_m^c)$
$= \mathbb{R}^2 - \{\{r_{n1}, r_{n2}, \cdots, r_{nn}\} \times \{r_{m1}, r_{m2}, \cdots, r_{mm}\}\}$
$= \mathbb{R}^2 - \{(r_{n1}, r_{m1}), (r_{n1}, r_{m2}), \cdots, (r_{nn}, r_{mm})\}$
$= \mathbb{R}^2 - \{(r_{ni}, r_{mj})\}_{i,j=1}^{n,m}$,
the whole plane excluding an $n \times m$ matrix of coordinate points.

In fact all kinds of matrices (column, row vectors, and others) of coordinate points are closed sets of this topological space. (This may be contrasted with the X-topology induced weak topology on the plane in which matrices are actually open sets. See Subsection 4.1.1 again.) Finally, we state that by considering the semi-cofinite topologies on $\mathbb{R}$ we are bound to have similar weak topologies on $\mathbb{R}^2$. For example, if we use the semi-cofinite topology $T_{C\mathbb{N}}$ on $\mathbb{R}$ induced by the set of natural numbers, then all the vertical and the horizontal lines mentioned above would pass through points or positions of natural numbers. If we use the semi-cofinite topology $T_{C\mathbb{Q}^c}$ then the lines will all pass through points of irrational numbers. And so on.

4.3 Complement of a Topology and Complement Topologies

4.3.1 Definition, Properties and Implications

Definition 4.40 *Let (X, τ) be a topological space. Then the complement τ^c of the topology τ on X is the family*

$$\tau^c = \{G^c : G \in \tau\}$$

of complements of τ-open sets.

Definition 4.41 *If (X, τ) is a topological space and the complement τ^c of τ is itself also a topology on X, then τ is called a complement topology, on X.*

REMARK 4.18
Since $\tau = (\tau^c)^c$, it follows that τ is a complement topology on X if and only if τ^c is also a complement topology on X. It turns out that large classes of topologies are complement topologies.

Theorem 4.14 *Every topology on a finite set is a complement topology.*

Proof:
Let τ be a topology on a finite set X and let τ^c be its complement. Then

1. Clearly both $\emptyset$ and X belong to τ^c.

2. Let $\{G_i\}_{i=1}^n \subset \tau^c$. Then $\bigcap_{i=1}^n G_i = (\bigcup_{i=1}^n G_i^c)^c$. But $G_i \in \tau^c \Rightarrow G_i^c \in \tau$. $\Rightarrow \bigcup_{i=1}^n G_i^c \in \tau$. $\Rightarrow \bigcap_{i=1}^n G_i = (\bigcup_{i=1}^n G_i^c)^c \in \tau^c$. $\Rightarrow \tau^c$ is closed under finite intersections.

3. Let $\{G_\alpha\}_{\alpha\in\Delta} \subset \tau^c$. Then $\bigcup_{\alpha\in\Delta} G_\alpha = (\bigcap_{\alpha\in\Delta} G_\alpha^c)^c$. Now, $G_\alpha \in \tau^c \Rightarrow G_\alpha^c \in \tau$. $\Rightarrow \bigcap_{\alpha\in\Delta} G_\alpha^c \in \tau$, as finite intersections of sets of τ belong to τ. (We observe that the intersection cannot be infinite since X is finite.) Hence, since the complement of every set in τ is collected in τ^c, it follows that $\bigcup_{\alpha\in\Delta} G_\alpha = (\bigcap_{\alpha\in\Delta} G_\alpha^c)^c \in \tau^c$. This implies that τ^c is also closed under arbitrary unions. Hence the complement of every topology on a finite set is a topology on the set.

EXAMPLE 4.63
Let $X = \{x_1, x_2, x_3, \cdots, x_n\}$ be any non-empty finite set and let

$$\tau = \{\emptyset, X, \{x_1\}, \{x_1, x_2\}\}$$

be a topology on X. Then $\tau^c = \{X, \emptyset, \{x_2, x_3, \cdots, x_n\}, \{x_3, x_4, \cdots, x_n\}\}$ is clearly a topology on X.
EXAMPLE 4.64
Let $X = \{x_1, x_2, x_3, \cdots, x_n\}$ be a non-empty finite set and let

$$\tau_k = \{\emptyset, X, \{x_1\}, \{x_1, x_2\}, \cdots, \{x_1, x_2, \cdots, x_k\}\}; 1 \leq k < n.$$

Then τ_k is a topology on X, for all k. Now

$$\tau_k{}^c = \{X, \emptyset, \{x_2, \cdots, x_n\}, \{x_3, \cdots, x_n\}, \cdots, \{x_{k+1}, \cdots, x_n\}\}$$

is also a topology on X, $1 \leq k < n$. (This example illustrates the remark after Definition 4.41 above. More of such examples appear at the end of this section.)
The proof of theorem 4.14 above points the way to a more general result.

Theorem 4.15 *Let X be any nonempty set and let τ be a finite topology (topology with finite cardinality) on X. Then τ is a complement topology on X.*

Corollary 4.8 *Let X be an infinite set and let τ be a topology on X. Then the complement τ^c of the topology τ is itself a topology on X if τ contains only a finite number of open sets.*

EXAMPLE 4.65
Let $a, b \in \mathbb{R}$ be any two real numbers. Then $\tau = \{\emptyset, \mathbb{R}, \{a\}, \{b\}, \{a, b\}\}$ is a topology on $\mathbb{R}$. Without loss of generality, we can let $a < b$. Then the complement
$\tau^c = \{\emptyset, \mathbb{R}, \mathbb{R} - \{a\}, \mathbb{R} - \{b\}, \mathbb{R} - \{a, b\}\}$
$= \{\emptyset, \mathbb{R}, (-\infty, a) \bigcup (a, +\infty), (-\infty, b) \bigcup (b, +\infty), (-\infty, a) \bigcup (a, b) \bigcup (b, +\infty)\}$
of τ is easily seen to be a topology on $\mathbb{R}$.

Let $G_0 = \mathbb{N} = \{0, 1, 2, \cdots\}$, $G_1 = \{1, 2, 3, \cdots\}$, $G_2 = \{2, 3, 4, \cdots\}$. Then $\tau = \{\emptyset, G_k\}_{k=0}^{2}$ is easily seen to be a topology on $\mathbb{N}$. The complement of τ, $\tau^c = \{\emptyset, \mathbb{N}, \{0\}, \{0, 1\}\}$ is also a topology on $\mathbb{N}$. In general if $G_0 = \mathbb{N}$, $G_1 = \mathbb{N} - \{0\}$, $G_2 = \mathbb{N} - \{0, 1\}$, $G_3 = \mathbb{N} - \{0, 1, 2\}, \cdots$, $G_n = \mathbb{N} - \{0, 1, 2, \cdots, n - 1\}$, then $\tau = \{\emptyset, G_k\}_{k=0}^{n}$ is a topology on $\mathbb{N}$, and its complement τ^c is also a topology on $\mathbb{N}$.

Now, every topology τ on a finite set is necessarily finite. Hence theorem 4.15 asserts, relative to theorem 4.14, that every finite topology on an infinite set is a complement topology. This raises the following interesting question: *Are the finite topologies the only topologies on infinite sets that are complement topology?* That is, is a complement topology on an infinite set necessarily finite? The next theorem which answers the above question in the negative provides a characterization of complement topologies.

Theorem 4.16 *A topology τ on a set X is a complement topology if, and only if τ is closed under arbitrary intersections.*

Proof:
$\Longrightarrow$ Clearly if τ is a complement topology then it is closed under arbitrary intersections.
$\Longleftarrow$. Let τ be closed under arbitrary intersections and let τ^c be the complement of τ. We show that τ^c is a topology on X. We need only show that τ^c is closed under arbitrary unions, as the other properties of a topology are easily seen to be satisfied by τ^c. So, let $\{A_\alpha : \alpha \in \Delta\} \subset \tau^c$ be any family of sets of τ^c. We consider

$$(\bigcup_{\alpha\in\Delta} A_\alpha)^c = \bigcap_{\alpha\in\Delta} A_\alpha^c \quad . \ . \ . \ . \ . \ . \ . \ (1)$$

Clearly $A_\alpha^c \in \tau$, for all $A_\alpha \in \tau^c$. Since τ is, by hypothesis, closed under arbitrary intersections $\bigcap_{\alpha\in\Delta} A_\alpha^c \in \tau$. Hence the left side of (1) is an element of τ; implying that $[(\bigcup_{\alpha\in\Delta} A_\alpha)^c]^c = (\bigcup_{\alpha\in\Delta} A_\alpha) \in \tau^c$.

From theorem 4.16, it follows that every discrete topology is a complement topology; and in particular it follows that discrete topologies of infinite sets (which necessarily contain infinitely many open sets) are complement topologies. And there are other complement topologies, with infinitely many open sets, which are not discrete topologies.

Lemma 4.10 (Comparison) *Let τ_1 and τ_2 be any two complement topologies on a set X such that (say) τ_1 is weaker than τ_2. Then τ_1^c is weaker than τ_2^c.*

4.3.2 Application of the Comparison Lemma

Definition 4.42 *A family $C = \{\tau_\alpha\}_{\alpha\in\Delta}$ of topologies on a set, X, is called a chain of topologies, on X, if elements of C are pair-wise comparable, in* *that for any two topologies, τ_α and τ_r, in C, either τ_α is weaker than τ_r or vice versa.*

Theorem 4.17 *Let $X = \{x_1, x_2, x_3, \cdots, x_n\}$ be any non-empty finite set. There exists a finite family of topologies on X forming a chain, such that the family of their complement topologies is also a chain.*

Proof:
Let

$$G_0 = X;$$

$$G_1 = X - \{x_1\} = \{x_2, x_3, \cdots, x_n\}.$$

Then $\tau_1 = \{\emptyset, G_0, G_1\}$ is a topology on X.
Let

$$G_0 = X;$$

$$G_1 = X - \{x_1\} = \{x_2, x_3, \cdots, x_n\};$$

$$G_2 = X - \{x_1, x_2\} = \{x_3, x_4, \cdots, x_n\}.$$

Then $\tau_2 = \{\emptyset, G_k\}_{k=0}^{2}$ is a topology on X, stronger than τ_1.

$$\vdots$$

Let

$$G_0 = X;$$

$$G_1 = X - \{x_1\} = \{x_2, x_3, \cdots, x_n\};$$

$$G_2 = X - \{x_1, x_2\} = \{x_3, x_4, \cdots, x_n\};$$

$$G_3 = X - \{x_1, x_2, x_3\} = \{x_4, x_5, \cdots, x_n\};$$

$$\vdots$$

$$G_k = X - \{x_1, x_2, \cdots, x_k\} = \{x_{k+1}, x_{k+2}, \cdots, x_n\},$$

$1 \leq k \leq n$. Then $\tau_k = \{\emptyset, G_t\}_{t=0}^{k}$ is a topology on X finer than τ_{k-1}. Hence $\{\tau_k\}_{k=1}^{n}$ is a (finite) family of topologies on X forming a chain in that

$$\tau_1 < \tau_2 < \cdots < \tau_n.$$

We also see that

$\tau_1^c = \{\emptyset, X, \{x_1\}\}$;

$\tau_2^c = \{\emptyset, X, \{x_1\}, \{x_1, x_2\}\}$, etc.

are topologies (in chain) on X.

Proof of Theorem 4.17 can be extended to any set—even if infinite—with a chain of complement topologies (using lemma 4.10). The next corollary states this.

Corollary 4.9 *Let $C = \{\tau_\alpha\}_{\alpha \in \Delta}$ be a chain of complement topologies on any set X. Then the family $C^* = \{\tau_\alpha^c : \tau_\alpha \in C\}_{\alpha \in \Delta}$ of complements of the topologies in C is also a chain of complement topologies on X. Conversely, the family of the complements of the topologies in a chain of complement topologies on any set X is itself also a chain of complement topologies on X.*

MORE EXAMPLES

EXAMPLE 4.66

The usual topology u on the set $\mathbb{R}$ of real numbers is not closed under arbitrary intersections and is thus not a complement topology.

EXAMPLE 4.67

The usual topology on the Cartesian plane is not closed under arbitrary intersections and is, hence, not a complement topology.

EXAMPLE 4.68

The lower limit (or Sorgenfrey) topology on $\mathbb{R}$ is not closed under arbitrary intersections and is also not a complement topology.

EXAMPLE 4.69

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set, and let $k \in \mathbb{N}$ be such that $2k-1 \leq n$. Then

$$\tau_{2k-1} = \{\emptyset, X, \{x_1\}, \{x_1, x_3\}, \cdots, \{x_1, x_3, \cdots, x_{2k-1}\}\}$$

is a topology on X, for $1 \leq k \leq \left[\frac{n+1}{2}\right]$. We see also that

$$\tau_{2k-1}^c = \{X, \emptyset, \{x_2, \cdots, x_n\}, \{x_2, x_4, \cdots, x_n\}, \cdots, \{x_2, x_4, \cdots, x_n\}\}$$

is a topology on X.

EXAMPLE 4.70

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set, and let $k \in \mathbb{N}$ be such that $2k-1 \leq n$. Then $\tau_{2k-1} = \left\{\emptyset, X, \bigcup_{t=1}^{k} \{x_{2t-1}\}\right\}$ is a topology on X, for $1 \leq k \leq \left[\frac{n+1}{2}\right]$. Also $\tau_{2k-1}^c = \left\{X, \emptyset, X - \bigcup_{t=1}^{k} \{x_{2t-1}\}\right\}$ is a topology on X; $1 \leq k \leq \left[\frac{n+1}{2}\right]$.

EXAMPLE 4.71
Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set, $n = mt + r, 0 \leq r < m$. Let $\tau_m = \{\emptyset, X, \{x_m\}, \{x_m, x_{2m}\}, \cdots, \{x_m, x_{2m}, \cdots, x_{tm}\}\} = \left\{\emptyset, X, \bigcup_{i=1}^{k} \{x_{im}\}\right\}; 1 \leq k \leq t$. Then τ_m is a topology on X. And we see that $\left\{X, \emptyset, X - \bigcup_{i=1}^{k} \{x_{im}\}\right\}; 1 \leq k \leq t$ is a topology on X.
REMARK 4.18
It is known that a topological space (X, τ) is a T_1-space if, and only if, singletons are τ-closed subsets of X. It is observable from the foregoing that if a topology τ on a set X is a complement topology then the very sets which are seen as τ-closed are the sets which constitute the open sets of another topology on X, with equal cardinality as τ. These imply the following.

Corollary 4.10 *If (X, τ) is a T_1 topological space, then τ is a complement topology if, and only if, τ is the discrete topology of X.*

Proof:
Since (X, τ) is T_1, singletons of X are τ-closed. Since τ is a complement topology on X and singletons of X are τ-closed, it follows that singletons are among the τ^c-open sets. Hence every subset of X is τ^c-open, implying that τ^c is the discrete topology of X. Since $(\tau^c)^c = \tau$, it follows that τ is the discrete topology of X.

REMARK 4.19
That a topology is a complement topology does not imply that it is T_1. Also, every T_1-space is not a complement topology. By Corollary 4.10, a T_1-space which is a complement topology must be a discrete topology. It follows that if a T_1-space is not discrete then it cannot be a complement topology. For example, the set $\mathbb{R}$ of real numbers with its usual topology u is T_1 but u is not a complement topology. Hence all complement topologies are not T_1 and all T_1-spaces are not complement topologies.

