GENERALIZED MULTIVARIATE MOMENT GENERATING FUNCTIONS FOR PROBABILITY DISTRIBUTION OF SOME RANDOM VARIABLES

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CERTIFICATION PAGE

This is to certify that this dissertation here presented for the award of the Doctor of Philosophy in Statistics is the original work of Matthew Chukwuma Michael, 2011557001F, and that it has not been submitted in any other institution for the award of any Degree, Diploma of Certificate.

……………………………………………….. …………………………..

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APPROVAL PAGE

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DEDICATION

This work is dedicated to my God-sent children; Eminence Nwannebuikem, Prominence Chikelueze, my wife, Anita Chidimma, my parents, Chief and Mrs Okolie and my sisters, Okolie Chukwunonso, Nwachokor Nonye and Onwuanishia Nkeakam.

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ABSTRACT

This dissertation developed the generalized multivariate moment generating function for some random vectors/matrices and their probability distribution functions with the intention to replace the traditional/conventional moment generating functions due to its simplicity and versatility. The new function was developed for the multivariate gamma family of distributions, the multivariate normal and the dirrichlet distributions as a binomial expansion of the expected value of an exponent of a random vector/matrix about an arbitrarily chosen constant. The function was used to generate moments of random variables and their probability distribution functions; it was applied to data analysis and results obtained were compared with those from existing traditional/conventional methods. It was observed that the function generated same results as the traditional/conventional methods; in addition, it generated both central and non-central moments in the same simple way without requiring further tedious manipulations; it gave more information about the distribution, for instance while the traditional method gives skewness and kurtosis values of 0 and 3 respectively for p-variate multivariate normal distribution, the new methods gives $\left((0)\right)_{p\times 1}$ and 3^p respectively and; it could generate moments of integral and real powers of random vectors/matrices.

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CHAPTER ONE

INTRODUCTION

1.1 BACKGROUND OF THE STUDY

The significance of moments in explaining the characteristics of random variables and their probability distributions cannot be over emphasized in statistics. Basically, there are two types of moments namely moments about zero and moments about arbitrary points. Moments about zero are called crude moments while moment about an arbitrary point is called the central moment if the arbitrarily chosen point is mean or average of the distribution (Pearson, 1900; Kenney and Keeping, 1962 and; Weisstein, 2002). The central moments are fundamental to the determination of such characteristics of probability distributions as the variance, skewness and kurtosis (Arua *et al*, 1997). Higher moments can be obtained in theoretical statistics but their applications in practical cases are yet to be uncovered. Central moments of any order can be obtained by the mathematical combination of crude moments. A very important method of obtaining moments of a random variable and its probability distribution is the moment generating function (Chukwu and Amuji, 2012). However, very serious setbacks of this method of generating moments are that it does not always exist for all probability distributions and that it can only generate crude moments. To obtain the crude moment of a random variable and its distribution, its moment generating function is differentiated the required number of times and evaluated at a zero value of some real coefficient of the variable in the transformation that determines the function (Chukwu and Amuji, 2012; William and Richard, 1973). To obtain the central moments from crude moments, some mathematical combinations of the crude moments of required order are applied. The process of differentiating to obtain crude moments through moment generating functions and eventually having to obtain central moments from crude moments by the mathematical combination of the crude moments is tedious and cumbersome (Oyeka *et al* 2010).

In a bid to seek easier and quicker methods of finding the moments of random variables and their probability distributions, Oyeka *et al* (2008, 2010 and 2012) developed the univariate alternative methods for finding the moments of random variables and their probability distributions and bivariate alternative methods for finding the moments of two jointly distributed random variables and their probability distributions respectively. The researchers of these methods showed that these methods yield the same results as the traditional methods of generating moments of random variables and their distributions, and that they are easier and quicker to apply without requiring any further modifications, can handle cases of non-negative powers that are not necessarily integers and, in special cases, can handle non positive real powers enabling the generation of moments of negative powers.

In order to validate the findings made by the researchers of the Alternative methods of generating moments of random variables and their distributions, this study will review their works and eventually use them as springboard for the development of the multivariate generalized moment generating functions.

1.2 STATEMENT OF PROBLEM

Though the traditional methods for generating moments of random variables and their probability distributions have proved to be viable tools, they do not always exist, their application is limited to single power of random variables and they are used to generate only crude moments (Bulmer, 1979). Thus, they are not suitable for the generation of moments where non-integer and non-positive powers are considered. The alternative methods of generating moments of random variables and their probability density functions as obtained by Oyeka *et al* (2008, 2010 and

2012) for the univariate and bivariate distributions have such beautiful qualities as being easier and quicker to apply than the traditional methods and being able to generate moments of non-integer and non-positive powers of random variables. These findings however, have not been validated by an independent study. More so, the Generalized Multivariate Moment Generating Function of Random Vectors/matrices and their probability distributions has not been developed.

1.3 AIM AND OBJECTIVES OF THE STUDY

The aim of this study is to develop generalized multivariate moment generating functions for probability distributions of some random variables, while the specific objectives are to;

1. validate alternative methods of generating moments for the univariate and bivariate distributions;
2. develop the function for specific multivariate probability distribution functions;
3. use the function to generate moments for their respective probability distribution functions;
4. apply the method in data analysis;
5. expose the advantages of the new method over the traditional/conventional ones.

1.4 SIGNIFICANCE OF THE STUDY

This study will be significant in developing functions that can generate both crude moments, central moments and moments about arbitrarily chosen constants for some multivariate random vectors and their probability density functions which are more versatile, easier and quicker to apply than the traditional methods.

1.5 SCOPE OF THE STUDY

This work develops the Generalized Multivariate Moment Generating Function of random variables and their probability distributions. It specifically develops the function for the Multivariate Gamma Family of distributions, the Multivariate Normal distribution and the Dirrichlet (Multivariate Beta) distribution. It uses the developed functions to generate moments of the distributions, applies the methods to data analysis with the hope of exposing the advantages of the new methods over the traditional/conventional ones.

1.6 LIMITATION OF THE STUDY

This study develops generalized multivariate moment generating functions of multivariate random variables and their probability distributions; specifically for the multivariate gamma, the multivariate normal and the dirrichlet distributions. However, these methods can only be used for continuous distributions. Thus, the limitation of this study is that the functions it developed are not applicable to discrete random variables and their probability distributions.

So many works have been done in the areas of Moments of Random Variables, Moments Generating Functions, Characteristic Functions, Factorial Moment Generating Functions, Multivariate Moment Generating Function, Univariate and Bivariate Alternative Moment Generating Functions. Those works shall be reviewed in the next chapter.

CHAPTER TWO

LITERATURE REVIEW

2.1: LITEREATURE REVIEW

Pearson (1900) stated that the r^{th} moment of a random variable about the origin of its distribution is the expected value of the r^{th} power of the random variable. This assertion was supported by William and Richard (1973) and Oyeka (2013). Arizona (2009) stated that the expected value of X^m is called the m^{th} moment of X. Grimmet and Stirzaker (2001) stated that there are two types of moments of random variables namely, crude or uncorrected moments and central or corrected moments. He stated further that the k^{th} crude or uncorrected moment of a random variable, X, with probability distribution, $p(x)$ is given by the expectation of X to the power of k and that, the k^{th} central or corrected moment of a random variable, X, with mean, μ, and probability distribution, $p(x)$, is given by the expectation of the k^{th} power of the difference between the random variable, X, and its mean, μ.

Feller (1966) stated that, as in the case of discrete random variables, we define the k^{th} moment of a continuous random variable, X, by the expected value of the k^{th} power of X provided the integral exists.

Ross (1993) stated that the expected value of a random variable, X, $E(X)$, is also referred to as the mean or the first moment of X. He states further that the quantity $E(X^n), n \geq 1$, is called the n^{th} moment of X.

Lukacs (1972) defined the moment of random variables as follows; let X be a random variable and let k be a positive integer and suppose that the expectation of X^k exists, then, this expectation is called the moment (algebraic moment) of order k of the random variable, X. He stated further that the expectation of the k^{th} power of the absolute value of the random variable, X, is called the absolute moment of order k of X.

The moment generating function of a random variable is a special form of expectation of a random variable that generates moment of the random variable and its distribution.

In support of the above assertion, Onyeka (2000) posited that there are some functions that can easily generate some parameters of a random variable adding that such functions are called generating functions. He gave some examples as the moment generating function, (mgf), the probability generating function, (pgf), and the characteristic function, (cf). He argued further that direct computation of central moments may be quite cumbersome in which case one could resort to first obtaining the first k crude moments and then using them to compute the k^{th} central moment and, that a function that facilitates quick generation of crude moments is known as the moment generating function which he defined as the expected value of e^{tx}.

Grimmett and Welsh (1986) emphasized that the moment generating function is the expectation of a function of the random variable.

Arizona (2009) stated that the expected value of exponential tx is called the Laplace transformation or the moment generating function.

Vidyadhar (1995) defined the Laplace-Steiltjes transform of a nonnegative random variable, X, as $\varphi_X(s) = E(e^{-sx})$, assuming that it exists for some complex s with $R_e(s) > 0$. He highlighted the properties of the Laplace transform to include that it uniquely identifies a distribution function; the r^{th} moment of the random variable, X, is obtained from its Laplace transform as $E(X^r) = (-1)^r \frac{d^r}{ds^r} \varphi_X(s)\Big|_{s=0}$;

computing the cumulative density function and probability density function of X from $\varphi_X(s)$ is a complicated affair; if X_1 and X_2 are independent, $\varphi_{X_1+X_2}(s) = \varphi_{X_1}(s).\varphi_{X_2}(s)$.