One axiom of a generic topology τ is that τ is closed under finite intersections. If a topology τ is closed under arbitrary intersections (hence a complement topology) then we may theoretically feel more at ease to work with it; since then we would not really care so much about the type of intersections of τ-open sets we may have before us. Therefore we may now pose the following question: Since every topology is not a complement topology, can we find for each topology τ another topology τ_s which is a complement topology so specially related to τ that it (τ_s) would coincide with τ if τ was a complement topology, and (if τ was not a complement topology) no other complement topology would be strictly stronger than τ and strictly weaker than τ_s? Among other things, the following expositions shall establish the answer to this question.

4.3.3 The Supra of a Topology—Constructive Approach

Construction 4.7 *Let (X, τ) be a topological space and let $S = \{B = \bigcap_{\alpha \in \Delta} A_\alpha : A_\alpha \in \tau\}$ be the collection of arbitrary intersections of τ-open sets. Let $\tau_s = \{U = \bigcup_{\delta \in \omega} B_\delta : B_\delta \in S\}$ be the collection of all unions (i.e. any possible union) of the sets in S.*

Then we have the following observations.

1. Every τ-open set is in S and hence in τ_s; that is, $\tau \subset S \Rightarrow \tau \subset \tau_s$.

2. Hence $\emptyset \in \tau_s$ and $X \in \tau_s$.

3. Any union of sets in τ_s is simply a union of sets in S and is hence a set in τ_s.

4. Sets U in τ_s are of the form $U = \bigcup B$, $B \in S$ and $B = \bigcap_{\alpha \in \Delta} A_\alpha$, where $A_\alpha \in \tau$. Therefore any intersection of the sets in τ_s is a union of intersections of some sets $A_\alpha \in \tau$. Since the intersections of $A_\alpha \in \tau$ are collected in S, it follows that any intersection of the sets in τ_s is a union of some sets in S and is, hence, in τ_s. That is

$$\bigcap_{r \in w} U_r = \bigcup B,\ B \in S.$$

   Hence $\bigcap_{r \in w} U_r \in \tau_s$; implying that τ_s is closed under arbitrary intersections.

What we have proved is the following.

Theorem 4.18 (The Supra-topology Theorem) *Let (X, τ), S and τ_s be as introduced above. Then*

1. τ_s is a topology on X;

2. *τ is weaker than τ_s;*

3. *τ_s is closed under arbitrary intersections and hence by theorem 4.16 is a complement topology on X;*

4. *If τ was closed under arbitrary intersections, hence a complement topology itself, then $\tau = S = \tau_s$;*

5. *If τ is not a complement topology, then τ is strictly weaker than τ_s.*

Note
We observe that τ_s is well defined and is uniquely related to τ. This assertion will be proved formally in theorem 4.19 below.

Definition 4.43 *Let (X, τ) and τ_s be as introduced above. Then we shall call τ_s the supra of the topology τ on X, because τ is weaker than, and specially related to τ_s on X.*

OBSERVATIONS

1. To every topology τ on a set X there corresponds a supra topology τ_s (with $\tau \leq \tau_s$) on X.

2. Every supra topology τ_s is a complement topology, since τ_s is closed under arbitrary intersections.

3. $\tau = \tau_s$ if, and only if, τ is a complement topology. In fact, τ_s is the weakest complement topology finer than τ. (see Theorem 4.19 below) Hence τ_s is the complement topology generated by τ.

4. Every τ_s-open set is either an intersection of τ-open sets or a union of such intersections.

5. Every complement topology is a supra topology. (Proved next in the proposition below)

Proposition 4.18 *Every complement topology on a set X is a supra topology.*

Proof:
Let τ on a set X be a complement topology. Then (from Theorem 4.16) τ is closed under arbitrary intersections. It follows that $S = \{B = \bigcap\limits_{\alpha \in \Delta} A_\alpha : A_\alpha \in$

$\tau\} = \tau$. Therefore $\tau_s = \{U = \bigcup_{\delta \in \omega} B_\delta : B_\delta \in S\} = \tau$. That is, τ is equal to its own supra and is, hence, a supra topology.

EXAMPLE 4.72
Every discrete topology is a supra-topology.
EXAMPLE 4.73
Each indiscrete topology is a supra-topology.
EXAMPLE 4.74
Any topology that has only a finite number of open sets (hence every topology on a finite set) is a supra-topology.
EXAMPLE 4.75
The usual topology u of the set $\mathbb{R}$ of real numbers is not a supra-topology.
EXAMPLE 4.76
The lower limit (or Sorgenfrey) topology on $\mathbb{R}$ is not a supra-topology.
EXAMPLE 4.77
The usual topology of the Cartesian plane is not a supra-topology.
EXAMPLE 4.78
The supra u_s of the usual topology u of the set $\mathbb{R}$ of real numbers is the discrete topology of $\mathbb{R}$. This is because the S for u_s has singletons as elements and u_s is closed under arbitrary unions.

Theorem 4.19 *τ_s is the smallest complement topology, on X, stronger than τ.*

Proof:
Let γ be another complement topology stronger than τ but strictly weaker than τ_s on X. If $\tau = \gamma$, implying that τ is itself a complement topology, it would follow from theorem 4.18 (4) that $\tau_s = \tau = \gamma$, a contradiction. Hence τ must be strictly weaker than γ. That is

$$\tau < \gamma < \tau_s. \dots\dots\dots (*)$$

It follows from $(*)$ that the family $S = \{B = \bigcap_{\alpha \in \Delta} A_\alpha : A_\alpha \in \tau\}$ is a proper subfamily of the family $S_1 = \{B = \bigcap_{\alpha \in \Delta} A_\alpha : A_\alpha \in \gamma\}$. Hence again the family $\{\bigcup B : B \in S\} = \tau_s$ is a subfamily of the family $\{\bigcup B : B \in S_1\} = \gamma$ (this equality being true because γ is, by hypothesis, a complement topology). That is, τ_s is a subfamily of γ; a contradiction of $(*)$. Hence the proof is complete. Hence the supra τ_s of a topology τ is unique and is the smallest complement topology stronger than τ.

In fact, it is also true that no topology γ on a set X, exists between a topology τ and its supra τ_s if $\gamma_s \neq \tau_s$. This is proved next.

Proposition 4.19 *No other topology, say γ, on a set X, can strictly exist between a topology τ on X and its supra τ_s and have a different supra γ_s on X.*

Proof:
Let γ be another topology stronger than τ but strictly weaker than τ_s on X, and let γ_s be the supra of γ. We need to show that $\gamma_s = \tau_s$. Let

$$S = \{B = \bigcap_{\alpha \in \Delta} A_\alpha : A_\alpha \in \tau\} \text{ and } S_1 = \{B = \bigcap_{\alpha \in \Delta} A_\alpha : A_\alpha \in \gamma\}.$$

Then the supra τ_s of τ is

$$\tau_s = \{\bigcup_{\delta \in \omega} B_\delta : B_\delta \in S\}$$

and the supra γ_s of γ is

$$\gamma_s = \{\bigcup_{\delta \in \omega} B_\delta : B_\delta \in S_1\}.$$

If $\gamma_s < \tau_s$ we have a contradiction since τ_s is built up from τ and $\tau < \gamma$. If $\tau_s < \gamma_s$ we again have a contradiction, as γ is a proper subfamily of τ_s. Hence $\gamma_s = \tau_s$.

EXAMPLE 4.79
Let $(\mathbb{R}, u)$, $\mathbb{R}, l)$ and $(\mathbb{R}, d)$ denote the usual topological space of $\mathbb{R}$, the lower limit topological space of R and the discrete topological space of $\mathbb{R}$, respectively. Then $u < l < d$ and $u_s = l_s = d$.

Note: The supra of a topology and the complement of the topology are not the same. The supra will always be a topology while the complement may not be a topology. For example the usual topology u of $\mathbb{R}$ is such that u^c is not a topology but u_s is a topology, the discrete topology of $\mathbb{R}$.

4.3.4 The Supra of a Topology—Analytic Approach

So far, the foregoing process of developing and/or defining the supra τ_s of a topology τ on a set X may be said to be *constructive* in nature. Let us now explore an alternative approach which may rightly be seen as *analytic*.

Construction 4.8 *Let* (X, τ) *be any topological space and let* $\Psi = \{\gamma : \gamma$ *is a complement topology on* X *and* τ *is weaker than* $\gamma\}$.

Then

1. Ψ is nonempty, since at least Ψ contains the discrete topology of X which, as we have seen, is an easy example of complement topology; and every other topology on X is weaker than its discrete topology.

2. If τ is itself a complement topology on X, then $\tau \in \Psi$.

3. If τ is itself not a complement topology on X, then $\tau \notin \Psi$.

Theorem 4.20 *Let* (X, τ) *and* Ψ *be as introduced above. Let* $\tau_s = \bigcap_{\gamma \in \Psi} \gamma$ *be the intersection of all the topologies in* Ψ*. Then*

1. τ_s *is a complement topology on* X*, and* τ *is weaker than* τ_s*.*

2. *If* τ *is itself a complement topology on* X*, then* $\tau = \tau_s$*.*

3. *If* τ *is not a complement topology on* X*, then* τ *is strictly weaker than* τ_s*.*

4. *Hence* τ_s *is the supra of the topology* τ *on* X*.*

Proof:

1. It is clear that τ_s is a topology on X, being an intersection of topologies on X. We only need to show here that τ_s is closed under arbitrary intersections. So, let $\{A_\alpha\}_{\alpha \in \Delta} \subset \tau_s$ be any family of sets in τ_s; that is, τ_s-open sets. Then $\{A_\alpha\}_{\alpha \in \Delta} \subset \gamma$, for each $\gamma \in \Psi$. Since each $\gamma \in \Psi$ is a complement topology, it follows that $\bigcap_{\alpha \in \Delta} A_\alpha \in \gamma$, for each $\gamma \in \Psi$. Hence $\bigcap_{\alpha \in \Delta} A_\alpha \in \bigcap_{\gamma \in \Psi} \gamma = \tau_s$. Therefore τ_s is a complement topology on X.

2. Clearly if τ is itself a complement topology on X, then $\tau \in \Psi$ and $\bigcap_{\gamma \in \Psi} \gamma = \tau_s = \tau$.

3. Also if τ is not a complement topology, then $\tau \notin \Psi$ and, hence, τ is strictly weaker than τ_s.

4. All the results in 1 to 3 show that τ_s is the supra of the topology τ on X since the supra of a topology is unique.

Corollary 4.11 *The supra of an infinite dimensional product topology is a power set discrete topology if all the factor spaces are power set discrete topologies.*

NOTE
The outcome of the constructive process of subsection 4.3.3 corroborates (and is corroborated by) the analytic process of subsection 4.3.4 on the question of existence and uniquess of supra-topologies.

4.3.5 Exhaustive Topologies

Definition 4.44 *Let (X, τ) be a topological space. The topology τ is called an exhaustive topology on X if for each x in X there exists a τ-open proper subset of X[11] containing x. By notations, τ is exhaustive on X if $\forall x \in X$, $\exists G \in \tau$ such that $G \neq X$ and $x \in G$.*

EXAMPLE 4.80
Let $X = (-\infty, 0) \bigcup (0, +\infty) = \mathbb{R} - \{0\}$. Then any topology induced on $\mathbb{R}$ by X (i.e. X-topology on $\mathbb{R}$) cannot be exhaustive on $\mathbb{R}$ since no open proper subset of $\mathbb{R}$ in such a topology would contain the element 0.
EXAMPLE 4.81
Every discrete topology is exhaustive since for each $x \in X$, the singleton $\{x\}$ is an open proper subset of X and contains x.
EXAMPLE 4.82
Let $(\mathbb{R}, u)$ be the ususal topological space of $\mathbb{R}$. Then u is exhaustive on $\mathbb{R}$ since for each real number x, the interval $(x-1, x+1)$ is u-open and contains x.
EXAMPLE 4.83
Let $(\mathbb{R}, L)$ denote $\mathbb{R}$ with the lower limit topology L. Then L is exhaustive on $\mathbb{R}$.

Lemma 4.11 *Let (X, τ) be a topological space. If the supra τ_s of the topology τ is discrete, then τ is exhaustive on X. Conversely, if τ is not exhaustive then τ_s is not (or cannot be) discrete.*

[11] If X is itself a singleton then the only possible topologies on X are its discrete and indiscrete topologies—which are, in this case, coincident—and we shall then say that τ on X is exhaustive since every other discrete topology is exhaustive. Since a closer look also shows that the indiscrete topology is not exhaustive on any set, we may still say that any topology on a singleton is both exhaustive and non-exhaustive. For this, we shall assume that our topologies in further discussions are not defined on singletons.

Proof:
Recall: Each τ_s-open set is either an intersection of τ-open sets or a union of such intersections.
Let τ_s be discrete. Then τ_s is discrete if, and only if, each singleton $\{x\}$ is τ_s-open. Since a singleton cannot emerge as the union of two or more distinct sets (of different elements) we assume that singletons in τ_s emerged from intersections of τ-open sets. If $\{x\}$ emerges as the intersection of any number of sets, then the element x of X must belong to (at least) one proper subset of the set X—except of course the universal set X itself is a singleton, a case which has automatically been ruled out of our considerations. All these imply that each element x of X is contained in some τ-open proper subset of X; implying that τ is exhaustive on X.

Now, we prove the lemma by converse. Suppose (X, τ) is not exhaustive. We show that τ_s is not the discrete topology of X. Our assumption (that τ is not exhaustive) implies that there exists $x_0 \in X$ such that no τ-open, proper subset of X contains x_0. This further implies that the singleton $\{x_0\}$ cannot emerge from the intersection of any number of τ-open sets. (That is, $\{x_0\}$ is not in the subbase S for τ_s.) This proves that $\{x_0\} \notin \tau_s$; implying that τ_s is not the discrete topology of X.

REMARK 4.20
We know that two or more different topologies on one set can have one supra topology in common. The following two theorems state that even when this is so, and provided that the common-supra topologies are actually different from one another, these topologies nevertheless generate <u>other</u> distinct supra topologies in number as many as the topologies with common supra.

Theorem 4.21 *Let τ_1 and τ_2 be two non-exhaustive and non-complement topologies on X such that τ_1 is strictly weaker than τ_2. Then the supra τ_{s_1} of τ_1 is strictly weaker than the supra τ_{s_2} of τ_2.*

Proof:
Let

$$S_1 = \{\bigcap A_\alpha : A_\alpha \in \tau_1\} \text{ and } S_2 = \{\bigcap A_\alpha : A_\alpha \in \tau_2\}.$$

Then the supra τ_{s_1} of τ_1

$$\tau_{s_1} = \{\bigcup B_\delta : B_\delta \in S_1\} \text{ and the supra of } \tau_2 \text{ is } \tau_{s_2} = \{\bigcup B_\delta : B_\delta \in S_2\}.$$

Since τ_1 is strictly weaker than τ_2, and τ_2 is not a complement topology, $\tau_2 \neq \tau_{s_2}$ and also τ_2 contains at least one more set, say G, than τ_1. Hence S_2 contains at least one more set than S_1. It follows that τ_{s_2} contains at least one more set than τ_{s_1}. Since S_1 is a subfamily of S_2, τ_{s_1} a subfamily of τ_{s_2}, and the supra of τ_1 and τ_2 are, respectively, τ_{s_1} and τ_{s_2}, the proof is complete.

NOTE
[**1**] In this theorem τ_1 can be a complement topology but τ_2 should not be a complement topology.
[**2**] If τ_2 is a complement topology, then the supra of τ_2 does not get bigger than τ_2. And since the supra of τ_1 is an extension of τ_1 (when τ_1 is non-complement topology), the supra of τ_1 may coincide with the supra of τ_2 if τ_2 is a complement topology—though this will not always be the case as it depends on how far below τ_2 is τ_1.

EXAMPLE 4.84
Let $(\mathbb{R}, \tau_1)$ be a topological space such that each $r \neq 0$ has the usual Euclidean neighborhoods that do not contain 0, and the only open set containing 0 is the whole space $\mathbb{R}$. Let $(\mathbb{R}, \tau_2)$ be a topological space such that each $r \in R$ with $r \neq 0$ is an isolated point, and the only open set containing 0 is the whole space $\mathbb{R}$. Then τ_1 is strictly weaker than τ_2, and these spaces are not exhaustive because of 0. The supra τ_{s_1} of τ_1 is τ_2, and the supra τ_{s_2} of τ_2 is also τ_2. Therefore τ_{s_1} is not strictly weaker than τ_{s_2}. This happened because (a) τ_2 is a complement topology and hence equal to its own supra; and (b) τ_1 is very close in size as a topology (landscape) to τ_2. If τ_1 is not very close to τ_2, then τ_{s_1} would still be strictly weaker than τ_{s_2} even if τ_2 were a complement topology. The next example illustrates this.

EXAMPLE 4.85
Let $(\mathbb{R}, \tau_1)$ be a topological space such that each $r \in \mathbb{R}$ with $r \notin [-1, 1]$ has the usual Euclidean neighborhoods that do not contain points of $[-1, 1]$, and the only open set containing points of $[-1, 1]$ is the whole space $\mathbb{R}$. Let $(\mathbb{R}, \tau_2)$ be exactly as in example 4.84 above. Then $\tau_1 < \tau_2$ and these spaces are not exhaustive for obvious reasons. The interval (0,1) is open in the supra τ_{s_2} of τ_2 but this interval is not open in the supra (now) of τ_1. Hence $\tau_{s_1} < \tau_{s_2}$.

Theorem 4.22 *To every two incomparable topologies on a set X there* correspondingly *exist two incomparable complement topologies on X.*

Proof:
Let γ and η be two incomparable topologies on X. Then there exist $G_\gamma \in \gamma$ and $G_\eta \in \eta$ such that $G_\gamma \notin \eta$ and $G_\eta \notin \gamma$. Let

$$\tau(G_\gamma) = \{G \in \gamma : G \subset G_\gamma\} \bigcup \{X\} \text{ and } \tau(G_\eta) = \{G \in \eta : G \subset G_\eta\} \bigcup \{X\}.$$

Then both $\tau(G_\gamma)$ and $\tau(G_\eta)$ are topologies on X. Also $G_\gamma \in \tau(G_\gamma)$ and $G_\eta \in \tau(G_\eta)$ with $G_\gamma \notin \tau(G_\eta)$ and $G_\eta \notin \tau(G_\gamma)$. Hence these two topologies, on X, are not comparable. Let

$$S_1 = \{\bigcap A_\alpha : A_\alpha \in \tau(G_\gamma)\} \text{ and } S_2 = \{\bigcap A_\alpha : A_\alpha \in \tau(G_\eta)\}.$$

Then S_1 and S_2 are not comparable, as families of sets, since at least they are made distinct by G_γ and G_η. Hence the supra $\tau(G_\gamma)_s$ of $\tau(G_\gamma)$ given by $\tau(G_\gamma)_s = \{\bigcup B : B \in S_1\}$ is distinct from, and incomparable to the supra $\tau(G_\eta)_s$ of $\tau(G_\eta)$ given by $\tau(G_\eta)_s = \{\bigcup B : B \in S_2\}$—because for instance $G_\eta \in \tau(G_\eta)_s$ but $G_\eta \notin \tau(G_\gamma)_s$ and $G_\gamma \in \tau(G_\gamma)_s$ but $G_\gamma \notin \tau(G_\eta)_s$. Since the supra of each topology is a complement topology, the proof is complete.

1. The meaning of Theorem 4.21 is this: If two topologies are distinct, comparable and non-exhaustive, they generate two distinct and comparable complement topologies provided the stronger topology is not a complement topology.

2. The meaning of Theorem 4.22 is this: Provided two topologies are incomparable, even if they have common supra, they nevertheless generate two other incomparable complement topologies—which will not be their supras.

3. Theorem 4.22 necessarily also shows how to construct the distinct complement topologies as illustrated in the example below.

EXAMPLE 4.86
Let $\tau_l = \{[a, b) : a, b \in \mathbb{R}\}$ be the lower limit topology on $\mathbb{R}$ and $\tau_u = \{(a, b] : a, b \in \mathbb{R}\}$ the upper limit topology on $\mathbb{R}$. Then τ_l and τ_u are incomparable and have common supra—the discrete topology of $\mathbb{R}$. Any two topologies induced on $\mathbb{R}$ by the two subintervals $[a, b)$ and $(a, b]$ will have their supras incomparable since such supras will contain the subintervals $[a, b)$ and $(a, b]$ separately.

Theorem 4.23 *Let X be any nonempty and non-singleton set and let $X_1 = X - \{x_1\}$, where $x_1 \in X$. Then any topology induced on X by X_1 cannot be exhaustive; hence the supra of such a topology cannot be discrete.*

Proof:
Let τ be an X_1-topology on X. Then (since $x_1 \notin X_1$) no τ-open proper subset of X contains the element x_1 of X, as all τ-open proper subsets of X are subsets of X_1. Hence τ is not exhaustive and (by Lemma 4.11) the supra τ_s of τ is hence not discrete.

REMARK 4.21
Theorem 4.23 can be seen like this: If (X, τ) is a topological space and there

exists $x_0 \in X$ such that no τ-open proper subset of X contains x_0, then τ_s, the supra of τ, is not discrete. This may be seen as an alternative (to Lemma 4.11) way of proving that only an exhaustive topology can have discrete supra topology. These results can be used to know exactly whether the supra of a topology can or cannot be discrete.

So far, we have been notably silent on the converse of Lemma 4.11. Namely, if a topology τ on X is exhaustive, is the supra τ_s of τ *necessarily* the discrete topology on X? Answer: No. The supra of an exhaustive topology may not be discrete. For example, let $X = \{a, b, c, d\}$ and let τ on X be defined as $\tau = \{\emptyset, X, \{a, b\}, \{c, d\}\}$. Then τ is an exhaustive topology on X, since every element of X is contained in a τ-open proper subset of X. The supra τ_s of τ is equal to τ itself; and since τ is clearly not the discrete topology of X, this explanation is complete.

4.3.6 The Supra of a Weak Topology

Let X be a nonempty set. For each $\alpha \in \Delta$, let $f_\alpha : X \to (X_\alpha, \tau_\alpha)$ be a map of X into a topological space (X_α, τ_α). Let $\{f_\alpha\}_{\alpha\in\Delta}$ be the family of such maps of X into various topological spaces $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}$, and let $S = \{f_\alpha^{-1}(G_\alpha) : G_\alpha \in \tau_\alpha\}$ be the collection of inverse images of only (but all) the open sets of the, topological, range spaces. Let $B = \{\bigcap_\alpha f_\alpha^{-1}(G_\alpha) : f_\alpha^{-1}(G_\alpha) \in S\}$ be the collection of arbitrary intersections (across the range spaces) of the sets in S, in that f_α must vary within the family $\{f_\alpha\}_{\alpha\in\Delta}$ of functions on X. Let $\tau_b = \{\bigcup B_\delta : B_\delta \in B\}$ be the collection of arbitrary unions of the sets in B.

REMARK 4.22

1. If $X = \prod_{\alpha\in\Delta} X_\alpha$ is a Cartesian product of the family of range (topological) spaces, and the family $\{f_\alpha\}_{\alpha\in\Delta} = \{p_\alpha\}_{\alpha\in\Delta}$ of functions is the projection maps, then τ_b is nothing but what has been known before as the box topology. This time too the corresponding weak topology, say τ on X, is called product topology on X.

2. If X is not a Cartesian product set and (hence) the family $\{f_\alpha\}_{\alpha\in\Delta}$ of functions is not the projection maps, τ_b is still a topology on X, and we shall continue the use of the nomenclature already developed and simply then call τ_b the (generalized) box topology, on X, corresponding to, or generated by the family $\{f_\alpha\}_{\alpha\in\Delta}$ of functions. We shall make a

difference by specifying that τ_b is generated by the projection maps $\{p_\alpha\}_{\alpha\in\Delta}$ when the old box topology is assumed.

DEVELOPMENTS

Now let

$$D = \left\{ \bigcap_\alpha f_\alpha^{-1}(G_\alpha), \bigcap_r f_\alpha^{-1}(G_{\alpha_r}), \bigcap_\alpha f_\alpha^{-1}(\bigcap_r G_{\alpha_r}) : G_\alpha, G_{\alpha_r} \in \tau_\alpha \right\}$$

be the collection of all possible intersections[12] of the sets in S. And let $\tau_s = \{\bigcup_\delta D_\delta : D_\delta \in D\}$ be the collection of arbitrary unions of sets in D. Then

1. $\emptyset$ and X are both in τ_s.

2. τ_s is closed under arbitrary unions.

3. The intersection of any number of sets in τ_s is a union of some sets in D and is, hence, in τ_s. That is, τ_s is a topology on X, closed under arbitrary intersections. Since both τ_s and the weak topology τ on X have the same subbase S, it follows that τ_s is the supra of the weak topology τ on X.

Theorem 4.24 *Let τ, τ_b and τ_s, respectively, be the weak topology, the box topology, and the supra topology on X, generated by the family $\{f_\alpha\}_{\alpha\in\Delta}$ of functions. Then we have the following observations.*

1. *τ is always weaker than τ_b, and τ_b is always weaker than τ_s. That is, $\tau \leq \tau_b \leq \tau_s$.*

2. *If the number of range spaces $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}$, hence the family of functions, is finite, then $\tau = \tau_b$; and both will equal τ_s if, and only if, the topology τ_α of each of the range spaces is closed under arbitrary intersections.*

3. *If the family $\{f_\alpha\}_{\alpha\in\Delta}$ of functions is infinite, then τ will be strictly weaker than τ_b (i.e. $\tau < \tau_b$); and then τ_b will equal τ_s if, and only if, each τ_α is closed under arbitrary intersections.*

4. *If the family $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}$ of range spaces is infinite, and τ_α is not a complement topology (i.e. not closed under arbitrary intersections), for some $\alpha \in \Delta$, then τ is strictly weaker than τ_b and τ_b is strictly weaker than τ_s. That is, $\tau < \tau_b < \tau_s$.*

[12] Contrast this with the fact that the so-called arbitrary intersections collected in B, for τ_b, were taken only *across the range spaces.*

5. *If the family* $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}$ *of range spaces is finite, and* τ_α *is not closed under arbitrary intersections, for some* $\alpha \in \Delta$*, then both* τ *and* τ_b *will be strictly weaker than* τ_s*, and equal to each other (i.e.* $\tau = \tau_b < \tau_s$*).*

6. *If the family* $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}$ *of range spaces is finite, and each of the topologies* τ_α *is closed under arbitrary intersections, then the three topologies* τ*,* τ_b *and* τ_s *will coincide; i.e.* $\tau = \tau_b = \tau_s$*.*

7. *If the family* $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}$ *of range spaces is infinite, and* τ_α *is closed under arbitrary intersections, for each* $\alpha \in \Delta$*, then both* τ_b *and* τ_s *will coincide, and be (both) strictly stronger than* τ*. That is,* $\tau < \tau_b = \tau_s$*.*

Proof:

1. Since τ, τ_b, and τ_s all have the same subbase S, $S \subset \tau$, $S \subset \tau_b$, and $S \subset \tau_s$. That is, $S \subset \tau \bigcap \tau_b \bigcap \tau_s$. Each of τ, τ_b, and τ_s is closed under arbitrary unions of (particularly) the sets in S. The differences between τ, τ_b, and τ_s, if any, lie in how the base of each of these topologies is formulated as intersections of the sets in S. So, let B_τ be the base for τ. Then as seen already, B (as given above) is the base for τ_b, and D is the base for τ_s. We know that B_τ contains all finite intersections of sets in S, and B contains in addition to this the arbitrary intersections $\bigcap\limits_\alpha f_\alpha^{-1}(G_\alpha)$, where f_α varies over the family of functions. Hence $\tau \leq \tau_b$.
And $\tau = \tau_b$ if, and only if, the range spaces $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}$ and/or the family $\{f_\alpha\}_{\alpha\in\Delta}$ of functions is finite.
And D contains, in addition to elements of B, the arbitrary intersections (of sets of S) of the form $\bigcap\limits_r f_\alpha^{-1}(G_{\alpha_r}) = f_\alpha^{-1}[\bigcap\limits_r(G_{\alpha_r})]$, obtained by holding a function f_α fixed and varying the open sets of its range space; and D also contains such intersections as $\bigcap\limits_\alpha f_\alpha^{-1}(\bigcap\limits_r G_{\alpha_r})$ obtained by taking arbitrary intersections of open sets of the range spaces and varying the functions across these range spaces. These show that $\tau_b \leq \tau_s$. That is, $\tau \leq \tau_b \leq \tau_s$.

2. We have seen that $\tau = \tau_b$ if, and only if, the range spaces $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}$ and/or the family $\{f_\alpha\}_{\alpha\in\Delta}$ of functions is finite. If (in this case) τ_α is not closed under arbitrary intersections, for some $\alpha \in \Delta$, then it follows that $\bigcap\limits_r(G_{\alpha_r}) \notin \tau_\alpha$, if this intersection is infinite. This implies that $f_\alpha^{-1}[\bigcap\limits_r(G_{\alpha_r})] \notin \tau$ and (as $\tau = \tau_b$) $f_\alpha^{-1}[\bigcap\limits_r(G_{\alpha_r})] \notin \tau_b$. Since $\bigcap\limits_r f_\alpha^{-1}(G_{\alpha_r}) = f_\alpha^{-1}[\bigcap\limits_r(G_{\alpha_r})] \in \tau_s$ and $\bigcap\limits_r f_\alpha^{-1}(G_{\alpha_r}) = f_\alpha^{-1}[\bigcap\limits_r(G_{\alpha_r})] \notin \tau = \tau_b$, it follows

that both τ and τ_b are (not only equal but) strictly weaker than τ_s. That is, $\tau = \tau_b < \tau_s$.

3. If $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}$ is finite and each τ_α is a complement topology (i.e. closed under arbitrary intersections), then, from 2, $\tau = \tau_b = \tau_s$.

4. If $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}$ is infinite and each τ_α is a complement topology, then it is easy to see from 1 and 3 that $\tau_b = \tau_s$, and that τ is strictly weaker than both τ_b and τ_s. That is, $\tau < \tau_b = \tau_s$.

5. If $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}$ is infinite and, for some $\alpha \in \Delta$, τ_α is not a complement topology, then from the foregoing, $\tau < \tau_b < \tau_s$.

All the possible scenarios above are summarized in the following list.