These properties are the properties of the moment generating function, therefore, the moment generating function is a Laplace transform of its density function.

William and Richard (1973) stated that the moment generating function, $M(t)$, for a random variable, Y, is defined to be $E(e^{ty})$. They added that the moment generating function for, Y, exists if there exists a positive constant b such that $M(t)$ is finite for $|t| \leq b$.

Statlect (2015) asserted that if the expected value of $exp(tx)$ exists and is finite for all real numbers belonging to the closed interval $|t| \leq b$, then it could be said that the random variable, X, possesses a moment generating function and the function $M_X(t) = E(e^{tx})$ is called the moment generating function of X.

Hossein (2015) defined the moment generating function (MGF) of a random variable, X, as the expected value of $exp(\theta x)$ for all real θ for which the sum (in the case of discrete random variables) or the integral (in the case of continuous random variables) converges absolutely. He stated further that in some cases the existence of the moment generating function can be a problem for non-zero θ.

Ross (1993) defined the moment generating function $\varphi(t)$ of the random variable, X, for all values t as the expected value of e^{tx}. He stressed that $\varphi(t)$ is called the moment generating function because all of the moments of X can be obtained by successively differentiating $\varphi(t)$. He continued by asserting that an important property of moment generating function is that the moment generating function of the sum of independent random variables is just the product of individual moment generating functions.

Weisstein (2015) supported this assertion by showing that moment generating function of sum of random variables equals the product of moment generating functions of the individual random variables.

William and Richard (1973) summarily concluded that a moment generating function is simply a mathematical device that sometimes (but not always) provides an easy way to find the crude moment of random variables and to prove the equivalence of two probability distributions.

Grimmett and Welsh (1986) pointed out that a key problem with moment generating function is that moment generating function may not exist, as the integrals need not converge absolutely. They stated that an important property of the moment generating function is that if two distributions have the same moment generating function, then they are identical at almost all points. They added that in some cases, the moments exist and yet the moment generating function does not due to the fact that the limiting state may not exist. He gave the lognormal distribution as example of when this occurs.

Statlect (2015) stated that moment generating functions have great practical relevance not only because they can be used to easily derive moments, but also because a probability distribution is uniquely determined by its moment generating function. The source added that this fact, coupled with the analytical tractability of moment generating functions, makes them handy tool to solve several problems, such as deriving the distribution of sum of two or more random variables. In continuation of the discussion on the properties of the moment generating function, the source stressed that if X and Y are two random variables with distribution functions, $f(x)$ and $f(y)$, and moment generating functions, $M_X(t)$ and $M_Y(t)$, then X and Y have the same distribution if and only if they have the same moment generating functions. The source concluded by emphasizing that this proposition is extremely important and relevant from a practical viewpoint because it is much easier to prove equality of the moment generating function than prove equality of the distribution functions.

In a similar argument, Chukwu and Amuji (2012) stated that the moment generating function may not exist, citing the beta distribution as an example of such distributions where moment generating function does not exist.

Onyeka (2000) supported this argument by stating that there are some random variables whose moment generating function do not exist. He added that where it exists, there are two methods of generating crude moments from the moment generating function, emphasizing that one method is by expanding the moment generating function in Maclaurin's Series (expansion method); and the other is by differentiating the moment generating function (differentiation method). He, however, observed that there are cases where the expansion of the moment generating function in Maclaurin's Series is not possible and that the choice of which method to use is only a matter of convenience.

William and Richard (1973) stated that moment generating function possesses two important applications. The first being to find moments of random variables while the second is in proving that a random variable possesses a particular probability distribution. They highlighted that if moment generating function exists for a particular distribution, it is unique; meaning that, it is impossible for variables with different probability distributions to have the same moment generating functions.

Ross (1993) posited that another important result is that the moment generating function uniquely determines the distribution. This means that there exists a one-to-one correspondence between the moment generating function and the probability distribution function of random variables.

Weisstein (2015) asserted that the moment generating function of the uniform distribution is not differentiable at zero but that moments can be calculated by differentiating and then taking limit at zero.

A commonly used alternative to the moment generating function of random variables and their probability distributions is the characteristic function which unlike the moment generating function always exists.

Hossein (2014) posited that there are random variables for which the moment generating function does not exist on any real interval with positive length. He cited the Cauchy distributed random variable as a typical example. He defined the characteristic function as $\varphi_X(\omega) = E\left[e^{jwX}\right]$ where $j = \sqrt{-1}$ and ω is a real-valued random variable. He stressed that the advantage of characteristic function over the moment generating function is that it is defined for all real valued random variables.

Lukacs (1970) stated that if a random variable admits a density function, then the characteristic function is its dual, in the sense that each of them is a Fourier Transform of the other. He stated further that if a random variable has a moment generating function, then the domain of the characteristic function can be extended to the complex plane and $\varphi_X(-it) = M_X(t)$; this function always exists. He concluded by stating that if a random variable has a probability density function, then the characteristic function is its Fourier Transform with sign reversal in the complex exponential.

Feller (1966) discussed characteristic function as the expectation of $e^{-\lambda x}$, a useful tool made available for the study of arbitrary non-negative random variables. He added that this property is shared by the exponential function with a purely imaginary argument, that is, by the function defined for real x by $e^{i\theta x} = cos\theta x + isin\theta x$ where θ is a real constant and $i^2 = -1$. He argued further that t being bounded, the expectation of this function exists under any circumstance and provides a powerful and universally applicable tool but bought at the price of introducing complex-valued functions and random variables. He continued by stating that the characteristic function is the Fourier-Stieltjes transform of the distributions which are defined for all bounded measures and the term *characteristic function* emphasizes that the measure has a unit mass. He highlighted some properties of the characteristic function of a random variable and its distribution as follows: it is continuous, equals one at zero, its absolute value is less than or equal to one, the characteristic function of linear transformation of a random variable is a function of the random variable, the characteristic function of the convolution of two distributions has a characteristic function that is equal to the product of the characteristic functions of the two distributions and that if x_2 with characteristic function, φ, has the same distribution as $-x_1$ then, $|\varphi|^2$ is the characteristic function of the symmetrized distribution.

Onyeka (2000) in his discussion of characteristic function stated that the moment generating function exists only when $|e^{tx}| < 1$ for some t in the interval $-h < t < h$ where h is a positive number; a restriction that no doubt reduces the usefulness of the moment generating function. According to him, a closely related function which exists for all distributions is the characteristic function. In a brief definition, he stated that the characteristic function of a random variable, X, with probability distribution, $p(x)$, is given by $\varphi_X(t) = E\left(e^{itx}\right)$ where $i = \sqrt{-1}$ is a complex number. He concluded that the characteristic function exists for all real values of t and for all discrete and continuous distributions and, that characteristic function just like the moment generating function is a unique function of a given distribution.

Arizona (2009) asserted that $\varphi_X(\theta)$, equals the expectation of $e^{i\theta x}$, is called the Fourier Transform or the Characteristic Function and because the absolute value of the exponential $i\theta x$, $e^{i\theta x}$, equals 1, the expectation exists for any random variable.

By the definitions of the moment generating function and the characteristic function, one can infer that the characteristic function is obtained by replacing t in moment generating function of a random variable by it. That is, if the moment generating function, $M_X(t)$, is defined as the expectation of e^{tx} then the characteristic function can be defined as the expectation of e^{itx} implying that it is equal to $M_X(it)$.

The factorial moment generating function is a mathematical device that generates the factorial moments of a random variable and its distribution.

William and Richard (1973) stated that a mathematical device that is very useful in finding the probability distributions and other properties of integral-valued random variables is the probability generating function. In giving a mathematical definition to this function, they asserted that if Y is an integral-valued random variable for which $P[Y = i] = p_i, i = 0, 1, 2, \ldots$ the probability generating function $p(t)$ for Y is defined to be values of t such that $P(t)$ is finite. Here the coefficient of t^y is $P(t)$. They posited further that repeated differentiation of $P(t)$ yields factorial moments for the random variable Y which they defined as follows: "the k^{th} factorial moment for a random variable, Y, is defined to be $\mu_k = E[Y(Y-1)(Y-2)\ldots(Y-k+1)]$, where k is a positive integer."

Hogg *et al* (2013) stated that the factorial moment generating function of the probability distribution of a real-valued random variable, X, is defined as $M_X(t) = E[t^x]$ for all complex numbers t for which this expectation exists. They stated further that this is the case at least for all t on the unit circle $|t| = 1$. On the relationship between factorial moment generating functions and probability generating functions, they stated that, if X is a discrete random variable taking values only in the set $\{0, 1, 2, \ldots\}$ of nonnegative integers, then $M_X(t)$ is also called probability generating function of X and $M_X(t)$ is well-defined at least for all t on the closed disk $|t| \leq 1$. It affirmed that the factorial moment generating function generates the factorial moments of the probability distribution. Going on to the mathematical procedure of obtaining factorial moments from factorial moment generating functions, it affirmed that provided $M_X(t)$ exists in a neighbourhood of $t = 1$, the n^{th} factorial moment is given by $E[(X)_n] = M_X^{(n)} = \frac{d^n}{dt^n} M_X(t)|_{t=1}$.