1. In general $\tau \leq \tau_b \leq \tau_s$;

2. We can have $\tau = \tau_b < \tau_s$;

3. Or we have $\tau < \tau_b = \tau_s$;

4. Or we have $\tau < \tau_b < \tau_s$;

5. Or we have $\tau = \tau_b = \tau_s$.

EXAMPLE 4.87
Let $(\mathbb{R}, u)$ be the set $\mathbb{R}$ of real numbers with its usual topology. Then for each $n \in \mathbb{N}$, the interval $G_n = (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$ is u-open. We see that $\bigcap\limits_{n=1}^{\infty} G_n = \bigcap\limits_{n=1}^{\infty} (x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) = \{x_0\} \notin u$. Hence for projection maps,

$$\bigcap_{n=1}^{\infty} p_k^{-1}[(x_0 - \tfrac{1}{n}, x_0 + \tfrac{1}{n})] = p_k^{-1}\{x_0\} \ . \ . \ . \ .(*)$$

In $\mathbb{R}^2$, with the factor spaces given their usual topology, $(*)$ is an infinite (vertical or horizontal) line. Clearly the box topology τ_b on $\mathbb{R}^2$—even as it used to be known only in connection with projection maps on Cartesian product sets—does not contain $(*)$ when the factor spaces have the usual topology of $\mathbb{R}$. However, it is clear that $(*)$ is in τ_s; and in this case $\tau = \tau_b < \tau_s$, exemplifying item 2 of the summarized listing above.

EXAMPLE 4.88
Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ be the set of natural numbers. Let $G_0 = \mathbb{N}$, $G_1 = \mathbb{N} - \{0\}$, $G_2 = \mathbb{N} - \{0, 1\}$, etc. Then the semi-cofinite topology $C_{\mathbb{N}} = \{\emptyset, G_n\}_{n \in \mathbb{N}}$, on $\mathbb{N}$, is closed under arbitrary intersections. If we endow each factor space of the Cartesian product $\mathbb{N} \times \mathbb{N}$ with this semi-cofinite topology, then given

any finite family $\{f_\alpha\}$ of functions on $\mathbb{N} \times \mathbb{N}$ into its factor spaces, the bases B_τ, B, and D—hence the topologies τ, τ_b, and τ_s—are equal. This illustrates case 5.

EXAMPLE 4.89

However, if we give each factor space of the infinite dimensional product set $\prod_{k=1}^{\infty} \mathbb{N}_k$ the cofinite topology of $\mathbb{N}$, which is not closed under arbitrary intersections, then the weak topology τ would be strictly weaker than the box topology τ_b, and the box topology τ_b would be strictly weaker than the supra τ_s of τ; illustrating item 4.

EXAMPLE 4.90

If we give each factor space of the infinite dimensional product set $\prod_{k=1}^{\infty} \mathbb{N}_k$ the semi-cofinite topology (constructed above) of $\mathbb{N}$, which is closed under arbitrary intersections, then the weak topology τ would be strictly weaker than the box topology τ_b, and the box topology τ_b would be equal to the supra τ_s of τ; illustrating item 3.

4.4 Discrete Weak Topology of Cartesian Product Sets

4.4.1 Finite Dimensional Case

Let

$$X = \prod_{i=1}^{n} X_i$$

be the Cartesian product of a finite number of topological spaces. Then sets of the form

$$A_k = \bigcap_{i=1, i\neq k}^{m} p_i^{-1}(G_i) \bigcap p_k^{-1}(\{x_k^*\}),$$

where G_i is open in X_i and the p_is are the projection maps, will be of use in the proof of our first theorem. We see that

$$A_k = \{(x_1, x_2, \cdots, x_k^*, \cdots, x_n) : x_i \in G_i, 1 \leq i \leq m \leq n,\ i \neq k,\\ x_k = x_k^*, x_j \in X_j, j > m\} \subset X.$$

To see the fact of A more clearly, we note that for any subset (open or not) G_i of X_i, $p_i^{-1}(G_i) = \{\bar{x} \in X : p_i(\bar{x}) \in G_i\} = \{\bar{x} \in X : p_i(\bar{x}) = x_i \in G_i\} = \{\bar{x} \in X : x_i \in G_i \text{ and } x_j \in X_j, \forall j \neq i, 1 \leq i, j \leq n\} = \{\bar{x} \in X :$ the i-th coordinate of $\bar{x}$ is in the subset G_i of X_i and the other coordinates come from the other factor spaces without restriction }. Hence, if there are m open sets, $G_i, 1 \leq i \leq m \leq n$, from m factor spaces of X, the intersection of their inverse images $\bigcap\limits_{i=1}^{m} p_i^{-1}(G_i)$ is

$\bigcap\limits_{i=1}^{m} p_i^{-1}(G_i) = \{\bar{x} \in X :$ the i-th coordinate of $\bar{x}$ must come from the (open) subset G_i in $X_i, 1 \leq i \leq m \leq n$, and the other coordinates come freely from the remaining *unaffected* factor spaces$\}$.

If we consider a singleton $\{x_k^*\}$ in X_k, then it is easy to see that

$$p_k^{-1}(\{x_k^*\}) = \{\bar{x} \in X : x_i \in X_i \text{ and } x_k = x_k^*, i \neq k\} = \{\bar{x} = (x_1, x_2, \cdots, x_k^*, \cdots, x_n) : x_i \in X_i, i \neq k\}.$$

It follows from all these that

$$A_k = \bigcap_{i=1, i\neq k}^{m} p_i^{-1}(G_i) \bigcap p_k^{-1}(\{x_k^*\}) = \{\bar{x} \in X : x_i \in G_i \text{ and } x_k = x_k^*, 1 \leq i \leq m \leq n, x_j \in X_j, j > m\}, i, j \neq k,$$

where we mean by x_t the t-th coordinate of the tuple $\bar{x}$. We also observe that a fixed point $\bar{x}^*$ in X is one in which all the coordinates $x_1, x_2, \cdots, x_n$ are fixed to $x_1^*, x_2^*, \cdots, x_n^*$, respectively. So, such a fixed point may be denoted as $\bar{x}^* = (x_1^*, x_2^*, \cdots, x_n^*)$. As we have seen, to fix only one coordinate x_k^* of a tuple in X, we can take the inverse image

$$p_k^{-1}(\{x_k^*\})$$

of the singleton of x_k^* in X_k, under the k-th projection map. Therefore, a singleton $\bar{x}^*$ in X may be seen as the intersection

$$\{\bar{x}^*\} = \bigcap_{i=1}^{n} p_i^{-1}(\{x_i^*\})$$

of singletons from <u>all</u> the factor spaces, under the projection maps. This is true in finite- as well as infinite-dimensional Cartesian product sets. That is, if $X = \prod\limits_{i=1}^{\infty} X_i$ and $\bar{x}^* \in X$, then the singleton of $\bar{x}^*$ can be expressed (in this context) as

$$\{\bar{x}^*\} = \bigcap_{i=1}^{\infty} p_i^{-1}(\{x_i^*\}),$$

and this shows that the product topology on an infinite-dimensional Cartesian product set is not discrete if all the factor spaces are discrete topological spaces, because singletons in such a space can neither emerge as sub-basic sets nor basic sets.

The proof of the following theorem, which makes use of the set A described above, can best be seen as proof from first principles. It is both rigorous and cumbrous, even doubtful except that its prediction is impeccably butressed later by theorem 4.26. Theorem 4.25 represented the first idea that came to us about how to prove this lofty intuition while theorem 4.26 is the fortuitous second thought.

Theorem 4.25 *Let $\{(X_i, \tau_i) : i = 1, \cdots, n\}$ be a finite number of topological spaces, $X = \prod_{i=1}^{n} X_i$ the Cartesian product of these spaces and let τ_p be the product topology on X, induced by these spaces. If τ_p is the discrete topology on X, then each (X_i, τ_i) is a discrete topological space, $i \in \{1, \cdots, n\}$.*

Proof:
Let

$$X = X_1 \times X_2 \times \cdots \times X_n = \prod_{i=1}^{n} X_i$$

be the Cartesian product of a finite family of nonempty sets, each being a topological space. Suppose *the* product topology τ_p on X is discrete. We show that each X_i, $1 \leq i \leq n$, is a discrete topological space. Suppose one of the factor spaces, say X_k $(1 \leq k \leq n)$ is not a discrete topological space. Then $\exists x_k^* \in X_k \ni \{x_k^*\}$ is not an open set in X_k. We consider sets of the form

$$A_k = \bigcap_{i=1, i \neq k}^{m} p_i^{-1}(G_i) \bigcap p_k^{-1}(\{x_k^*\}),$$

where G_i is open in X_i and the p_is are the projection maps. We see that

$$A_k = \{(x_1, x_2, \cdots, x_k^*, \cdots, x_n) : x_i \in G_i, 1 \leq i \leq m \leq n \text{ and } \\ i, j \neq k, x_k = x_k^*, x_j \in X_j, j > m\} \subset X.$$

Since $\{x_k^*\} \subset X_k$ is not open in the topology of X_k, $p_k^{-1}(\{x_k^*\})$ is not in the subbase for this product topology τ_p on X which, by hypothesis, is discrete. Hence, in particular, sets of the form A are not open in X which, by hypothesis, is a discrete topological space. This is a contradiction. Hence every factor space of X must be a discrete topological space if X is a discrete product topological space. (Another way to look at sets of type A is to see them as subsets of $p_k^{-1}(\{x_k^*\})$ in X. Since $p_k^{-1}(\{x_k^*\})$ is not in the sub base, not all nonempty subsets of it are in the base. Conversely, if all subsets of $p_k^{-1}(\{x_k^*\})$ are open in X, then $p_k^{-1}(\{x_k^*\})$ must also be open in X.) That is, each $X_i (1 \leq i \leq n)$ is discrete if (X, τ) is discrete.

4.4.2 Generalizations—Infinite-dimensional Case, Weak and Box Topologies

Theorem 4.26 *Let $[(X,\tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. If (X,τ) is discrete, then any range space (X_α, τ_α) for which f_α is onto and an open map is discrete.*

Proof:
Let (X_α, τ_α) be a range space for which f_α is onto and an open map; and let $\{x_\alpha\} \subset X_\alpha$ be a singleton in X_α. We need to show that $\{x_\alpha\} \in \tau_\alpha$. Since $x_\alpha \in X_\alpha$ and f_α is onto, there exists an $x \in X$ such that $f_\alpha(x) = x_\alpha$. For this $x \in X$, the singleton $\{x\}$ is τ-open, since (X,τ) is discrete. It follows that $\{x_\alpha\} = f_\alpha(\{x\}) \in \tau_\alpha$, as f_α is an open map.

NOTE
Since the family of functions in theorem 4.25, the projection maps, are onto and open maps, theorem 4.26 aptly generalizes theorem 4.25, and it also serves as an alternative way of proving Theorem 4.25. It (theorem 4.26) does not only generalize theorem 4.25; it also extends it to situations in which the Cartesian product set (in 4.25) is infinite-dimensional. Hence this corollary.

Corollary 4.12 *If a product topology—on a finite- or infinite-dimensional Cartesian product set—is discrete, then each of the factor spaces is a discrete topological space.*

Corollary 4.13 *Let $[(X,\tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. If for some $\alpha_0 \in \Delta$, f_{α_0} is bijective (i.e. one-to-one and onto) then $(X_{\alpha_0}, \tau_{\alpha_0})$ is discrete if and only if (X,τ) is discrete.*

Definition 4.45 *Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}$ be any family of topological spaces and let $X = \prod_{\alpha\in\Delta} X_\alpha$ be the Cartesian product of these sets. Let τ_b be the box topology on X, induced by the projection maps $p_\alpha : X \to X_\alpha$.*

Corollary 4.14 *A box topology τ_b is discrete if, and only if, τ_α is discrete, $\forall \alpha \in \Delta$.*

Proof:
($\Rightarrow$) Suppose τ_b is discrete and that τ_{α_0} is not discrete, for some $\alpha_0 \in \Delta$. Then $\exists x^*_{\alpha_0} \in X_{\alpha_0}, \ni \{x^*_{\alpha_0}\} \notin \tau_{\alpha_0}$. So $p^{-1}_{\alpha_0}(\{x^*_{\alpha_0}\})$ is not a subbasic open set in τ_b. Hence in particular $p^{-1}_{\alpha_0}(\{x^*_{\alpha_0}\})$ is not open in τ_b. Contradiction! Hence each (X_α, τ_α) is discrete if (X, τ_b) is discrete.
($\Leftarrow$) Now let τ_α be discrete, $\forall \alpha \in \Delta$. Then $\{x_\alpha\} \in \tau_\alpha$, $\forall x_\alpha \in X_\alpha$. Hence $\forall \bar{x} \in X$, $\{\bar{x}\} = \bigcap\limits_{\alpha \in \Delta} p^{-1}_\alpha(\{x_\alpha\})$ is a basic open set in τ_b; so τ_b is discrete.

4.5 Weak Topological Systems; More Extensions and Generalizations

4.5.1 Introduction

Definition 4.46 *A function $f : (X, \tau) \to (Y, \gamma)$ from one topological space (X, τ) to another topological space (Y, γ) is called an open map if it maps open sets to open sets; in the sense that $f(U) \in \gamma$ for all $U \in \tau$.*

Definition 4.47 *A function $f : (X, \tau) \to (Y, \gamma)$ from one topological space (X, τ) to another topological space (Y, γ) is called a closed map if it maps closed sets to closed sets; in the sense that $[f(U^c)]^c \in \gamma$ for all $U \in \tau$.*

(References: Neumann (1935), Titchmarsh (1939), Friedrichs (1944), James (1951), Nakano (1951), Cooke (1953), Simmons (1963), James (1964), Schechter (1971), Pedersen (1989), Conway (1994), Munkres (2007), Albers and Frauenfelder (2009), Memarian (2009), Sandon (2009), Abbondandolo and Schwarz (2009), Ping (2013), Sunukjian and Baykur (2013), Medvedev and Zhuzhoma (2013), Carlson (2013), Valdez-Sanchez (2013), Wahl (2013), Murakami (2013), Toda (2013) and Morris (2016).)

4.5.2 The First Separation Axioms, and Discreteness

Lemma 4.12 *Any one-to-one function f_α in a weak topological system $[(X, \tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}, \{f_\alpha\}_{\alpha \in \Delta}]$ is an open map.*

Proof:
Let $U \in \tau$ and let $f_\alpha(U) = U_\alpha$, where f_α is a one-to-one function in the weak topological system. Then $U = f^{-1}_\alpha(U_\alpha)$. Since $U \in \tau$, it follows that $f^{-1}_\alpha(U_\alpha) \in \tau$. $\Rightarrow U_\alpha \in \tau_\alpha$. (Note: If $U_\alpha \notin \tau_\alpha$, then $f^{-1}_\alpha(U_\alpha)$ is not in τ. This

is because τ is built up from, or constructed by first collecting *all* sets of the form $f_\alpha^{-1}(U_\alpha)$, where $U_\alpha \in \tau_\alpha$, and leaving out *all other subsets* of X_α which are not τ_α-open. Conversely, the only subsets of X of the form $f_\alpha^{-1}(U_\alpha)$ which are directly τ-open are those for which U_α are τ_α-open; being the subbasic sets of τ.) That is, $f_\alpha(U) = U_\alpha \in \tau_\alpha$, if f_α is one-to-one and $U \in \tau$. Hence f_α is an open map.