Riordan (1958) and Daley and Vere-Jones (2003) both agreed that the factorial moments are useful for studying nonnegative integer-valued random variables and, that factorial moments serve as analytical tools in the mathematical field of combinatorics, which is the study of discrete mathematics structures.

Potts (1953) asserted that $E((X)_r) = E[X(X-1)(X-2)\ldots(X-r+1)]$ where $(X)_r = x(x-1)(x-2)\ldots(x-r+1) \equiv \frac{x!}{(x-r)!}$. He emphasized therefore that if X is binomially distributed with probability of success, $p\epsilon[0,1]$, n number of trials, then the factorial moments of X are $E[(X)_r] = \frac{n!}{(n-r)!} p^r \quad r\epsilon[0, 1, \ldots, n]$. He affirmed that for all $r > n$ the factorial moments are zero.

Vidyadhar (1995) defined the generating function, GF, of a nonnegative integer-valued random variable, X, as $g_X(z) = E(z^x)$. He affirmed that this function is defined for all complex z with $|z| \leq 1$. He stated as the properties of GF that a probability distribution is uniquely identified by its generating function; the probability density function can be derived (at least in theory) from its generating function by $P(X = k) = \frac{1}{k!} \frac{d^k}{dz^k} g_X(z)\Big|_{z=0}$; moments of X can be derived from its GF

as $E[(X)_r] = \frac{d^r}{dz^r} g_X(Z)|_{z=1}$ where $(X)_r = X(X-1) \dots (X-r+1)$ and; if X_1 and X_2 are independent random variables then $g_{X_1+X_2}(z) = g_{X_1}(z) g_{X_2}(z)$.

These properties are those of the factorial moment generating function of the random variable.

The moment generating function of random variables and probability densities can be extended to Multivariate Statistics.

As a build-up to the theory of moment generating function of multivariate random variables and their probability density, Grimmett and Welsh (1986) stated more generally that where $X = (X_1, \dots, X_n)^T$, an $n-$dimensional random vector, one uses $t.X = t^T X$ instead of tx.

Bulmer (1979) supported this statement by defining the moment generating function of $n-$dimensional random vector as $M_X(t) = E\left(e^{t^T X}\right)$ while adding that the reason for defining this function is that it can be used to find all moments of the distribution.

In reviewing the multivariate moment generating function of a multivariate Normal Distribution, Onyeagu (2003) stated that the moment generating function of a univariate normal random variable, X, is given as $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ but for the multivariate case where $X_{p\times 1} \sim N_p(\mu, \Sigma)$, the moment generating function, $M_X(t)$ of X is given as $E\left(e^{t'X}\right) = exp\left(t'\mu + \frac{1}{2} t'\Sigma \mathrm{t}\right)$.

Oyeka *et al* (2008) developed a method of finding moments or expected values of the distribution of a non-negative power of a continuous random variable based directly on the distribution of the random variable itself. They stated that $E(X)^{cn}$ is interpreted as the cn^{th} moment or expected value of X about zero with c and n

being non-negative integers. But if expressed as $E(X^c)^n$, this can be seen as also equal to the n^{th} moment of the distribution of X^c about zero where $c, (c \geq 0)$ is some non-negative real number not necessarily an integer.

Interest here is in finding $E(X^c)^n$, the n^{th} moment of the distribution of X^c about zero given the distribution of X where c is a non-negative real number and n is non-negative integer. The assumption here is that X is continuously differentiable on the real line or over its range of definition with probability density function, (pdf), $f(x)$.

In finding $E(X^c)^n$ one would need to first find, and then use, the distribution of X^c in the calculations. However, as seen from their illustrations, the results obtained using either the distribution of X^c or simply using the distribution of X are always the same. Hence, in finding the moments of the distribution of some functions such as X^c of a given random variable, X, it is not necessary to find and use the distribution of this function.

Use of the method was illustrated with continuous random variables taking on the pdf's: $f(x) = 2x; 0 < x < 1$; Beta, β, distribution, $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} . x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1$; the Gamma distribution, $f(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}, 0 < x < \infty$ and the normal distribution, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty$. Results were also presented of the Chi-Square distribution, Exponential distribution, Pareto distribution, Raleigh distribution, Uniform distribution and the Weibull distribution.

They stressed that calculations using the usual Moment Generating Function are relatively more difficult than using their new method.

In conclusion, they emphasized that their developed method always exists for all continuous probability distributions unlike the usual moment generating function which does not always exist.

Oyeka *et al* (2010) proposed the Alternative Moment Generating Function (AMGF). They defined the method as

$$M_X^c(t) = E\left(e^{tx^c}\right) = \int_{-\infty}^{\infty}\left(1 + \frac{(tx^c)}{1!} + \frac{(tx^c)^2}{2!} + \cdots + \frac{(tx^c)^r}{r!} + \cdots\right)f(x)dx$$

$$\therefore M_X^c(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_{-\infty}^{\infty} x^{cr} f(x)dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r(c)$$

where $\mu_r(c)$ is the r^{th} moment of the distribution of X^c about zero.

They asserted that $M_X^c(t)$ is called the Alternative Moment Generating Function (AMGF) of the distribution of the random variable, X^c.

They showed that the usual moment generating function is not always defined at $t = 0$, and therefore unlike the Alternative Moment Generating Function, $M_X^c(t)$ cannot always be used to find the moments of the distribution of the random variable, X.

They emphasized that this non-existence is a serious limitation which Alternative Moment Generating Function does not have. Hence, making it preferable and easier to use.

Illustration of the application of the Alternative Moment Generating Function (AMGF) was presented for some common continuous probability distributions including the beta distribution, the gamma distribution and the normal distribution.

Specifically, they presented the Alternative Moment Generating Functions of the Beta, Uniform, Gamma, Chi-square, Exponential and Normal distributions.

In a concluding statement, they asserted that the method is much more generalized and enables one to obtain the moments of the distribution of all non-negative real powers of a continuous random variable.

Oyeka *et al* (2012) developed and presented the Alternative Moment Generating Function (AMGF) of the joint distribution of some functions of powers of two continuous random variables when both powers are not necessarily whole numbers.

They defined the moment generating function of the joint distribution $u = X^c$ and $v = Y^d$ with pdf, $f(x, y)$ on the real line as

$$M_{u,v}(t_1, t_2) = M_{X^c Y^d}(t_1, t_2) = E\left(e^{t_1 x^c + t_2 x^c}\right), (t_1 \geq 0, t_2 \geq 0)$$

where $\mu_r(cd)$ is the r^{th} moment of the joint distribution of X^c and Y^d about zero.

They asserted that $M_{uv}(t_1 t_2)$ is called the Alternative Moment Generating Function of the distribution of the random variables X^c and Y^d and that it generates all conceivable moments of the joint distribution of the random variables, $u = X^c$ and $v = Y^d$.

They affirmed that the r^{th} moment of this joint distribution is the coefficient of $\frac{t_1^r t_2^r}{r!}$ or its r^{th} derivative with respect to t_1 and t_2 evaluated at $t_1 = t_2 = 0$.

They posited that the AMGF where $c = d = 1$; that is, for the joint distribution of the random variable X and Y is obtained as

$$M_{XY}(t_1 t_2) = \sum_{r=0}^{\infty} \frac{t_1^r t_2^r}{r!} \mu_r'(1,1)$$

The marginal distribution of the random variable $u = X^c$ is obtained by setting $d = 0$ and $t_2 = 1$ while the corresponding marginal distribution of the random variable $v = Y^d$ is obtained by setting $c = 0$ and $t_1 = 1$.

They illustrated the application of this method with some general distributions and the joint distribution of two independent normal variables. Subsequently, they concluded that the method is quicker and easier to apply than the usual or regular Moment Generating Function where it exists.

2.2 SUMMARY OF LITERATURE REVIEW

The literature reviewed shows that n^{th} moment of any given probability density function is obtained by differentiating the moment generating function, $M_X(t)$, n times and evaluating at $t = 0$. The process becomes tedious and cumbersome as n increases, this also applies to the characteristic function; the moment generating function does not always exist for all probability density function and at all points; the moment generating function generates only crude moments; the moment generating function is application in generating only positive integral moments of random variables; the Univariate Alternative Moment Generating Function (Oyeka et al, 2008 and 2010) and the Bivariate Alternative Moment Generating Function (Oyeka et al, 2012) have not been verified by an independent research and a method that applies in multivariate data has not been developed.

CHAPTER THREE

METHODOLOGY

3.1 UNIVARIATE GENERALIZED MOMENT GENERATING FUNCTIONS

The general interpretation of $E(X^c+\lambda)^n$ is that it is the n^{th} moment or expected value of the distribution of X^c about some real number λ where n and c are usually non-negative integers and λ is either 0 or $-\mu$ where μ is the mean of the random variable or probability distribution of X. This definition applies in the classical cases.