We remark that an open map in a weak topological system may not be one-to-one. For example, it is known that the projection maps (which generate the weak topology known as product topology) are open maps, and they are not one-to-one.

Lemma 4.13 *Any one-to-one function f_α in a weak topological system $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ maps closed sets to closed sets. That is, it is a closed map in the sense of definition 4.47.*

Proof:
Let U be a τ-closed subset of X, so that $U^c \in \tau$. Then by lemma 4.12, $f_\alpha(U^c) = U_\alpha^c \in \tau_\alpha$, so that $U_\alpha = (U_\alpha^c)^c$ is τ_α-closed. Now (by one-to-one) $f_\alpha^{-1}(U_\alpha) \subset U$. $\Rightarrow U_\alpha \subset f_\alpha(U)$. [NOTE: If $f_\alpha^{-1}(U_\alpha)\bigcap U^c \neq \emptyset$, then $\exists z \in f_\alpha^{-1}(U_\alpha)\bigcap U^c$. $\Rightarrow z \in f_\alpha^{-1}(U_\alpha)$ and $z \in U^c$. $\Rightarrow f_\alpha(z) \in U_\alpha$ and (as $f_\alpha(U^c) = U_\alpha^c$) $f_\alpha(z) \in U_\alpha^c$. This is a contradiction since $U_\alpha \bigcap U_\alpha^c = \emptyset$.]

Let $x_\alpha \in f_\alpha(U)$. Then $\exists x \in U$, $\ni f_\alpha(x) = x_\alpha$. If $x_\alpha \notin U_\alpha$, for this $x_\alpha \in f_\alpha(U)$, then it follows that $x_\alpha \in U_\alpha^c$. $\Rightarrow \exists x \in U$, $\ni f_\alpha(x) = f_\alpha(y)$, for some $y \in U^c$ (because $U_\alpha^c = f_\alpha(U^c)$). But $x \neq y$, $\forall x \in U$ and $y \in U^c$ and f_α is one-to-one. We have a contradiction. Hence $x_\alpha \in U_\alpha$, $\forall x_\alpha \in f_\alpha(U)$. That is, $U_\alpha \subset f_\alpha(U) \subset U_\alpha$; implying that $f_\alpha(U) = U_\alpha$. Since U is an arbitrary τ-closed subset of X and U_α is τ_α-closed, the proof is complete.[13]

Theorem 4.27 *Let $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. If there exists an $\alpha \in \Delta$ such that (X_α,τ_α) is discrete and (for this $\alpha \in \Delta$) f_α is one-to-one, then (X,τ) is discrete.*

Proof:
Let $\{x\} \subset X$ be a singleton in X. We show that $\{x\} \in \tau$. Since some X_α is discrete and its f_α is one-to-one, it follows that (for this $\alpha \in \Delta$)

[13] An alternative proof is this: Let U be a τ-closed subset of X, and let $f_\alpha(U) = U_\alpha$. Now U, τ-closed, implies that $U^c \in \tau$. And $x_\alpha \in f_\alpha(U^c) \iff f_\alpha^{-1}(x_\alpha) \notin U$. $\iff x_\alpha \notin f_\alpha(U)$. $\iff x_\alpha \in [f_\alpha(U)]^c = U_\alpha^c$ (as $f_\alpha(U) = U_\alpha$). So, $f_\alpha(U^c) = U_\alpha^c$ and, by lemma 4.12, $f_\alpha(U^c) = U_\alpha^c \in \tau_\alpha$; implying that $f_\alpha(U) = U_\alpha = (U_\alpha^c)^c$ is τ_α-closed.

$f_\alpha(\{x\}) = \{x_\alpha\} \in \tau_\alpha$, by Lemma 4.12; and that, as f_α is one-to-one, $f_\alpha^{-1}(\{x_\alpha\}) = \{x\}$ is a subbasic set of τ—and, hence, that $\{x\} \in \tau$. This proves that (X, τ) is a discrete topological space.

We observe that theorem 4.27 extends its counterpart in product topology which has been in existence. Theorem 4.27 is important because

1. Unlike its analogue which has since been obtained for product topologies, theorem 4.27 does not require all the range (or factor) spaces of a weak topology to be discrete;
2. It only requires that a function be one-to-one and its range space to be discrete;
3. Theorem 4.27 applies to general weak topological sytems.

Definition 4.48 *A function $f : X \to Y$ is said to be well-behaved if $x \neq y$ in X implies that $f(x) \neq f(y)$ in Y.*

Theorem 4.28 *Let $[(X, \tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. If (for some $\alpha \in \Delta$) f_α is well-behaved, and, for this $\alpha \in \Delta$, (X_α, τ_α) is T_0, then (X, τ) is T_0.*

Proof:
Let $x \neq y$ in X. Then $x_\alpha = f_\alpha(x) \neq f_\alpha(y) = y_\alpha$, for the $\alpha \in \Delta$. Since X_α is T_0 there exists $U_\alpha \in \tau_\alpha \ni x_\alpha \in U_\alpha, y_\alpha \notin U_\alpha$, say. Now, $f_\alpha^{-1}(U_\alpha) \in \tau$, by definition of τ, and clearly $x \in f_\alpha^{-1}(x_\alpha) \in f_\alpha^{-1}(U_\alpha) = U$, say. That is, $x \in U \in \tau$. Suppose $y \in U$ also. Then it follows That $y_\alpha = f_\alpha(y) \in f_\alpha(U) = U_\alpha$. This implies that $y_\alpha \in U_\alpha$, a contradiction to the earlier assumption that $y_\alpha \notin U_\alpha$. Hence X is T_0.

Theorem 4.29 *Let $[(X, \tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. (X, τ) is T_1 if one (X_α, τ_α) is T_1 and its associated function f_α is well-behaved.*

Proof:
Let $x \neq y$ in X. We find $U, V \in \tau$ such that $x \in U, y \in V$ and $x \notin V, y \notin U$. As $x \neq y$ and one of the functions, say f_α, is well-behaved it follows that $f_\alpha(x) = x_\alpha \neq y_\alpha = f_\alpha(y)$. Since in addition X_α is T_1 for this particular $\alpha \in \Delta$, and $x_\alpha \neq y_\alpha \in X_\alpha$, there exist two τ_α-open subsets U_α, V_α of X_α such that $x_\alpha \in U_\alpha, y_\alpha \in V_\alpha$ and $x_\alpha \notin V_\alpha, y_\alpha \notin U_\alpha$. Clearly $f_\alpha(x) = x_\alpha \in U_\alpha$ implies that $x \in f_\alpha^{-1}(x_\alpha) \in f_\alpha^{-1}(U_\alpha) = U$; that is, $x \in U = f_\alpha^{-1}(U_\alpha) \in \tau$.

Similarly $y \in V = f_\alpha^{-1}(V_\alpha) \in \tau$. If say $x \in V$, then it follows that $x_\alpha = f_\alpha(x) \in f_\alpha(V) = V_\alpha$, implying that $x_\alpha \in V_\alpha$. This is a contradiction to the assumption above. Hence $x \notin V$ and, similarly, $y \notin U$.

REMARK 4.23
We note that in a product topological system the product topology is T_0 (or T_1) if all the factor spaces are T_0 (or respectively T_1). In theorems 4.28 and 4.29 these ideas are extended to a general weak topological system by removing the condition that all the range spaces be T_0 (or T_1) and replacing it with the smaller requirement that a range space be T_0 (or T_1) and its associated function to be well-behaved. We also apply this approach to theorem 4.30 below.

Theorem 4.30 *Let $[(X,\tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. Then (X,τ) is T_2 if one (X_α,τ_α) is T_2 and its associated function f_α is well-behaved.*

Proof:
Let $x \neq y$ in X. We find $U, V \in \tau$ such that $U \bigcap V = \emptyset$ and $x \in U, y \in V$. As $x \neq y$ and one of the functions, say f_α, is well-behaved it follows that $f_\alpha(x) = x_\alpha \neq y_\alpha = f_\alpha(y)$. Since in addition X_α is T_2 for this particular $\alpha \in \Delta$, and $x_\alpha \neq y_\alpha$ in X_α, there exist two disjoint τ_α-open subsets U_α, V_α of X_α such that $x_\alpha \in U_\alpha, y_\alpha \in V_\alpha$. Clearly $f_\alpha(x) = x_\alpha \in U_\alpha$ implies that $x \in f_\alpha^{-1}(x_\alpha) \in f_\alpha^{-1}(U_\alpha) = U \in \tau$; that is, $x \in U = f_\alpha^{-1}(U_\alpha) \in \tau$. Similarly $y \in V = f_\alpha^{-1}(V_\alpha) \in \tau$. If $U \bigcap V \neq \emptyset$, then it follows that $\exists z \in U \bigcap V = f_\alpha^{-1}(U_\alpha) \bigcap f_\alpha^{-1}(V_\alpha)$. $\Rightarrow z \in f_\alpha^{-1}(U_\alpha) \bigcap f_\alpha^{-1}(V_\alpha) = f_\alpha^{-1}(U_\alpha \bigcap V_\alpha) = \emptyset$. That is, $f_\alpha(z) \in \emptyset$; a contradiction. Hence $U \bigcap V = \emptyset$ and, as $U, V \in \tau$, the proof is complete.

Theorem 4.31 *Let $[(X,\tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. Any range space (X_α,τ_α) whose function f_α is one-to-one and onto is T_0 if (X,τ) is T_0.*

Proof:
Let $x_\alpha \neq y_\alpha$ in X_α. We find $U_\alpha \in \tau_\alpha$ such that $x_\alpha \in U_\alpha$, say, and $y_\alpha \notin U_\alpha$. Since f_α is one-to-one and onto, there exist, in X, $x = f_\alpha^{-1}(x_\alpha)$ and $y = f_\alpha^{-1}(y_\alpha)$ such that $x \neq y$. As X is T_0, there exists $U \in \tau$ such that, say $x \in U$ and $y \notin U$. Clearly as $x = f_\alpha^{-1}(x_\alpha) \in U$, it follows that $f_\alpha(x) = x_\alpha \in f_\alpha(U) = U_\alpha \in \tau_\alpha$ as, also, f_α is open (by lemma 4.12). If $y_\alpha \in U_\alpha$ also, then it follows that $y = f_\alpha^{-1}(y_\alpha) \in f_\alpha^{-1}(U_\alpha) = U$. That is, $y \in U$; a contradiction. Hence (X_α, τ_α) is T_0.

Theorem 4.32 *Let $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. If (X,τ) is T_1 and f_α is onto and one-to-one (i.e. a bijection), for some $\alpha\in\Delta$, then (X_α,τ_α) is T_1, for this $\alpha\in\Delta$.*

Proof:
Let $x_\alpha \neq y_\alpha$ in X_α. We find $U_\alpha, V_\alpha \in \tau_\alpha$ such that $x_\alpha \in U_\alpha, y_\alpha \in V_\alpha$ and $x_\alpha \notin V_\alpha, y_\alpha \notin U_\alpha$. Since f_α is onto for this $\alpha \in \Delta$ there exist $x, y \in X$ such that $f_\alpha(x) = x_\alpha, f_\alpha(y) = y_\alpha$. If $x = y$, then it follows that $f_\alpha(x) = x_\alpha = f_\alpha(y) = y_\alpha$ (as f_α is 1-1). Since $x_\alpha \neq y_\alpha$, it follows that $x \neq y$. Since $x \neq y \in X$ and (X,τ) is T_1, there exist $U, V \in \tau$ such that $x \in U, y \in V$ and $x \notin V, y \notin U$. Since f_α is an open map (as a one-to-one function in a weak topological system), $f_\alpha(U) = U_\alpha$ and $f_\alpha(V) = V_\alpha$ are τ_α-open subsets of X_α. We know that $x_\alpha = f_\alpha(x) \in f_\alpha(U) = U_\alpha$. That is, $x_\alpha \in U_\alpha$ and, similarly, $y_\alpha \in V_\alpha$. Suppose, that $x_\alpha \in V_\alpha$ also. Then it follows that $x_\alpha \in V_\alpha = f_\alpha(V)$. Hence $x = f_\alpha^{-1}(x_\alpha) \in f_\alpha^{-1}(V_\alpha) = V$. Thus $x \in V$, a contradiction. Hence $x_\alpha \notin V_\alpha$ and, similarly, $y_\alpha \notin U_\alpha$. That is, (X_α,τ_α) is T_1.

Theorem 4.33 *Let $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. If (X,τ) is T_2 and f_α is a bijection, for some $\alpha\in\Delta$, then (X_α,τ_α) is T_2, for this $\alpha\in\Delta$.*

Proof:
Let $x_\alpha \neq y_\alpha$ in X_α. We find $U_\alpha, V_\alpha \in \tau_\alpha$ such that $x_\alpha \in U_\alpha, y_\alpha \in V_\alpha$ and $V_\alpha \cap U_\alpha = \emptyset$. Since f_α is onto for this $\alpha \in \Delta$ there exist $x, y \in X$ such that $f_\alpha(x) = x_\alpha, f_\alpha(y) = y_\alpha$. Since $x_\alpha \neq y_\alpha$, we know that $x \neq y$. And since $x \neq y \in X$ and (X,τ) is T_2, there exist $U, V \in \tau$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. Since f_α is an open map (as a one-to-one function in a weak topological system), $f_\alpha(U) = U_\alpha$ and $f_\alpha(V) = V_\alpha$ are τ_α-open subsets of X_α. We know that $x_\alpha \in U_\alpha$ and that $y_\alpha \in V_\alpha$. We need to show now that $U_\alpha \cap V_\alpha = \emptyset$. So, suppose that $U_\alpha \cap V_\alpha \neq \emptyset$. Then it follows that $f_\alpha^{-1}(U_\alpha \bigcap V_\alpha) \neq \emptyset$. But $f_\alpha^{-1}(U_\alpha \bigcap V_\alpha) = f_\alpha^{-1}(U_\alpha) \bigcap f_\alpha^{-1}(V_\alpha)$ and $f_\alpha^{-1}(U_\alpha) \bigcap f_\alpha^{-1}(V_\alpha) = U \cap V$ since f_α is one-to-one. Thus $U \cap V \neq \emptyset$, a contradiction to the earlier assumption about U and V. Hence $U_\alpha \cap V_\alpha = \emptyset$. That is, (X_α,τ_α) is T_2.