However, for the Alternative Method for Generating Moments of Continuous Distributions, while n may still be any non-negative integer, c and λ may be any real numbers that are not necessarily integers or whole numbers. The method is still based on the generalized definition of expected values of random variables. To differentiate this method from the conventional Moment Generating Functions (MGF) of the random variable X; usually designated by $M_X(t)=E(e^{tX})$, this method is referred to as the Alternative Moment Generating Function (AMGF) designated by $g_n(c,\lambda)$ read $g\ n$ of c about λ and termed the n^{th} moment of X^c about λ for $n=0,1,2,\dots,;\ -\infty\le c\le\infty$ and $-\infty\le\lambda\le\infty$.

$$g_n(c;\lambda)=E(X^c+\lambda)^n \qquad (3.1)$$

$$g_n(c;\lambda)=E(X^c+\lambda)^n=\int_{-\infty}^{\infty}(x^c+\lambda)^n f(x)dx=\int_{-\infty}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}(x^c)^r f(x)dx$$

$$=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\int_{-\infty}^{\infty}x^{cr}f(x)dx=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\mu_r'(c) \qquad (3.2)$$

where

$$\mu_r'(c) = \int_{-\infty}^{\infty} x^{cr} f(x) dx \qquad (3.3)$$

is the r^{th} moment of X^c about zero, or the $(cr)^{th}$ moment of X about zero. The $g_n(c;\lambda)$ as defined in Equation 3.2 generates all conceivable moments of the distribution of X^c for all real values of c.

From the definition of the generalized moment generating function; $g_n(c;\lambda)$, some of its properties include:

$$g_0(c;\lambda) = 1 \qquad (3.4)$$

$$g_1(c;\lambda) = E(X^c + \lambda) = \lambda + \mu_1'(c) \qquad (3.5)$$

Equation 3.5 is the first moment of the distribution of X^c about λ.

If $\lambda = 0$; that is, if the n^{th} moment of X^c is taken about the origin (zero) then,

$$g_n(c;0) = E(X^c - 0)^n = \mu_n'(c) \qquad 3.6$$

Equation 3.6 gives the n^{th} crude moment of the distribution.

If $\lambda = -\mu$, the n^{th} moment or mean value of X^c about the mean then

$$g_n(c;-\mu) = E(X^c - \mu)^n = \mu_n(c) \qquad 3.7$$

That is, the n^{th} moment of X^c about its mean μ, $\mu = \mu_1(c)$

Under specified conditions $g_n(c;\lambda)$ may be used to obtain all possible moments of the distribution of X^c where c is some non-positive-real-numbers thereby enabling one to obtain moments of random variables with negative and fractional indices.

The above properties of $g_n(c;\lambda)$ are quite consistent with existing theories of probability distributions. For example, from Equation 3.4, the sum of all probability values over its range of definition is always 1. Equation 3.5 in particular also conforms with the known fact that first moments of distributions about their mean $\lambda = -\mu_1(c)'$, is always zero. If in Equation 3.7 we let $n = 2$, that is, if the second moment of a distribution is taken about its mean, the resulting value is the variance of that distribution.

As noted above, generalized moment generating functions, $g_n(c;\lambda)$ may be used to obtain all conceivable moments of a continuous distribution. For example, the variance, third and fourth moments of the distribution of $Y = X^c$ are obtained from equation 3.2 by setting $\lambda = -\mu_1(c)' = -\mu$ where μ is the mean of X^c. Thus,

$$\mu_2(c) = g_2(c;-\mu) \qquad 3.8$$

variance of X^c.

$$\mu_3(c) = g_3(c;-\mu) \qquad 3.9$$

the third moment of X^c about its mean; and

$$\mu_4(c) = g_4(c;-\mu) \qquad 3.10$$

the fourth moment of the distribution of X^c about its mean. Hence, the skewness, $sk(c)$ and kurtosis, $ku(c)$ of the distribution of X^c are obtained respectively as

$$sk(c) = \frac{\mu_3(c)}{\left(\mu_2(c)\right)^{\frac{3}{2}}} = \frac{g_3(c;-\mu)}{\left(g_2(c;-\mu)\right)^{\frac{3}{2}}} \qquad 3.11$$

$$ku(c) = \frac{\mu_4(c)}{\left(\mu_2(c)\right)^2} = \frac{g_4(c;-\mu)}{\left(g_2(c;-\mu)\right)^2} \qquad 3.12$$

Suppose the random variable X has the probability density function, (pdf);

$$f(x) = 2x, 0 < x < 1 \qquad 3.13$$

Interest is to find an expression for the estimation of all conceivable moments of the random variable $Y = X$. Conventionally, the mean and variance of $Y = X$ is by definition

$\mu = \mu_1{}' = \frac{2}{3}$ and $\sigma^2 = \frac{1}{18}$ obtained as follows;

$$E(X) = \int_0^1 xf(x)dx \qquad 3.14$$

$$= \int_0^1 x.2xdx = 2\int_0^1 x^2\, dx$$

$$= \frac{2x^3}{3}\Big]_0^1$$

$$\therefore E(x) = \frac{2}{3} = \mu$$

$$Var(x) = E(x^2) - [E(x)]^2 \qquad 3.15$$

Now,

$$E(x^2) = \int_0^1 x^2 . f(x)dx$$

$$= \int_0^1 x^2 . 2x. dx$$

$$= 2\int_0^1 x^3 \,.\, dx$$

$$= 2\left[\frac{x^4}{4}\right]_0^1$$

$$\therefore E(x^2) = \frac{1}{2}$$

$$\Rightarrow Var(x) = \frac{1}{2} - \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{2} - \frac{4}{9}$$

$$\therefore Var(x) = \frac{1}{18}$$

To develop a more generalized expression for obtaining these moments and more, we have from Equation 3.2 that

$$g_n(c;\lambda) = \sum_{r=0}^{n} \binom{n}{r} . \lambda^{n-r} \frac{2}{cr+2}$$

where

$$\mu_r(c)' = \frac{2}{cr+2}$$

that is,

$$E(X^{cr}) = 2\int_0^1 x^{cr+1}\, dx$$

$$= \frac{2x^{cr+2}}{cr+2}\Bigg]_0^1$$

$$\therefore E(x^{cr}) = \frac{2}{cr+2}$$

For $c = 1$,

$$\mu_r(1)' = \frac{2}{r+2}$$

Since $c = 1$ in this present case.

The first moment $(n = 1)$ of $Y = X$ about λ is from above expression

$$g_1(1;\lambda) = \lambda + \frac{2}{1+2} = \lambda + \frac{2}{3}$$

Obtained as follows;

$$g_n(c;\lambda) = \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} \frac{2}{r+2}$$

Thus for

$$g_1(c;\lambda) = \sum_{r=0}^{1} \binom{1}{r} \lambda^{1-r} \frac{2}{r+2}$$

$$= \binom{1}{0} \lambda^{1-0} \frac{2}{0+2} + \binom{1}{1} \lambda^{1-1} \frac{2}{1+2}$$

$$\therefore g_1(c;\lambda) = \lambda + \frac{2}{3}$$

If now $\lambda = 0$, then $g_1(1;0) = 0 + \frac{2}{3} = \frac{2}{3} = \mu_1' = \mu$, the mean of $Y = X$ as earlier obtained. Hence if $\lambda = \frac{-2}{3}$ then as expected, $g_1\left(1;-\frac{2}{3}\right) = 0$. If now we set $n = 2$, that is, if interest is in determining the second moment of $Y = X$ about λ, we have

$$g_2(1;\lambda) = \lambda^2 + 2\lambda * \frac{2}{3} + \frac{2}{4} = \lambda^2 + \frac{4\lambda}{3} + \frac{1}{2}$$

If we now let $\lambda = -\mu_1(1) = -\mu = \frac{-2}{3}$, then we would have that

$$g_2\left(1;\frac{-2}{3}\right) = \left(\frac{-2}{3}\right)^2 + \frac{4}{3}\left(\frac{-2}{3}\right) + \frac{1}{2} = \frac{4}{9} - \frac{8}{9} + \frac{1}{2} = \frac{1}{18} = \sigma^2$$

that is, the variance of $Y = X$ as earlier obtained.

If we had chosen $c = \frac{1}{2}$, that is, if interest is in determining the moments of $Y = X^{\frac{1}{2}}$, then we would have that

$$\mu_r(c)' = \mu_r\left(\frac{1}{2}\right)' = \frac{2}{cr+2} = \frac{2}{\frac{1}{2}r+2}$$

so that,

$$g_n\left(\frac{1}{2};\lambda\right) = \sum_{r=0}^{n}\binom{n}{r}.\lambda^{n-r}\frac{2}{\frac{1}{2}r+2}$$

Hence the first moment of $Y = X^{\frac{1}{2}}$ about λ is

$$g_1\left(\frac{1}{2};\lambda\right) = \lambda + \frac{2}{\frac{1}{2}+2} = \lambda + \frac{4}{5}$$

so that if $\lambda = 0$ then,

$$g_1\left(\frac{1}{2};0\right) = \frac{4}{5}$$

If $\lambda = -\mu_1\left(\frac{1}{2}\right)' = -\mu = \frac{-4}{5}$ then $g_1\left(\frac{1}{2};\frac{-4}{5}\right) = 0$

The second moment of $Y = X^{\frac{1}{2}}$ about λ is

$$g_2\left(\frac{1}{2};\lambda\right) = \lambda^2 + 2\lambda\left(\frac{4}{5}\right) + \frac{2}{3}$$

Hence, if $\lambda = -\mu_1\left(\frac{1}{2}\right)' = -\mu = \frac{-4}{5}$, then

$$g_2\left(\frac{1}{2};-\mu\right) = g_2\left(\frac{1}{2};\frac{-4}{5}\right) = \left(\frac{-4}{5}\right)^2 + 2\left(\frac{-4}{5}\right)\left(\frac{4}{5}\right) + \frac{2}{3} = \frac{2}{3} - \frac{16}{25} = \frac{2}{75} = \sigma^2$$

the variance of the distribution of $Y = X^{\frac{1}{2}}$ as would have been obtained using the traditional method.

Using the moment generating function would have that the corresponding moment generating function for $Y = X$ is

$$M_Y(t) = M_X(t) = \frac{te^t - e^t}{t^2} = \left(\frac{t-1}{t^2}\right)e^t$$

Obtained as follows:

$$M_Y(t) = M_X(t) = E(e^{tx})$$

$$= \int_0^1 e^{tx} * x * dx = \int_0^1 xe^{tx}dx$$

Integrating by parts,

Let $u = x$ and $dv = e^{tx}$

$$\therefore \int_0^1 xe^{tx}dx = \left[\left[x\frac{e^{tx}}{t}\right]_0^1 - \int_0^1 \frac{e^{tx}}{t}dx\right]$$

$$= \frac{e^t}{t} - \left.\frac{e^{tx}}{t^2}\right]_0^1$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} = \frac{te^t - e^t}{t^2}$$

$$\therefore M_X(t) = \frac{(t-1)e^t}{t^2}$$

which is fairly cumbersome to obtain and even if differentiable with respect to t, the resulting derivatives do not exist at $t = 0$. Hence the method of generating function cannot possibly be used to obtain the moments of distribution of the random variable $Y = X$ and other similarly specified distributions.

3.2 GENERALIZED MOMENT GENERATING FUNCTION (GMGF) FOR THE BETA FAMILY OF DISTRIBUTIONS

Suppose interest is in finding the $gmgf$ of the distribution of the random variable, $Y = X^c$, where X has the beta distribution with parameters α and β and pdf, $f(x)$ given as

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1, \alpha > 0, \beta > 0$$

To obtain the required $gmgf$ we have from Equation 3.3 that

$$\mu_r'(c) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{cr}\, x^{\alpha-1}(1-x)^{\beta-1} dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{cr+\alpha-1}(1-x)^{\beta-1} dx$$

or

$$\mu_r'(c) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(cr+\alpha)\Gamma(\beta)}{\Gamma(cr+\alpha+\beta)} \tag{3.16}$$

Hence from Equation 3.2 we have that the $gmgf$ of the beta family of distributions represented by the random variable $Y = X^c$ is

$$g_n(c;\lambda) = \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} \frac{\Gamma(\alpha+\beta)\Gamma(cr+\alpha)}{\Gamma(\alpha)\Gamma(cr+\alpha+\beta)} \tag{3.17}$$

All desired moments of the beta family of distributions may be obtained using Equation 3.17. For instance the first moment of X^c about λ is

$$g_1(c;\lambda) = \lambda + \mu_1(c)' = \lambda + \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{\Gamma(c+\alpha)}{\Gamma(c+\alpha+\beta)}$$

If $c = 1$; that is, if interest is in the first moment or mean of $Y = X$ then, we have

$$g_1(1;\lambda) = \lambda + \frac{\alpha}{\alpha+\beta}$$

so that if $\lambda = 0$; that is, if the moment is taken about zero, then

$$g_1(1;0) = \mu_1(1) = \mu_1 = \frac{\alpha}{\alpha+\beta}$$

the mean of the beta distribution.

If $n = 2$ and $c = 1$ then,

$$g_2(1;\lambda) = \lambda^2 + 2\lambda\left(\frac{\alpha}{\alpha+\beta}\right) + \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

Hence if

$$\lambda = -\mu_1(1)' = -\mu_1 = \frac{-\alpha}{\alpha+\beta}$$

then we have,

$$g_2\left(1;\frac{-\alpha}{\alpha+\beta}\right) = \left(\frac{-\alpha}{\alpha+\beta}\right)^2 + 2\left(\frac{-\alpha}{\alpha+\beta}\right)\left(\frac{\alpha}{\alpha+\beta}\right) + \frac{\alpha}{(\alpha+\beta)} \frac{(\alpha+1)}{(\alpha+\beta+1)}$$

$$= \frac{\alpha}{(\alpha+\beta)} \frac{(\alpha+\beta)}{(\alpha+\beta+1)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 = \sigma^2$$

the variance of the beta distribution.

This is obtained as

$$g_2\left(1;\frac{-\alpha}{\alpha+\beta}\right)=\left(\frac{-\alpha}{\alpha+\beta}\right)^2+2\left(\frac{-\alpha}{\alpha+\beta}\right)\left(\frac{\alpha}{\alpha+\beta}\right)+\frac{\alpha}{(\alpha+\beta)}\frac{(\alpha+1)}{(\alpha+\beta+1)}$$

$$=\left(\frac{\alpha}{\alpha+\beta}\right)^2-2\left(\frac{\alpha}{\alpha+\beta}\right)^2+\frac{\alpha}{(\alpha+\beta)}\frac{(\alpha+1)}{(\alpha+\beta+1)}$$

$$=\frac{\alpha}{(\alpha+\beta)}\frac{(\alpha+1)}{(\alpha+\beta+1)}-\left(\frac{\alpha}{\alpha+\beta}\right)^2=\sigma^2$$

If we set $\alpha=\beta=1$ in Equation 3.17, then we obtain the $gmgf$ of the uniform distribution in a generalized form as

$$g_n(c;\lambda)=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\frac{\Gamma(cr+1)}{\Gamma(cr+2)}=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\frac{1}{cr+1} \qquad (3.18)$$

The moments of the beta family of distributions which are easily obtained using the $gmgf$ are more difficult to obtain with the traditional moment generating function. In fact, it can only be obtained by the introduction of the De L'Hospital's Rule of differentiation (Chukwu and Amuji 2012; 27).

3.3 GENERALIZED MOMENT GENERATING FUNCTION ($GMGF$) FOR THE GAMMA FAMILY OF DISTRIBUTIONS

Suppose the random variable, X, has the gamma distribution with parameters α and β, and pdf, $f(x)$, given as

$$f(x)=\frac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-\frac{x}{\beta}};x\geq 0,\alpha>0,\beta>0 \qquad (3.19)$$

To determine the $gmgf$ of the random variable $Y=X^c$ where X has the gamma distribution from Equation 3.2, we have that

$$\mu_r'(c)=\frac{1}{\beta^\alpha\Gamma(\alpha)}\int_0^\infty x^{cr}\,x^{\alpha-1}e^{-\frac{x}{\beta}}dx$$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_{0}^{\infty} x^{cr+\alpha-1}\, e^{-\frac{x}{\beta}} dx$$

Letting $v = \frac{x}{\beta}$, integrating and simplifying, we have that

$$\mu_r'(c) = \frac{\beta^{cr}\Gamma(cr+\alpha)}{\Gamma(\alpha)} \qquad (3.20)$$

Equation 3.20 is obtained a follows:

$$\mu_r'(c) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_{0}^{\infty} x^{cr+\alpha-1}\, e^{-\frac{x}{\beta}} dx$$

Let

$$\frac{x}{\beta} = v$$

$$\Rightarrow \beta v = x$$

$$\therefore \frac{dx}{dv} = \beta$$

$$\therefore dx = \beta dv$$

$$\therefore \mu_r'(c) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_{0}^{\infty} (\beta v)^{cr+\alpha-1}\, e^{-v} \beta dv$$

$$= \frac{\beta^{cr+\alpha-1}\beta}{\beta^{\alpha}\Gamma(\alpha)} \int_{0}^{\infty} v^{cr+\alpha-1}\, e^{-v} dv$$

$$= \frac{\beta^{cr}\beta^{\alpha}\beta}{\beta^{\alpha}\beta\Gamma(\alpha)} \int_{0}^{\infty} v^{cr+\alpha-1}\, e^{-v} dv$$

$$\therefore \mu_r'(c) = \frac{\beta^{cr}}{\Gamma(\alpha)} \int_0^\infty v^{cr+\alpha-1}\, e^{-v} dv$$

$$\therefore \mu_r'(c) = \frac{\beta^{cr}}{\Gamma(\alpha)} \Gamma(cr + \alpha)$$

Hence using equation 3.20 in equation 3.2 yields the $gmgf$ of the gamma family of distributions represented by the random variable $Y = X^c$, as

$$g_n(c; \lambda) = \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} \frac{\beta^{cr}\Gamma(cr + \alpha)}{\Gamma(\alpha)} \qquad (3.21)$$

As usual, all conceivable moments of the gamma family of distributions are obtained using Equation 3.21. For example, the variance of $Y = X^c$ is

$$g_2(c; \lambda) = \lambda^2 + 2\lambda \frac{\Gamma(c + \alpha)\beta^c}{\Gamma(\alpha)} + \frac{\Gamma(2c + \alpha)\beta^{2c}}{\Gamma(\alpha)}$$

If $c = 1$ then

$$g_2(1; \lambda) = \lambda^2 + 2\lambda\alpha\beta + \alpha(\alpha + 1)\beta^2$$

Hence if $\lambda = -\mu = -\alpha\beta$ where $\mu = \alpha\beta$ is the mean of the usual gamma distribution, then

$$g_2(1; -\alpha\beta) = (-\alpha\beta)^2 + 2(-\alpha\beta)(\alpha\beta) + \alpha(\alpha + 1)\beta^2$$

$$= \alpha\beta^2 = \sigma^2$$

that is the variance of the usual gamma distribution.