Definition 4.49 *Let $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. If $f_\alpha(x) = 0$, $\forall \alpha \in \Delta$, $\Rightarrow$ $x = 0$, then we say that the family $\{f_\alpha\}_{\alpha\in\Delta}$ of functions is nonvanishing. If each element of the family $\{f_\alpha\}_{\alpha\in\Delta}$ of functions is a linear map, then we call the family linear, or a linear family of functions.*

Theorem 4.34 *Let* $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If each* (X_α,τ_α) *is* T_0*, then* (X,τ) *is* T_0 *if the family* $\{f_\alpha\}_{\alpha\in\Delta}$ *of functions is linear and nonvanishing.*

Proof:
Let $x \neq y$ in X. Then $x - y \neq 0$. If $f_\alpha(x) = f_\alpha(y)$, $\forall\alpha \in \Delta$, then it follows that $f_\alpha(x) - f_\alpha(y) = 0$, $\forall\alpha \in \Delta$. Again by linearity of each f_α, we have $f_\alpha(x-y) = f_\alpha(x) - f_\alpha(y) = 0$, $\forall\alpha \in \Delta$. Hence $f_\alpha(x-y) = 0$, $\forall\alpha \in \Delta$. It follows that, as the family of functions is nonvanishing, $x - y = 0$. That is, $x = y$; a contradiction. Hence $\exists\alpha \in \Delta$, $\ni f_\alpha(x) \neq f_\alpha(y)$. Let $x_\alpha = f_\alpha(x)$ and $y_\alpha = f_\alpha(y)$, for this $\alpha \in \Delta$. Then $x_\alpha \neq y_\alpha$, in X_α. Since, in particular, this X_α is T_0, $\exists G_\alpha \in \tau_\alpha$, $\ni x_\alpha \in G_\alpha$, $y_\alpha \notin G_\alpha$, say. Now, $f_\alpha^{-1}(G_\alpha) \in \tau$ and (as $f_\alpha(x) = x_\alpha \in G_\alpha$) $x \in f_\alpha^{-1}(G_\alpha)$. If also $y \in f_\alpha^{-1}(G_\alpha)$, then it follows that $y_\alpha = f_\alpha(y) \in G_\alpha$, a contradiction to the assumption made before. Hence $y \notin f_\alpha^{-1}(G_\alpha) \in \tau$ and $x \in f_\alpha^{-1}(G_\alpha)$. That is, X is T_0.

Theorem 4.35 *Let* $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If each* (X_α,τ_α) *is* T_1*, then* (X,τ) *is* T_1 *if the family* $\{f_\alpha\}_{\alpha\in\Delta}$ *of functions is linear and nonvanishing.*

Proof:
Let $x \neq y$ in X. Then (without loss of details) $\exists\alpha \in \Delta$, $\ni f_\alpha(x) = x_\alpha \neq y_\alpha = f_\alpha(y)$, in X_α. Since X_α is T_1, there exist two τ_α-open subsets U_α, V_α of X_α such that $x_\alpha \in U_\alpha, y_\alpha \in V_\alpha$ and $x_\alpha \notin V_\alpha, y_\alpha \notin U_\alpha$. Clearly $f_\alpha(x) = x_\alpha \in U_\alpha$ implies that $x \in f_\alpha^{-1}(x_\alpha) \subset f_\alpha^{-1}(U_\alpha) = U$; that is, $x \in U = f_\alpha^{-1}(U_\alpha) \in \tau$. Similarly $y \in V = f_\alpha^{-1}(V_\alpha) \in \tau$. If say $x \in V$, then it follows that $x_\alpha = f_\alpha(x) \in f_\alpha(V) = V_\alpha$, implying that $x_\alpha \in V_\alpha$. This is a contradiction to the assumption above. Hence $x \notin V$ and, similarly, $y \notin U$. That is, (X,τ) is T_1.

Theorem 4.36 *Let* $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If each* (X_α,τ_α) *is* T_2*, then* (X,τ) *is* T_2 *if the family* $\{f_\alpha\}_{\alpha\in\Delta}$ *of functions is linear and nonvanishing.*

Proof:
Let x, y in X be such that $x \neq y$. We find $U, V \in \tau \ni U \bigcap V = \emptyset, x \in U, y \in V$. Then (no loss of details) $\exists\alpha \in \Delta$, $\ni f_\alpha(x) = x_\alpha \neq y_\alpha = f_\alpha(y)$, in X_α. As X_α is T_2 there exist $U_\alpha, V_\alpha \in \tau_\alpha$ such that $U_\alpha \bigcap V_\alpha = \emptyset$ and $x_\alpha \in U_\alpha, y_\alpha \in V_\alpha$. We see that (as $x_\alpha = f_\alpha(x) \in U_\alpha \in \tau_\alpha$) $x \in f_\alpha^{-1}(x_\alpha) \in f_\alpha^{-1}(U_\alpha) = U \in \tau$; and, similarly, $y \in f_\alpha^{-1}(y_\alpha) \in f_\alpha^{-1}(V_\alpha) = V \in \tau$. If $U \bigcap V \neq \emptyset$, then it

follows that $f_\alpha^{-1}(U_\alpha) \bigcap f_\alpha^{-1}(V_\alpha) \neq \emptyset$. $\Rightarrow f_\alpha^{-1}(U_\alpha \bigcap V_\alpha) \neq \emptyset$. $\Rightarrow f_\alpha^{-1}(\emptyset) \neq \emptyset$, a contradiction. Hence $U \bigcap V = \emptyset$, and (X, τ) is T_2.

REMARK 4.24
The last three theorems (4.34, 4.35 and 4.36) are important generalizations of existing results on product topologies, as projection maps are linear and nonvanishing. In particular, theorem 4.36 explains why (as well as indicates when) a weak topology may or may not be Hausdorff.
EXAMPLE 4.91
As an example, we consider again the weak topology on the Cartesian product of a Sierpinski space with itself, with six open sets. This Sierpinski weak topology, as it were, is not Hausdorff. To see this, take two distinct points of $X \times X = \{(0,0),(0,1),(1,0),(1,1)\}$ to be $x = (0,0)$ and $y = (0,1)$. In the Sierpinski weak topology $\tau = \{\emptyset, \{(0,0),(0,1),(1,0),(1,1)\}, \{(0,0),(0,1)\}, \{(0,0),(1,0)\}, \{(0,0)\}, \{(0,0),(0,1),(1,0)\}\}$, on $X \times X$,
there are no disjoint open sets containing $x = (0,0)$ and $y = (0,1)$. The reason for this is that the range spaces, each the Sierpinski topological space $\{X, \emptyset, \{0\}\}$, are not Hausdorff.

4.5.3 On Normality

Definition 4.50 *A topological space (X, τ) is said to be normal if for any two disjoint τ-closed subsets A, B of X there exist two disjoint τ-open subsets U, V of X such that $A \subset U$ and $B \subset V$.*

Theorem 4.37 (Normal Weak Topologies) *Let $[(X,\tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. If for some $\alpha \in \Delta$, f_α is one-to-one then (X, τ) is normal if and only if (X_α, τ_α) is normal, for the fixed $\alpha \in \Delta$.*

Proof:
$\Longrightarrow$ Suppose (X_α, τ_α) is normal and that f_α is one-to-one, for some $\alpha \in \Delta$. We show that (X, τ) is normal. So, let A and B be two disjoint τ-closed subsets of X. We find two disjoint τ-open subsets U and V of X such that $A \subset U$ and $B \subset V$. From the hypothesis $f_\alpha(A) = A_\alpha$ and $f_\alpha(B) = B_\alpha$ are two disjoint τ_α-closed subsets of X_α, as 1-1 functions are closed maps in a weak topological system. Since (X_α, τ_α) is normal, there exist two disjoint τ_α-open subsets, say U_α and V_α, of X_α such that $A_\alpha \subset U_\alpha$ and $B_\alpha \subset V_\alpha$. By τ-continuity of each f_α, or from the definition of τ, both $f_\alpha^{-1}(U_\alpha)$ and $f_\alpha^{-1}(V_\alpha)$ are τ-open. As $f_\alpha(A) = A_\alpha \subset U_\alpha$, it follows that $A = f_\alpha^{-1}(A_\alpha) \subset f_\alpha^{-1}(U_\alpha)$.

That is, $A \subset f_\alpha^{-1}(U_\alpha)$; and similarly $B \subset f_\alpha^{-1}(V_\alpha)$.

Next, we show that $f_\alpha^{-1}(U_\alpha) \bigcap f_\alpha^{-1}(V_\alpha) = \emptyset$. Suppose that $f_\alpha^{-1}(U_\alpha) \bigcap f_\alpha^{-1}(V_\alpha) \neq \emptyset$. Then $\exists z \in f_\alpha^{-1}(U_\alpha) \bigcap f_\alpha^{-1}(V_\alpha)$. This implies that $z \in f_\alpha^{-1}(U_\alpha)$ and $z \in f_\alpha^{-1}(V_\alpha)$. This implies that $f_\alpha z \in U_\alpha$ and $f_\alpha z \in V_\alpha$. This implies that $f_\alpha z \in U_\alpha \bigcap V_\alpha = \emptyset$. A contradiction. Putting $U = f_\alpha^{-1}(U_\alpha)$ and $V = f_\alpha^{-1}(V_\alpha)$ we see that $A \subset U, B \subset V$ and that $U \bigcap V = \emptyset$. Hence X is normal.

$\Longleftarrow$ Let (X, τ) be normal and let f_α be one-to-one, for some $\alpha \in \Delta$. Let A_α and B_α be two disjoint τ_α-closed subsets of X_α, for this $\alpha \in \Delta$. By τ-continuity of each f_α, $A = f_\alpha^{-1}(A_\alpha)$ and $B = f_\alpha^{-1}(B_\alpha)$ are two τ-closed subsets of X. They (A and B) are also disjoint, for otherwise we would get a contradiction to the assumption that A_α and B_α are disjoint (as f_α is 1-1). As X is normal, A and B closed and disjoint in X, there exist two disjoint τ-open sets U and V in X such that $A \subset U$ and $B \subset V$. Since this f_α is an open map (as every 1-1 function in a weak topological system is an open map), $f_\alpha(U)$ and $f_\alpha(V)$ are τ_α-open subsets of X_α. $f_\alpha^{-1}(A_\alpha) = A \subset U \Rightarrow A_\alpha \subset f_\alpha(U) \in \tau_\alpha$. Similarly $f_\alpha^{-1}(B_\alpha) = B \subset V \Rightarrow B_\alpha \subset f_\alpha(V) \in \tau_\alpha$. If $f_\alpha(U) \bigcap f_\alpha(V) \neq \emptyset$, then $\exists x_\alpha \in f_\alpha(U) \bigcap f_\alpha(V)$. It follows that $x_\alpha \in f_\alpha(U)$ and $x_\alpha \in f_\alpha(V)$. $\Rightarrow f_\alpha^{-1}(x_\alpha) \in U$ and $f_\alpha^{-1}(x_\alpha) \in V$. $\Rightarrow f_\alpha^{-1}(x_\alpha) \in U \bigcap V$. $\Rightarrow U \bigcap V \neq \emptyset$. A contradiction to the earlier assumption about U and V. Hence the proof is complete.

Definition 4.51 *A topological space (X, τ) is said to be perfectly normal if it has exact Urysohn functions; in the sense that if A and B are two disjoint closed subsets of X, there exists a τ-continuous real-valued function $g : X \to [0, 1]$ such that $g^{-1}(\{0\}) = A$ and $g^{-1}(\{1\}) = B$.*

Theorem 4.38 *Let $[(X, \tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}, \{f_\alpha\}_{\alpha \in \Delta}]$ be a weak topological system. If f_α is one-to-one and (X_α, τ_α) is perfectly normal, for some $\alpha \in \Delta$ then (X, τ) is perfectly normal.*

Proof:

Let a range space (X_α, τ_α) be perfectly normal and let A and B be two disjoint τ-closed subsets of X. We need to show that there is an Urysohn function F on X, for A and B. From the hypothesis $f_\alpha(A) = A_\alpha$ and $f_\alpha(B) = B_\alpha$ are two disjoint τ_α-closed subsets of X_α. Since X_α is perfectly normal, there exists a continuous function $g_\alpha : X_\alpha \to [0, 1]$, a Urysohn function, such that $g_\alpha(A_\alpha) = \{0\}$ and $g_\alpha(B_\alpha) = \{1\}$.

Define $F : X \to R$ by $F(x) = (g_\alpha \circ f_\alpha)(x) = g_\alpha(f_\alpha(x)) = g_\alpha(x_\alpha)$, where g_α is the Urysohn function on X_α corresponding to $f_\alpha(A) = A_\alpha$ and $f_\alpha(B) =$

B_α and $x_\alpha = f_\alpha(x)$.[14] We prove that F is an Urysohn function on X, corresponding to A and B.

If $x \in A$, then $f_\alpha(x) \in A_\alpha$; hence $g_\alpha(f_\alpha(x)) = 0, \forall x \in A$, as $g_\alpha(A_\alpha) = \{0\}$. Therefore $F(x) = (g_\alpha \circ f_\alpha)(x) = 0 \forall x \in A$. That is $F(A) = \{0\}$. Similarly $F(B) = \{1\}$.

We also see that

$$F^{-1}(\{0\}) = (g_\alpha \circ f_\alpha)^{-1}(\{0\})$$

$$= (f_\alpha^{-1} \circ g_\alpha^{-1})(\{0\})$$

$$= f_\alpha^{-1}(g_\alpha^{-1}(\{0\}))$$

$$\equiv f_\alpha^{-1}(A_\alpha), \text{ as } g_\alpha^{-1}(\{0\}) = A_\alpha \text{ from hypothesis.}$$

$$= A, \text{ as } f_\alpha(A) = A_\alpha \text{ and } f_\alpha \text{ is 1-1.}$$

Hence $F^{-1}(\{0\}) = A$. And in a similar way, $F^{-1}(\{1\}) = B$.

Next we show that F is continuous. Let U be an open set in R. Then

$$F^{-1}(U) = (g_\alpha \circ f_\alpha)^{-1}(U);$$

$$\equiv (f_\alpha^{-1} \circ g_\alpha^{-1})(U);$$

$$= f_\alpha^{-1}(g_\alpha^{-1}(U)). \ . \ . \ . \ . \ . \ (2.3)$$

Continuity of g_α ensures that $g_\alpha^{-1}(U)$ is τ_α-open; and τ-continuity of f_α, or the definition of τ, guarantees that $f_\alpha^{-1}(g_\alpha^{-1}(U))$ is an open set in (X, τ). That is, (2.3) is τ-open, implying that F is τ-continuous; and hence an Urysohn function (with respect to A and B) on X. That is, (X, τ) is perfectly normal.

Theorem 4.39 *Let* $[(X,\tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If* f_α *is an open map, for some* $\alpha \in \Delta$ *then, for this* $\alpha \in \Delta$, (X_α, τ_α) *is perfectly normal if* (X, τ) *is perfectly normal.*

Proof:
Let (X, τ) be perfectly normal and let A_α and B_α be any two disjoint τ_α-closed subsets of X_α. We need to find a Urysohn function g_α (with respect to A_α and B_α) on X_α.

Since f_α is τ-continuous and A_α and B_α are τ_α-closed, $f_\alpha^{-1}(A_\alpha) = A$

[14] If two or more range spaces of X are perfectly normal and their corresponding functions f_α are one-to-one, then the function F on X can be defined using any one Urysohn function g_α on X_α.

and $f_\alpha^{-1}(B_\alpha) = B$ are τ-closed subsets of X. If $f_\alpha^{-1}(A_\alpha) \bigcap f_\alpha^{-1}(B_\alpha) \neq \emptyset$, then $\exists z \in f_\alpha^{-1}(A_\alpha) \bigcap f_\alpha^{-1}(B_\alpha) = f_\alpha^{-1}(A_\alpha \bigcap B_\alpha)$. $\Rightarrow f_\alpha(z) \in A_\alpha \bigcap B_\alpha$, a contradiction to the earlier assumption that A_α and B_α are disjoint. Hence $f_\alpha^{-1}(A_\alpha) \bigcap f_\alpha^{-1}(B_\alpha) = A \bigcap B = \emptyset$.