That is,

$$g_2(1; \lambda) = \lambda^2 + 2\lambda\alpha\beta + \alpha(\alpha + 1)\beta^2$$

If $\lambda = -\mu = -\alpha\beta$

$$\therefore g_2(1;-\alpha\beta) = (-\alpha\beta)^2 + 2(-\alpha\beta)(\alpha\beta) + \alpha(\alpha+1)\beta^2$$

$$= (\alpha\beta)^2 - 2(\alpha\beta)^2 + (\alpha^2+\alpha)\beta^2$$

$$= (\alpha\beta)^2 - 2(\alpha\beta)^2 + (\alpha\beta)^2 + \alpha\beta^2$$

$$= 2(\alpha\beta)^2 - 2(\alpha\beta)^2 + \alpha\beta^2$$

$$\therefore g_2(1;-\alpha\beta) = \alpha\beta^2$$

The third moment of the gamma family of distributions about λ is obtained from Equation 3.9 as

$$g_3(c;\lambda) = \lambda^3 + 3\lambda^2\beta^c\frac{\Gamma(c+\alpha)}{\Gamma(\alpha)} + 3\lambda\beta^{2c}\frac{\Gamma(2c+\alpha)}{\Gamma(\alpha)} + \beta^{3c}\frac{\Gamma(3c+\alpha)}{\Gamma(\alpha)}$$

If in particular $c = 1$ and $\lambda = -\alpha\beta$ where $\alpha\beta$ is the mean of the gamma distribution, then we have that

$$g_3(1;-\alpha\beta) = (-\alpha\beta)^3 + 3(-\alpha\beta)^2(\alpha\beta) + 3(-\alpha\beta)\alpha(\alpha+1)\beta^2 + \alpha(\alpha+1)(\alpha+2)\beta^3 = 2\alpha\beta^3$$

Obtained as follows;

$$g_3(1;-\alpha\beta) = (-\alpha\beta)^3 + 3(-\alpha\beta)^2(\alpha\beta) + 3(-\alpha\beta)\alpha(\alpha+1)\beta^2 + \alpha(\alpha+1)(\alpha+2)\beta^3$$

$$= -(\alpha\beta)^3 + 3(\alpha\beta)^3 + 3(-\alpha\beta)\alpha(\alpha+1)\beta^2 + \alpha(\alpha+1)(\alpha+2)\beta^3$$

$$= -(\alpha\beta)^3 + 3(\alpha\beta)^3 - 3(\alpha\beta)[(\alpha\beta)^2 + \alpha\beta^2] + (\alpha^2+\alpha)(\alpha\beta^3 + 2\beta^3)$$

$$g_3(1;\ -\alpha\beta) = -(\alpha\beta)^3 + 3(\alpha\beta)^3 - 3(\alpha\beta)^3 - 3\alpha^2\beta^3 + (\alpha\beta)^3 + 2\alpha^2\beta^3 + \alpha^2\beta^3 + 2\alpha\beta^3$$

$$= -3\alpha^2\beta^3 + 3\alpha^2\beta^3 + 2\alpha\beta^3$$

$$\therefore g_3(1;-\alpha\beta) = 2\alpha\beta^3$$

Hence, the skewness of the gamma distribution is easily obtained using Equation 3.11 as

$$sk(1) = \frac{g_3(1; -\alpha\beta)}{\left(g_2(1; -\alpha\beta)\right)^{\frac{3}{2}}} = \frac{2\alpha\beta^3}{(\alpha\beta^2)^{\frac{3}{2}}} = \frac{2}{\alpha^{\frac{1}{2}}}$$

Obtained as follows;

$$sk(1) = \frac{g_3(1; -\alpha\beta)}{\left(g_2(1; -\alpha\beta)\right)^{\frac{3}{2}}} = \frac{2\alpha\beta^3}{(\alpha\beta^2)^{\frac{3}{2}}}$$

$$= \frac{2\alpha\beta^3}{\alpha^{\frac{3}{2}}\beta^3} = \frac{2\alpha}{\alpha^{\frac{3}{2}}}$$

$$= 2\alpha^{1-\frac{3}{2}}$$

$$\therefore sk(1) = \frac{2}{\alpha^{\frac{1}{2}}}$$

Similarly, the fourth moment of the gamma distribution about its mean is

$$g_4(1; -\alpha\beta) = (-\alpha\beta)^4 + 4(-\alpha\beta)^3(\alpha\beta) + 6(-\alpha\beta)^2\alpha(\alpha+1)\beta^2 + 4(-\alpha\beta)\alpha(\alpha+1)(\alpha+2)\beta^3 + \alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4 = 6\alpha\beta^4$$

Hence the corresponding kurtosis is

$$ku(1; -\alpha\beta) = \frac{g_4(1; -\alpha\beta)}{\left(g_2(1; -\alpha\beta)\right)^2} = \frac{6\alpha\beta^4}{(\alpha\beta^2)^2} = \frac{6}{\alpha}$$

(Oyeka *et al* 2008)

Setting $\alpha = 1$ in Equation 3.19 gives the $gmgf$ of all forms of the exponential distribution as

$$g_n(c;\lambda) = \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} \beta^{cr} \Gamma(cr+1)$$

$$= \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} \beta^{cr} (cr)\Gamma(cr) \qquad (3.22)$$

Similarly, setting $\beta = 2$ and $\alpha = \frac{k}{2}$ where $k = 1, 2, \ldots$ gives the $gmgf$ of the chi-square distribution with k degrees of freedom as

$$g_n(c;\lambda) = \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} 2^{cr} \frac{\Gamma\left(cr + \frac{k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \qquad (3.23)$$

As noted earlier, $gmgf$s can be used to obtain moments of powers of random variables with negative indices. For example, the gamma density in Equation 3.21. $cr + \alpha > 0$; that is, if the real number c is such that $c \geq \frac{-\alpha}{r}$; $r = 1, 2, \ldots$, and some specified value of $\alpha > 0$. For example, if in equation 3.21 we choose $c = \frac{-3}{2}$ and $k = 10$ and, interest is in determining all possible moments of the random variable $Y = X^{\frac{-3}{2}}$, where X has the Chi-Square distribution with 10 degrees of freedom then it is possible to generate moments up to the third moment of the random variable.

Specifically, the possible moments of $Y = X^{\frac{-3}{2}}$ are obtained from Equation 3.23 as

$$g_1\left(\frac{-3}{2};\lambda\right) = \lambda + 2^{\frac{-3}{2}} \frac{\Gamma\left(\frac{-3}{2} + 5\right)}{\Gamma(5)} = \lambda + 2^{\frac{-3}{2}} \frac{\Gamma\left(\frac{7}{2}\right)}{24}$$

$$= \lambda + 2^{-\frac{3}{2}} * \frac{5}{2} * \frac{3}{2} * \frac{1}{2} * \frac{\Gamma\left(\frac{1}{2}\right)}{24}$$

$$= \lambda + 2^{-\frac{3}{2}} \frac{5\sqrt{\pi}}{64}$$

Hence setting $\lambda = 0$, we have that the mean of the random variable $Y = X^{-\frac{3}{2}}$ where X has the chi-square distribution with 10 degrees of freedom is

$$\mu = \frac{5 * 2^{-\frac{3}{2}}\sqrt{\pi}}{64} = \frac{5\sqrt{2\pi}}{256}$$

Obtained as follows:

$$\mu = \frac{5 * 2^{-\frac{3}{2}}\sqrt{\pi}}{64} = \frac{5\sqrt{\pi}}{2^{\frac{3}{2}} * 64}$$

$$= \frac{5 * \sqrt{\pi} * 2^{\frac{1}{2}}}{2^{\frac{3}{2}} * 64 * 2^{\frac{1}{2}}} = \frac{5\sqrt{2\pi}}{2^{\frac{3}{2}+\frac{1}{2}} * 64} = \frac{5\sqrt{2\pi}}{2^2 * 64}$$

$$\therefore \mu = \frac{5\sqrt{2\pi}}{256}$$

If $n = 2$ and $c = 1$, the variance of $Y = X^{-\frac{3}{2}}$ is obtained from equation 3.23 as

$$g_2\left(-\frac{3}{2};\lambda\right) = \lambda^2 + 2\lambda\left(\frac{5 * 2^{-\frac{3}{2}}\sqrt{\pi}}{64}\right) + \frac{2^{-3}\Gamma(-3+5)}{\Gamma(5)}$$

$$= \lambda^2 + 2\lambda\left(\frac{5 * \sqrt{\pi} * 2^{-\frac{3}{2}}}{64}\right) + \frac{2^{-3}}{24}$$