As (X, τ) is perfectly normal, there exists a Urysohn function $F : X \to [0, 1]$ such that $F(A) = \{0\}$ and $F(B) = \{1\}$. Define $g_\alpha : X_\alpha \to R$ by $g_\alpha(x_\alpha) = (F \circ f_\alpha^{-1})(x_\alpha)$; where F is the Urysohn function on X, corresponding to $f_\alpha^{-1}(A_\alpha) = A$ and $f_\alpha^{-1}(B_\alpha) = B$.

We need to show that g_α is a Urysohn function on X_α corresponding to A_α and B_α. Clearly as $f_\alpha^{-1}(A_\alpha) = A$, it follows that if $x_\alpha \in A_\alpha$ then $f_\alpha^{-1}(x_\alpha) \in A$ and that (as $F(A) = \{0\}$) $F(f_\alpha^{-1}(x_\alpha)) \equiv (F \circ f_\alpha^{-1})(x_\alpha) = 0$. Hence $g_\alpha(x_\alpha) = 0$, $\forall x_\alpha \in A_\alpha$. $\Rightarrow g_\alpha(A_\alpha) = \{0\}$. In a similar way, $g_\alpha(B_\alpha) = \{1\}$.

Now,

$$g_\alpha^{-1}(\{0\}) = (F \circ f_\alpha^{-1})^{-1}(\{0\})$$

$$= (f_\alpha \circ F^{-1})(\{0\})$$

$$= f_\alpha(F^{-1}(\{0\}))$$

$\equiv f_\alpha(A)$, as F is, from hypothesis, the Urysohn function corresponding to A and B.

$= A_\alpha$, as $f_\alpha(A) = A_\alpha$.

Therefore $g_\alpha^{-1}(\{0\}) = A_\alpha$. And in a similar way, $g_\alpha^{-1}(\{1\}) = B_\alpha$.

Next, we show that g_α is continuous. Let U be an open subset of R. Then

$$g_\alpha^{-1}(U) = (F \circ f_\alpha^{-1})^{-1}(U);$$

$$\equiv (f_\alpha \circ F^{-1})(U);$$

$$= f_\alpha(F^{-1}(U)). \quad \ldots\ldots\ldots\ldots \quad (2.4)$$

Continuity of F on X ensures that $F^{-1}(U)$ is a τ-open set. The fact that f_α is an open map (hypothesis) ensures that (2.4), $f_\alpha(F^{-1}(U))$, is a τ_α-open set; and hence that $g_\alpha^{-1}(U)$ is τ_α-open. So g_α is continuous and, considering the other properties it satisfies, an Urysohn function (with respect to A_α and B_α) on X_α. Hence this (X_α, τ_α) is perfectly normal.

Corollary 4.15 *If a product topology is perfectly normal, then all the factor spaces are perfectly normal.*

Definition 4.52 *A topological space (X, τ) is completely normal if every subspace of X is normal.*

Before now, no specific theorem or statement has been made concerning perfect normality of weak topologies on Cartesian product sets; or, for that matter, general weak topological systems. However, it is known (see e.g. Morris (2016), page 213; 6(b)) that a perfectly normal space is completely normal. Hence we have yet another result on product topologies.

Corollary 4.16 *If a product topology is perfectly normal, then each of the factor spaces is completely normal.*

Proof:
By Theorem 4.39 and/or Corollary 4.15 each factor space is perfectly normal. By the remark just made, the result follows.

Theorem 4.40 *Let $[(X, \tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}, \{f_\alpha\}_{\alpha \in \Delta}]$ be a weak topological system. If f_α is a one-to-one function and (X_α, τ_α) is perfectly normal, for an $\alpha \in \Delta$ then (X, τ) is completely normal.*

Proof:
By theorem 4.38, (X, τ) is perfectly normal as (X_α, τ_α) is perfectly normal. Since every perfectly normal space is completely normal (Morris 2016), (X, τ) is completely normal as a perfectly normal space.

Theorem 4.41 *Let $[(X, \tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}, \{f_\alpha\}_{\alpha \in \Delta}]$ be a weak topological system. If f_α is an open map, for some $\alpha \in \Delta$ then (for this $\alpha \in \Delta$) (X_α, τ_α) is completely normal if (X, τ) is perfectly normal.*

Proof:
From theorem 4.39, (X_α, τ_α) is perfectly normal and, hence, completely normal.

4.5.4 Regularity

Definition 4.53 *A topological space (X, τ) is said to be regular if for any τ-closed set G, that is $G^c \in \tau$, and $x \notin G$ there exist two disjoint τ-open subsets U and V of X such that $x \in U$ and $G \subset V$.*

Theorem 4.42 *Let* $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If* f_α *is a one-to-one map, for some* $\alpha\in\Delta$ *then* (X,τ) *is regular if, for this* $\alpha\in\Delta$, (X_α,τ_α) *is regular.*

Proof:
Let a range space (X_α,τ_α) be regular and let $G^c\in\tau, x\notin G$ and $x\in X$. That is, G is a τ-closed subset of X and x is not an element of G.
Since f_α is one-to-one and $x\notin G$, it follows that $f_\alpha(x)=x_\alpha\notin f_\alpha(G)$, for this $\alpha\in\Delta$. Also, as f_α is a closed map, $f_\alpha(G)=G_\alpha$ is a τ_α-closed subset of X_α. So G_α is closed in X_α and $f_\alpha(x)=x_\alpha\notin f_\alpha(G)=G_\alpha$. As X_α is regular, there exist two disjoint τ_α-open subsets U_α and V_α of X_α such that $x_\alpha\in U_\alpha$ and $G_\alpha\subset V_\alpha$.
Clearly $G\subset f_\alpha^{-1}(V_\alpha)\in\tau$, as $f_\alpha(G)=G_\alpha\subset V_\alpha$. Also $x=f_\alpha^{-1}(x_\alpha)\in f_\alpha^{-1}(U_\alpha)\in\tau$. If $f_\alpha^{-1}(U_\alpha)\bigcap f_\alpha^{-1}(V_\alpha)\neq\emptyset$, then $\exists z\in f_\alpha^{-1}(U_\alpha)\bigcap f_\alpha^{-1}(V_\alpha)=f_\alpha^{-1}(U_\alpha\bigcap V_\alpha)$. This implies that $f_\alpha(z)\in(U_\alpha\bigcap V_\alpha)$; a contradiction to the assumption that $(U_\alpha\bigcap V_\alpha)=\emptyset$. Hence $f_\alpha^{-1}(U_\alpha)\bigcap f_\alpha^{-1}(V_\alpha)=\emptyset$, and so X is regular.

Theorem 4.43 *Let* $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If* f_α *is a bijection, then for this* $\alpha\in\Delta$, X_α *is regular if* (X,τ) *is regular.*

Proof:
Let (X,τ) be regular and let $G_\alpha^c\in\tau_\alpha, x_\alpha\notin G_\alpha\subset X_\alpha$. Then by τ-continuity of f_α, $f_\alpha^{-1}(G_\alpha)$ is a τ-closed subset of X. As f_α is onto, $\exists x_0\in X\ni f_\alpha(x_0)=x_\alpha$ and $x_0\notin T_\alpha^{-1}(G_\alpha)=G$; for otherwise it would follow that $x\in f_\alpha^{-1}(G_\alpha)\forall x\in X\ni f_\alpha(x)=x_\alpha$; this will imply that $f_\alpha^{-1}(x_\alpha)\in f_\alpha^{-1}(G_\alpha)$ and, hence, that $x_\alpha\in G_\alpha$, which is a contradiction.
As X is regular, there exist two disjoint τ-open sets U and V such that $x_0\in U$ and $G\subset V$. It is easy to see that $U_\alpha=f_\alpha(U)$ and $V_\alpha=f_\alpha(V)$ are two disjoint τ_α-open subsets of X_α and that $x_\alpha\in U_\alpha, G_\alpha\subset V_\alpha$.

4.5.5 T_3, T_4, Completely Regular and Tychonoff Spaces

Definition 4.54 *A topological space* (X,τ) *is called a* T_3*-space if it is a regular* T_1*-space.*

Theorem 4.44 *Let* $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If, for some* $\alpha\in\Delta$, f_α *is a one-to-one function, and for this* $\alpha\in\Delta$, (X_α,τ_α) *is* T_3 *(i.e. regular and* T_1*) then* (X,τ) *is* T_3.

Proof:
We have proved in theorem 4.42 above that (X, τ) is regular. We only prove now that (X, τ) is also T_1.

So let $x \neq y$ in X. We find $U, V \in \tau$ such that $x \in U, y \in V$ and $x \notin V, y \notin U$. Since $x \neq y$ and the function f_α, is one-to-one it follows that $f_\alpha(x) = x_\alpha \neq y_\alpha = f_\alpha(y)$. Since in addition X_α is T_1 (as a T_3-space) for this particular $\alpha \in \Delta$, and $x_\alpha \neq y_\alpha \in X_\alpha$, there exist two τ_α-open subsets U_α, V_α of X_α such that $x_\alpha \in U_\alpha, y_\alpha \in V_\alpha$ and $x_\alpha \notin V_\alpha, y_\alpha \notin U_\alpha$. Clearly $f_\alpha(x) = x_\alpha \in U_\alpha$ implies that $x \in f_\alpha^{-1}(x_\alpha) \in f_\alpha^{-1}(U_\alpha) = U$; that is, $x \in U = f_\alpha^{-1}(U_\alpha) \in \tau$. Similarly $y \in V = f_\alpha^{-1}(V_\alpha) \in \tau$. If say $x \in V$, then it follows that $x_\alpha = f_\alpha(x) \in f_\alpha(V) = V_\alpha$, implying that $x_\alpha \in V_\alpha$. This is a contradiction to the assumption above. Hence $x \notin V$ and, similarly, $y \notin U$. That is, (X, τ) is T_1 and, hence, T_3.

Theorem 4.45 *Let $[(X, \tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}, \{f_\alpha\}_{\alpha \in \Delta}]$ be a weak topological system. If, for some $\alpha \in \Delta$, f_α is a bijection, then for this $\alpha \in \Delta$, (X_α, τ_α) is T_3 if (X, τ) is T_3.*

Proof:
We have proved in theorem 4.43 (above) that (X_α, τ_α) is regular if f_α is onto and one-to-one. We only need to prove now that (X_α, τ_α) is T_1 also. From theorem 4.32 the proof is complete.

Definition 4.55 *A topological space (X, τ) is called a T_4-space if it is a normal T_1-space.*

Theorem 4.46 *Let $[(X, \tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}, \{f_\alpha\}_{\alpha \in \Delta}]$ be a weak topological system. If, for some $\alpha \in \Delta$, f_α is a bijective function, then for this $\alpha \in \Delta$, (X_α, τ_α) is T_4 (i.e. normal and T_1) if (X, τ) is T_4.*

Proof:

From the results of theorem 4.37, (X_α, τ_α) is normal as (X, τ) is normal. From theorem 4.32 (X_α, τ_α) is T_1 as (X, τ) is T_1.

$\odot$

Theorem 4.47 *Let $[(X, \tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}, \{f_\alpha\}_{\alpha \in \Delta}]$ be a weak topological system. If, for some $\alpha \in \Delta$, f_α is a one-to-one function, and for this $\alpha \in \Delta$, (X_α, τ_α) is T_4, then (X, τ) is T_4.*

Proof:
By theorem 4.37 (X, τ) is normal and by theorem 4.44 it is also T_1.

Definition 4.56 *A topological space (X, τ) is said to be completely regular if for any τ-closed subset A of X and an element $x \in X$, not in A, there exists a real-valued τ-continuous function $f : X \to [0, 1]$ on X into $[0, 1]$ such that $f(A) = \{1\}$ and $f(x) = 0$.*

Theorem 4.48 *Let $[(X, \tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. If, for some $\alpha \in \Delta$, f_α is a one-to-one function, and for this $\alpha \in \Delta$, (X_α, τ_α) is completely regular, then (X, τ) is completely regular.*

Proof:
Let (given the hypothesis) $A^c \in \tau, x \notin A, x \in X$. Then, for the satisfying $\alpha \in \Delta$, $f_\alpha(A) = A_\alpha$ and $f_\alpha(\{x\}) = \{x_\alpha\}$ are disjoint subsets of X_α. It then follows that $x_\alpha \notin A_\alpha$. Also it is clear that A_α is a τ_α-closed subset of X_α. As (X_α, τ_α) is completely regular, there exists a continuous function $g_\alpha : X_\alpha \to [0, 1]$ on X_α into $[0, 1]$ such that $g_\alpha(A_\alpha) = \{1\}$ and $g_\alpha(x_\alpha) = 0$.

Define $F : X \to [0, 1]$ by $F(x) = (g_\alpha \circ f_\alpha)(x) = g_\alpha(f_\alpha(x))$; where g_α and f_α are as already introduced above. We prove that, for the given pair of x and A above, $F(A) = \{1\}$ and $F(x) = 0$. Clearly, for any $a \in A$, $f_\alpha(a) \in A_\alpha$ as $f_\alpha(A) = A_\alpha$. It follows that $F(a) = (g_\alpha \circ f_\alpha)(a) = g_\alpha(f_\alpha(a)) = 1$, for any element a of A (since $g_\alpha(a) = 1, \forall a \in A_\alpha$). Hence $F(A) = \{1\}$. It is easy to see that $F(x) = 0$, for $x \notin A$.

Next, we show that F is continuous. Let U be an open set in $\mathbb{R}$. Then

$$F^{-1}(U) = (g_\alpha \circ f_\alpha)^{-1}(U);$$

$$\equiv (f_\alpha^{-1} \circ g_\alpha^{-1})(U);$$

$$= f_\alpha^{-1}(g_\alpha^{-1}(U)). \;.\;.\;.\;.\;.\; (6.9)$$

Continuity of g_α ensures that $g_\alpha^{-1}(U)$ is τ_α-open; and τ-continuity of f_α, or the definition of τ, guarantees that $f_\alpha^{-1}(g_\alpha^{-1}(U))$ is an open set in (X, τ). That is, (6.9) is τ-open, implying that F is τ-continuous. That is, (X, τ) is completely regular.

Theorem 4.49 *Let $[(X, \tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. If, for some $\alpha \in \Delta$, f_α is an open map, then for this $\alpha \in \Delta$, (X_α, τ_α) is completely regular if (X, τ) is completely regular.*

Proof:
Let $x_\alpha \notin A_\alpha, A_\alpha^c \in \tau_\alpha$. Then by τ-continuity, $f_\alpha^{-1}(A_\alpha)$ is τ-closed. If $f_\alpha^{-1}(\{x_\alpha\}) \bigcap f_\alpha^{-1}(A_\alpha) \neq \emptyset$, then it follows that $f_\alpha^{-1}(\{x_\alpha\} \bigcap A_\alpha) \neq \emptyset$. This implies that (as $\{x_\alpha\} \bigcap A_\alpha = \emptyset$) $f_\alpha^{-1}(\emptyset) \neq \emptyset$; a contradiction. Hence $f_\alpha^{-1}(\{x_\alpha\}) \bigcap f_\alpha^{-1}(A_\alpha) = \emptyset$. Let x be any element of $f_\alpha^{-1}(\{x_\alpha\}$ (assuming f_α is not one-to-one), and let $A = f_\alpha^{-1}(A_\alpha)$. Then clearly $x \notin A$ and A is τ-closed.