So that setting $\lambda = -\mu = \frac{5\sqrt{2\pi}}{256}$ gives

$$g_2\left(-\frac{3}{2};-\frac{5\sqrt{2\pi}}{256}\right) = \left(\frac{-5\sqrt{\pi} * 2^{-\frac{3}{2}}}{64}\right)^2 + 2\left(\frac{-5 * 2^{-\frac{3}{2}}\sqrt{\pi}}{64}\right)\left(\frac{5 * 2^{-\frac{3}{2}}\sqrt{\pi}}{64}\right) + \frac{2^{-3}}{24}$$

$$= \frac{2^{-3}}{24} + \left(\frac{5 * 2^{-\frac{3}{2}}\sqrt{\pi}}{64}\right)^2 - 2\left(\frac{5 * 2^{-\frac{3}{2}}\sqrt{\pi}}{64}\right)^2$$

$$g_2\left(-\frac{3}{2}; -\frac{5\sqrt{2\pi}}{256}\right) = \frac{2^{-3}}{24} - \left(\frac{5 \times 2^{-\frac{3}{2}}\sqrt{\pi}}{64}\right)^2$$

$$= \frac{1}{192} - \frac{25 * \pi}{8 * 64^2} = 0.0281148835 \simeq 0.003$$

3.4 GENERALIZED MOMENT GENERATING FUNCTION ($GMGF$) FOR THE NORMAL DISTRIBUTION

To obtain the $gmgf$ of the random variable $Y = X^c$ where X has the normal distribution with parameters μ and σ^2 with pdf, $f(x)$, given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2}; -\infty < x < \infty; -\infty < \mu < \infty; \sigma^2 > 0$$

we have from Equation 3.3 that the r^{th} moment of X^c about the origin or zero is

$$\mu_r'(c) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{cr}\, e^{-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2} dx$$

Letting $v = \left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2$, solving for x, expanding binomially, integrating and simplifying gives

$$\mu_r'(c) = \sum_{t=0}^{cr} \binom{cr}{t} \mu^{cr-t} (2\sigma^2)^{\frac{t}{2}} \frac{\Gamma\left(\frac{t}{2} + \frac{1}{2}\right)}{\sqrt{\pi}} \qquad (3.24)$$

Equation 3.24 is obtained as follows:

$$f(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2} & ; -\infty < x < \infty; -\infty < \mu < \infty; \sigma^2 > 0 \\ 0 \quad elsewhere \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} xf(x)\, dx$$

(Arua *et al* 1997)

$$\therefore E(X^{cr}) = \int_{-\infty}^{\infty} x^{cr} f(x) dx$$

$$= \int_{-\infty}^{\infty} x^{cr} \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2} dx \qquad (3.25)$$

Now letting

$$v = \left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2$$

$$\Rightarrow x = v^{\frac{1}{2}}(\sigma\sqrt{2}) + \mu$$

$$\therefore \frac{dx}{dv} = \frac{1}{2}(\sigma\sqrt{2})v^{-\frac{1}{2}} = \frac{\sigma v^{-\frac{1}{2}}}{\sqrt{2}}$$

$$\therefore dx = \frac{\sigma v^{-\frac{1}{2}}}{\sqrt{2}} dv$$

Substituting these in equation 3.25 gives

$$E(x^{cr}) = \int_{-\infty}^{\infty} \left(\mu + \sqrt{2}\sigma v^{\frac{1}{2}}\right)^{cr} \frac{1}{\sigma\sqrt{2\pi}} e^{-v} \frac{\sigma v^{-\frac{1}{2}}}{\sqrt{2}} dv$$

But

$$\left(\mu + \sqrt{2}v^{\frac{1}{2}}\right)^{cr} = \sum_{t=0}^{cr} \binom{cr}{t} \mu^{cr-t} \left(\sqrt{2}\sigma v^{\frac{1}{2}}\right)^{t} = \sum_{t=0}^{cr} \binom{cr}{t} \mu^{cr-t} (2\sigma^2)^{\frac{t}{2}} * v^{\frac{t}{2}}$$

$$\therefore E(x^{cr}) = \sum_{t=0}^{cr} \binom{cr}{t} \mu^{cr-t} (2\sigma^2)^{\frac{t}{2}} \int_{-\infty}^{\infty} v^{\frac{t}{2}} \frac{1}{\sigma\sqrt{2\pi}} e^{-v} \frac{\sigma v^{-\frac{1}{2}}}{\sqrt{2}} dv$$

$$= \sum_{t=0}^{cr} \binom{cr}{t} \mu^{cr-t} (2\sigma^2)^{\frac{t}{2}} \frac{1}{\sigma\sqrt{2\pi}} \frac{\sigma}{\sqrt{2}} \int_{-\infty}^{\infty} v^{\frac{t}{2}} e^{-v} v^{-\frac{1}{2}} dv$$

$$= \sum_{t=0}^{cr} \binom{cr}{t} \mu^{cr-t} (2\sigma^2)^{\frac{t}{2}} \frac{1}{2\sqrt{\pi}} 2 \int_{0}^{\infty} v^{\frac{t}{2}-\frac{1}{2}} e^{-v} dv$$

$$= \sum_{t=0}^{cr} \binom{cr}{t} \mu^{cr-t} (2\sigma^2)^{\frac{t}{2}} \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} v^{\frac{t}{2}+\frac{1}{2}-1} e^{-v} dv$$

$$= \sum_{t=0}^{cr} \binom{cr}{t} \mu^{cr-t} (2\sigma^2)^{\frac{t}{2}} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{t}{2} + \frac{1}{2}\right)$$

$$\therefore E(X^{cr}) = \sum_{t=0}^{cr} \binom{cr}{t} \mu^{cr-t} (2\sigma^2)^{\frac{t}{2}} \frac{\Gamma\left(\frac{t}{2} + \frac{1}{2}\right)}{\sqrt{\pi}}$$

Observe here that should the value of t be zero, that is, $t = 0$ in Equation 3.24 there would be the need to evaluate $\Gamma\left(\frac{1}{2}\right)$. This may be done using the standard normal distribution as follows;

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}; -\infty < x < \infty; \mu = 0; \sigma^2 = 1 \\ 0\ elsewhere \end{cases}$$

Recall

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

Let $\frac{x^2}{2} = v \Rightarrow x^2 = 2v$ and $x = (2v)^{\frac{1}{2}}$

Hence,

$$\frac{dx}{dv} = \frac{1}{2} 2^{\frac{1}{2}} v^{-\frac{1}{2}} = 2^{-\frac{1}{2}} v^{-\frac{1}{2}}$$

$$\therefore dx = 2^{-\frac{1}{2}} v^{-\frac{1}{2}} dv$$

$$\therefore \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} = \int_{-\infty}^{\infty} e^{-v}\, 2^{-\frac{1}{2}} v^{-\frac{1}{2}} dv = \sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} v^{-\frac{1}{2}} e^{-v} dv = 2\sqrt{\pi}$$

$$\therefore 2 \int_{0}^{\infty} v^{\frac{1}{2}-1} e^{-v} dv = 2\sqrt{\pi}$$

$$\int_{0}^{\infty} v^{\frac{1}{2}-1} e^{-v} dv = \sqrt{\pi}$$

But

$$\int_{0}^{\infty} v^{\frac{1}{2}-1}\, e^{-v} dv = \Gamma\left(\frac{1}{2}\right)$$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad \dots\dots\dots\dots\dots..(4.25b)$$

Equation 3.24 is evaluated at even numbers of t. That is $t = 0, 2, 4$, etc.

From Equation 3.2,

$$g_n(c;\lambda) = \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} \sum_{t=0}^{cr} \binom{cr}{t} \mu^{cr-t} (2\sigma^2)^{\frac{t}{2}} \frac{\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)}{\sqrt{\pi}} \qquad 3.26$$

for even numbers of t. This implies that we set

$$(2\sigma^2)^{\frac{t}{2}} \frac{\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)}{\sqrt{\pi}} = 0$$

for all odd values of t.

If

$$v = \frac{x-\mu}{\sigma}$$

we have that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v^t\, e^{-\frac{v^2}{2}} dv = 0$$

for odd values of t; that is for $t = 1, 3, 5, \dots$.

Equation 3.26 is used to generate all conceivable moments of all forms of the normal distribution represented by the random variable $Y = X^c$ for all real-valued c.

For example, the second moment of $Y = X^c$ for $c = 1$ where $X \sim N(\mu, \sigma^2)$ is

$$g_2(1;\lambda) = \sum_{r=0}^{2} \binom{2}{r} \lambda^{2-r} \sum_{t=0}^{r} \binom{r}{t} \mu^{cr-t} (2\sigma^2)^{\frac{t}{2}} \frac{\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)}{\sqrt{\pi}}$$

$$r = 0; t = 0$$

$$\binom{2}{0} \lambda^2 = \lambda^2$$

$$r = 1; t = 0$$

$$\binom{2}{1} \lambda(\mu) = 2\lambda\mu$$

$$r = 2; t = 0, 2$$

$$\binom{2}{2} \lambda^0 \left[\binom{2}{0} \mu^2 (2\sigma^2)^{\frac{0}{2}} + \binom{2}{2} \mu^0 (2\sigma^2) \frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{\pi}} \right] = \mu^2 + \frac{2\sigma^2}{2} = \mu^2 + \sigma^2$$

$$\therefore g_2(1;\lambda) = \lambda^2 + 2\lambda\mu + \mu^2 + \sigma^2$$

Let $\lambda = -\mu$

$$\therefore g_2(1;-\mu) = (-\mu)^2 + 2(-\mu)\mu + \mu^2 + \sigma^2$$

$$= \mu^2 - 2\mu^2 + \mu^2 + \sigma^2$$

$$= 2\mu^2 - 2\mu^2 + \sigma^2$$

$$\therefore g_2(1;-\mu) = \sigma^2$$

Also, the fourth moment of the random variable $Y = X^c$ about λ, where X has the normal distribution with parameters μ and σ^2 and $c = 1$, is from Equation 3.2 obtained as

$$g_4(1;\lambda) = \sum_{r=0}^{4} \binom{4}{r} \lambda^{4-r} \sum_{t=0}^{r} \binom{r}{t} \mu^{cr-t} (2\sigma^2)^{\frac{t}{2}} \frac{\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)}{\sqrt{\pi}}$$