As (X, τ) is completely regular, there exists a continuous function $F : X \to [0,1]$ such that $F(A) = \{1\}$ and $F(x) = 0$. Define $g_\alpha : X_\alpha \to [0,1]$ by $g_\alpha(x) = (F \circ f_\alpha^{-1})(x)$; where F and f_α are as already given. If $x \in A_\alpha$, then $f_\alpha^{-1}(x) \subset f_\alpha^{-1}(A_\alpha) = A$. Hence for any $x \in A_\alpha$, $g_\alpha(x) = (F \circ f_\alpha^{-1})(x) = 1$, since $F(A) = \{1\}$. It follows that $g_\alpha(A_\alpha) = \{1\}$. Similarly for the given $x_\alpha \notin A_\alpha$, $g_\alpha(x_\alpha) = 0$.

Next, we show that g_α is continuous. Let U be an open subset of $\mathbb{R}$. Then

$$g_\alpha^{-1}(U) = (F \circ f_\alpha^{-1})^{-1}(U);$$

$$\equiv (T_\alpha \circ F^{-1})(U);$$

$$= f_\alpha(F^{-1}(U)). \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6.10)$$

Continuity of F on X ensures that $F^{-1}(U)$ is a τ-open set. The fact that f_α is an open map (hypothesis) ensures that (6.10) $f_\alpha(F^{-1}(U))$ is a τ_α-open set; and hence that $g_\alpha^{-1}(U)$ is τ_α-open. So g_α is continuous. Hence this (X_α, τ_α) is completely regular.

Corollary 4.17 *If the product topology is completely regular, then all the factor spaces are completely regular*

Definition 4.57 *A completely regular T_1-space is called a* Tychonoff space*.*

Theorem 4.50 *Let $[(X, \tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}, \{f_\alpha\}_{\alpha \in \Delta}]$ be a weak topological system. If, for some $\alpha \in \Delta$, f_α is a one-to-one and onto map, then for this $\alpha \in \Delta$, (X_α, τ_α) is a Tychonoff space if (X, τ) is a Tychonoff space.*

Proof:
From Lemma 4.12 f_α is an open map. Hence by theorem 4.49 (X_α, τ_α) is completely regular. Finally, by theorem 4.32 (X_α, τ_α) is T_1 and, hence, Tychonoff

Theorem 4.51 *Let $[(X, \tau), \{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}, \{f_\alpha\}_{\alpha \in \Delta}]$ be a weak topological system. If, for some $\alpha \in \Delta$, f_α is a one-to-one function, and for this $\alpha \in \Delta$, (X_α, τ_α) is Tychonoff, then (X, τ) is Tychonoff.*

Proof:
Theorems 4.44 and 4.48 guarantee this.

4.5.6 Compactness

Compactness is widely known to be one of the most important topics in the whole of general topology. In the words of Professor Sidney Allen Morris, "The most important topological property is compactness." (Morris (2016)) No wonder, what is known as the famous theorem of Tychonoff is a statement about compactness. And it simply says that arbitrary product of compact spaces is compact; also its converse has also been known to hold in product topological systems. But a question now is: What obtains in general weak topological systems in terms of compactness; that is, how can we extend or generalize the property of compactness to arbitrary weak topological systems?

Theorem 4.52 *Let* $[(X,\tau),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If, for some* $\alpha\in\Delta$, f_α *is one-to-one and onto, and for this* $\alpha\in\Delta$, (X_α,τ_α) *is compact, then* (X,τ) *is compact.*

Proof:
Let $\{A_r\}_{r\in w}$ be any open cover for X. Then $X\subset\bigcup\limits_{r\in w}A_r$, $A_r\in\tau$. By onto $X_\alpha=f_\alpha(X)$. Also $f_\alpha(A_r)=B_r\in\tau_\alpha$, $\forall r\in w$ since (by Lemma 7.1) every one-to-one function in a weak topological system is an open map. Then $X_\alpha=f_\alpha(X)\subset f_\alpha(\bigcup\limits_{r\in w}A_r)\subset\bigcup\limits_{r\in w}f_\alpha(A_r)=\bigcup\limits_{r\in w}B_r$, each $B_r\in\tau_\alpha$.
Therefore $X_\alpha\subset\bigcup\limits_{r\in w}B_r$, each $B_r\in\tau_\alpha$. Since (X_α,τ_α) is compact, we have $X_\alpha\subset\bigcup\limits_{i=1}^{n}B_{r_i}$. Hence $X=f_\alpha^{-1}(X_\alpha)\subset f_\alpha^{-1}(\bigcup\limits_{i=1}^{n}B_{r_i})=\bigcup\limits_{i=1}^{n}f_\alpha^{-1}(B_{r_i})$. $\Rightarrow X\subset\bigcup\limits_{i=1}^{n}f_\alpha^{-1}(B_{r_i})=\bigcup\limits_{i=1}^{n}A_{r_i}$. That is $X\subset\bigcup\limits_{i=1}^{n}A_{r_i}$; implying that (X,τ) is compact.

REMARK 4.25
The last theorem extends (but does not generalize) the famous theorem of Andrey Nikolayevich Tychonoff to general weak topological systems. It shows that a weak topology would be compact if just one of its range spaces is compact and the function associated with the compact space is bijective.

Chapter 5

SUMMARIES, CONCLUSIONS AND SUGGESTIONS

5.1 Summaries and Conclusions

5.1.1 On Section 3.2

Vertical- and horizontal-line-open weak topologies were constructed on the Cartesian plane. In order not to trivialize this development and to actually appreciate its importance, one may have to consider the facts that

- Just as there is no topology strictly weaker than the discrete topology of the plane $\mathbb{R}^2$ in which all parabolas, or all cubic curves, or all quartic curves, or all quintic curves, etc. are open, no other topology on $\mathbb{R}^2$ strictly weaker than the discrete topology has all vertical lines or all horizontal lines as open sets. The construction of the line-open weak topologies throws up the challenge for someone to try and construct topologies (even if not weak topologies, that are smaller than the discrete topology) on $\mathbb{R}^2$ in which (at least) one of the other family of curves are open.
- And closely related to this is the construction of the hyperplane-open weak topologies on $\mathbb{R}^n$ in which lines, planes, etc. can be made to be open.

5.1.2 On Section 4.1

1. The method of the point-open weak topologies on $\mathbb{R}^n$ (a) can be used to make any point in $\mathbb{R}^n$ open, and in particular (b) led to the emergence of a matrix-open weak topology on $\mathbb{R}^2$. This particular weak topology may be compared or contrasted later with a weak topology, constructed on $\mathbb{R}^2$, in which matrices are actually closed sets.

2. Subset-induced topologies can be used to solve problems similar to the one given in the illustrative example—and more. We observe that this idea was used in the X-topology approach to hyperplane-open weak topologies. It is also used crucially in our exposition on seminorm topologies. Many more applications of this concept will certainly come up in future.

3. We can use reducible topologies to obtain minimal topologies that would be used instead in some analysis. And the idea of extensible topologies can be used to expand a topology in a way that may enhance analysis.

4. The concept of base reducibility or base extension of topologies can be used to construct topologies which compare with one another in terms of bases.

5. The discrete topology of any set X cannot be reduced in the strong sense if the cardinality of X is greater than 2. (Theorem 4.1 (a))

6. Every non-indiscrete topology on X can be reduced in some sense. (Theorem 4.1 (b))

7. The comparison theorems are again an important development. The summary of the results here is that a nontrivial weak topology actually sits in the middle of a chain of strictly comparable weak topologies each of which makes the fixed family of functions continuous. This implies that we should always clarify the reason(s) why a given nontrivial weak topology is adopted in a context of analysis.

8. Any non-indiscrete weak topology has a strictly weaker weak topology in its system. (Theorem 4.4)

9. An indiscrete weak topology can have cardinality greater than 2; however, it is not reducible to a strictly weaker weak topology in its own system. (Proposition 4.9)

10. Three equivalent statements for an indiscrete weak topology are made. (Theorem 4.5)

11. A discrete weak topology on a set X may not be equal to the power set of X. (Proposition 4.12)

12. Equivalent statements for a discrete weak topology are made. (Theorem 4.6)

13. Lemma 4.8 and theorem 4.11 are insightful results on seminorm topologies. In particular theorem 4.11 shows in clearest terms that every seminorm topology T generated by a family P of seminorms is nothing but a locally convex weak topology τ_w at the top of a sequence of non locally convex weak topologies generated by the family P of seminorms.

14. Our cursory look (from the constructive approach) at one existing result revealed that all weak topologies are not Hausdorff—modifying the existing result.

All the topological developments in this section emerged as a result of our attempt to construct weak topologies in real and rudimentary terms. While some of them throw up new challenges and opened up new areas of research, some are amenable to immediate exploitation for application and use. The overall value of these developments (like that of any new development) will only become clearer with time.

5.1.3 On Section 4.2

1. We showed that every infinite set has infinitely many *cofinite-like* topologies—and these we called *semi-cofinite* topologies.

2. Further investigations showed that this infinity of semi-cofinite topologies can be constructed or arranged to form a chain of topologies at which peak sits the cofinite topology. For a finite set, the semi-cofinite topologies forming a chain will only be finite in number and at the peak of their chain is the discrete topology of the finite set as the cofinite topology.

3. In our Cofinite Topology Lemma it was stated and proved that any pair of comparable, infinite and proper subsets of an infinite set induces a pair of comparable semi-cofinite topologies on their superset. This result leads to what we then later called the Cofinite Topology Theorem.

4. The whole episode climaxed into what we finally called the Branching Theorem. In a lay man's language, the Branching Theorem means that the cofinite topology of every (finite or infinite) set stands like a tree from which other cofinite-like topologies branch out. If the set under study is finite, this branching will terminate or have an end; and if the set is infinite, the branching will be endless or infinite.

5. Finally, the cofinite topology-induced weak topology on $\mathbb{R}^2$ is interesting in its contrasting property as a topology on $\mathbb{R}^2$ in which matrices (of coordinate points) come as closed sets.

5.1.4 On Section 4.3

From item 4 of the summarized listing—illustrated with example 4.89—we see that a topology τ_b can *strictly* exist between a weak topology τ and its supra τ_s; in the sense that τ_b is strictly stronger than τ and strictly weaker than τ_s. However, in all the situations considered in theorem 4.24, proposition 4.20 would explain that the supra of a box topology will always be equal to the supra of its associated weak topology.
POSTSCRIPT

1. It used to be generally accepted that many topologies are not closed under arbitrary intersection—in fact, none of the so-called standard (or Euclidean) topologies of $\mathbb{R}^n$ is closed under arbitrary intersections. We have a fundamentally very radical development here: namely (1) from the exposition of this dissertation it is clear that there is—related to any topology τ—a very small topology τ_s which is closed under arbitrary intersections and which is so close in size to τ that no other topology can exist between it and τ and have a different supra from τ_s; hence (2) supra topologies can be used to measure the closeness (in size) of comparable topologies; (3) any two incomparable topologies nevertheless generate two distinct complement topologies even if they have common supra (Theorem 4.22); hence (4) if we also consider the fact that comparable topologies which are not very close in size will generate distinct supras, we may now conclude that indeed more topologies are closed under arbitrary intersections! By this development, any inhibition inherent in a topology due to its lack of closure under arbitrary intersections can now be remedied, at least in the weak sense, through the exploitation now possible of its supra.

2. It is now clear that there are more complement topologies in existence

than any other known class of topologies; and the big question is: why has humanity not identified this before, in order to at least give such topologies a name? The answer to this question is simple: The underlying principle behind every research effort is the fact that all is never known. This is why every true research effort is respectable since it has the potential to bump into new discovery (big or small). There is always an unknown, an undiscovered out there waiting for someone to discover, and it is only a privilege (not a prerogative) to be able to see something profoundly different.

3. The relationship between a topology τ and its supra τ_s is a very interesting one, and only time will tell the extent to which this connection will be utilized in analysis and other applications.

4. The concept of exhaustive topology is another important idea which may not only be very helpful in future but has in this work certainly shown to be crucial in the way it has helped to indicate or suggest the kind of topologies which might have discrete supra.

5.1.5 On Section 4.4

The two important expositions of this section are theorems 4.25 and 4.26. Theorem 4.26 (with its very short but succint proof) is an extension and generalization of theorem 4.25; and its immediate application is corollary 4.13. As suggested in the abstract, theorem 4.26 is a sufficient condition for a discrete weak topology to induce the discreteness property on a range space, and it is considered the most important contribution of this section.

5.1.6 On Section 4.5

1. Any one-to-one function in a weak topological system is an open map; Lemma 4.12. And any one-to-one function in a weak topological system is a closed map; Lemma 4.13. These lemmas are important results unknown before this work. Their statement and proof facilitated much of the other results of the chapter—and they will certainly continue to facilitate the statement and proof of other results by researchers in future, we believe.

2. It has been known that if all the factor spaces are discrete, the product topology (in finite dimensions) would be discrete. As an extension, theorem 4.27 shows that only one discrete range space of a general weak topology can induce discreteness on the weak topology.

3. Theorems 4.28, 4.29 and 4.30 show how a weak topology can inherit the separation axioms of, respectively, T_0, T_1 and T_2. These results extend their analogues which have been in existence for only product topologies.

4. The conditions for a weak topology to induce the properties of T_0, T_1 and T_2 on its range spaces are established in theorems 4.31, 4.32 and 4.33. These results are extensions to other weak topologies since it is known that if a product topology is Hausdorff then all the factor spaces are Hausdorff.

5. It has been known that if all the factor spaces are T_0, T_1 or T_2, then the product topology is respectively T_0, T_1 or T_2. Theorems 4.34, 4.35 and 4.36 generalize and extend the existing results to any weak topological system in which the stated conditions are met.

6. In theorem 4.37 we established the conditions which guarantee exchange of *normality* between a weak topology and its range spaces. This particular result, like some other results in this chapter, found very useful application in other results here.

7. Theorem 4.38 showed how a perfectly normal range space can induce this property on the weak topology; and theorem 4.39 showed how a perfectly normal weak topology can induce this property on its range spaces.

8. An important corollary result is obtained immediately from theorem 4.39. This is corollary 4.27.

9. Corollary 4.16 is another very important implication of theorem 4.39.

10. Theorem 4.40 is a corollary result of theorem 4.38, and theorem 4.41 is still an imporatnt outcome of theorem 4.39.

11. In theorem 4.42 we showed that a regular range space can induce regularity on the weak topology, and in theorem 4.43 we showed how a regular weak topology can induce regularity on a range space.

12. In theorem 4.44 we showed how a T_3 range space can induce the T_3 property on the weak topology, and in theorem 4.45 we showed how a T_3 weak topology can induce this property on a range space.

13. In theorem 4.46 we showed how a range space can inherit the T_4 property from the weak topology, and in theorem 4.47 we showed how a weak topology can inherit this property from a range space.

14. In theorem 4.48 we showed how a completely regular range space can induce this property on the weak topology, and in theorem 4.49 we showed how a completely regular weak topology can induce this property on a range space.
15. Corollary 4.17 is an important fallout of theorem 4.49.
16. In theorems 4.50 and 4.51 we showed how the Tychonoff space property is exchanged between a weak topology and its range spaces.
17. It has been known that arbitrary product of compact spaces is compact. We showed that all the range spaces of a weak topology do not have to be compact in order that compactness be inherited by the weak topology from its range spaces. Theorem 4.52 is the statement and proof of this result.

5.2 Suggestions

More intensive and extensive research is needed to focus on *Weak Topology* via the constructive approach. We sincerely believe that further results that can be achieved by such research will likely relegate our results to being only a tip of the iceberg.

THE END

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