Let $r = 0; t = 0$

$$\therefore \binom{4}{0} \lambda^4 = \lambda^4$$

Let $r = 1; t = 0$

$$\binom{4}{1} \lambda^3 \binom{1}{0} \mu' (2\sigma^1)^0 = 4\lambda^3 \mu$$

Let $r = 2; t = 0, 2$

$$\binom{4}{2} \lambda^2 \left[\sum_{t=0}^{2} \binom{2}{t} \mu^{2-t} (2\sigma^2)^{\frac{t}{2}} \frac{\Gamma\left(\frac{t}{2} + \frac{1}{2}\right)}{\sqrt{\pi}} \right]$$

$$= \binom{4}{2} \lambda^2 \left[\binom{2}{0} \mu^2 + \binom{2}{2} \mu^0 (2\sigma^2) \frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{\pi}} \right]$$

$$= 6\lambda^2 \left[\mu^2 + \frac{2\sigma^2}{2} \right] = 6\lambda^2 (\mu^2 + \sigma^2)$$

Let $r = 3; t = 0, 2$

$$\binom{4}{3} \lambda \left[\sum_{t=0}^{3} \binom{3}{t} \mu^{3-t} (2\sigma^2)^{\frac{t}{2}} \frac{\Gamma\left(\frac{t}{2} + \frac{1}{2}\right)}{\sqrt{\pi}} \right]$$

$$= 4\lambda \left[\binom{3}{0} \mu^3 (2\sigma^2)^0 + \binom{3}{2} \mu (2\sigma^2) \frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{\pi}} \right]$$

$$4\lambda \left[\mu^3 + 3\mu \frac{2\sigma^2}{2} \right] = 4\lambda (\mu^3 + 3\mu\sigma^2)$$

Let $r = 4; t = 0, 2, 4$

$$\binom{4}{4}\lambda^0\left[\sum_{t=0}^{4}\binom{4}{t}\mu^{4-t}(2\sigma^2)^{\frac{t}{2}}\frac{\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)}{\sqrt{\pi}}\right]$$

$$=\binom{4}{0}\mu^4+\binom{4}{2}\mu^2(2\sigma^2)\frac{1}{2}+\binom{4}{4}\mu^0(2\sigma^2)^2\frac{\Gamma\left(\frac{5}{2}\right)}{\sqrt{\pi}}$$

$$=\mu^4+6\mu^2\sigma^2+3\sigma^4$$

$$\therefore g_4(1;\lambda)=\lambda^4+4\lambda^3\mu+6\lambda^2(\mu^2+\sigma^2)+4\lambda(\mu^3+3\mu\sigma^2)+\mu^4+6\mu^2\sigma^2+3\sigma^2$$

The 4^{th} moment about the mean; that is, where $\lambda=-\mu$

$$g_4(1;-\mu)=(-\mu)^4+4(-\mu)\mu+6(-\mu)^2(\mu^2+\sigma^2)+4(-\mu)(\mu^3+3\mu\sigma^2)+\mu^4+6\mu^2\sigma^2+3\sigma^4$$

$$=\mu^4+4\mu^4+6\mu^4+6\mu^2\sigma^2-4\mu^4-12\mu^2\sigma^2+\mu^4+6\mu^2\sigma^2+3\sigma^4$$

$$=8\mu^4-8\mu^4+12\mu^2\sigma^2-12\mu^2\sigma^2+3\sigma^4$$

$$\therefore g_4(1;-\mu)=3\sigma^4$$

Now from Equation 3.12 the kurtosis of the normal distribution is

$$k_u(1)=\frac{g_4(1;-\mu)}{[g_2(1;-\mu)]^2}=\frac{3\sigma^4}{(\sigma^2)^2}=\frac{3\sigma^4}{\sigma^4}=3$$

Also, the skewness of the normal distribution can be obtained with Equation 3.11

$$sk(1)=\frac{\mu_3(1)}{[\mu_2(1)]^{\frac{3}{2}}}=\frac{g_3(1;-\mu)}{[g_2(1;-\mu)]^{\frac{3}{2}}}$$

$$g_3(1;\lambda)=\sum_{r=0}^{3}\binom{3}{r}\lambda^{3-r}\sum_{t=0}^{r}\binom{r}{t}\mu^{r-t}(2\sigma^2)^{\frac{t}{2}}\frac{\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)}{\sqrt{\pi}}$$

for $r=0,t=0$

$$\binom{3}{0}\lambda^3 = \lambda^3$$

for $r = 1, t = 0$

$$\binom{3}{1}\lambda^2\binom{1}{0}\mu = 3\lambda^2\mu$$

for $r = 2; t = 0, 2$

$$\binom{3}{2}\lambda\left[\binom{2}{0}\mu^2(2\sigma^2)^0 + \binom{2}{2}\mu^0(2\sigma^2)\frac{1}{2}\right]$$

$$= 3\lambda(\mu^2 + \sigma^2)$$

$r = 3; t = 0, 2$

$$\binom{3}{3}\lambda^0\left[\binom{3}{0}\mu^3 + \binom{3}{2}\mu(2\sigma^2)\frac{1}{2}\right]$$

$$= \mu^3 + 3\mu\sigma^2$$

$$\therefore g_3(1;\lambda) = \lambda^3 + 3\lambda^2\mu + 3\lambda\mu^2 + 3\lambda\sigma^2 + \mu^3 + 3\mu\sigma^2$$

Let $\lambda = -\mu$

$$\therefore g_3(1;-\mu) = (-\mu)^3 + 3\mu^3 - 3\mu^3 - 3\mu\sigma^2 + \mu^3 + 3\mu\sigma^2$$

$$g_3(1;-\mu) = 4\mu^3 - 4\mu^3 - 3\mu\sigma^2 + 3\mu\sigma^2 = 0$$

$$\therefore sk(1) = \frac{g_3(1;-\mu)}{[g_2(1;-\mu)]^{\frac{3}{2}}} = \frac{0}{(\sigma^2)^{\frac{3}{2}}} = 0$$

Where the value of $sk(1)$ equals 0 implies that the distribution is symmetric (Arua, *et al* 1997).

This shows that the normal distribution is symmetric as expected.

3.5 THE BIVARIATE GENERALIZED MOMENT GENERATING FUNCTIONS

The Bivariate generalized moment generating function (BGMGF) is the expected value of $X^{cn}Y^{dn}$; that is, $E[X^{cn}Y^{dn}]$ (Oyeka *et al*, 2012). It is interpreted as the $(cn, dn)^{th}$ moment of the joint distribution of the random variables X and Y about zero, where c, d and n are all non-negative integers. Also, c and d may be non-negative real numbers but not necessarily integers. It is assumed that both X and Y are continuously differentiable on the real line or over their range of definition with joint probability density, $f(x, y)$. The n^{th} moment or expected value of the joint distribution of c^{th} power of the random variable X and the d^{th} power of the random variable Y about zero is denoted by $\mu_n(c, d)$.

$$\mu_n(c, d) = E(X^c Y^d)^n; c \geq 0, d \geq 0, n = 0, 1, 2, \ldots \qquad 3.27$$

That is,

$$\mu_n(c, d) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (X^c)^n (Y^d)^n f(x, y) dxdy$$

$$\iint_{-\infty}^{\infty} x^{cn} y^{dn} f(x, y) dxdy = \mu(cn, dn)$$

where $\mu(cn, dn)$ is the $(cn, dn)^{th}$ moment of the joint distribution of X and Y about zero.

Therefore,

$$\mu_n(c, d) = \mu(cn, dn) = \iint_{-\infty}^{\infty} x^{cn} y^{dn} f(x, y) dxdy \qquad 3.28$$

$$\mu_0(c, d) = 1 \; \forall \; (c \geq 0, d \geq 0)$$

That is,

$$\mu_0(c,d) = \iint_{-\infty}^{\infty} x^0 y^0 f(x,y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy$$

$$= \int_{-\infty}^{\infty} f(x)\,dx \int_{-\infty}^{\infty} f(y)\,dy = 1$$

$$\therefore \mu_0(c,d) = 1$$

The first moment; $n = 1$ is, as any other moment, obtained from Equation 3.28

That is,

$$\mu(x^c y^d) = \mu_1(c,d) \qquad 3.29$$

where $\mu_1(c,d) = \mu(c,d)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^c y^d f(x,y)dxdy$$

The joint variation of the distribution of X^c and Y^d is

$$Var(X^c Y^d) = \mu_2(c,d) - \mu_1^2(c,d) \qquad 3.30$$

This is the second moment of the joint distribution of X^c, Y^d about its mean.

The skewness (sk) and kurtosis (ku) of this distribution can also be easily obtained with Equation 3.28

$$sk(x^c y^d) = \frac{\mu_3(c,d) - 3\mu_2(c,d)\mu_1(c,d) + 2\mu_1(c,d)^3}{(\mu_2(c,d) - \mu_1(c,d)^2)^{\frac{3}{2}}} \qquad 3.31$$

Obtained as follows:

$$E[X^cY^d - \mu_1(c,d)]^2 = E[X^{2c}Y^{2d} - 2X^cY^d\mu_1(c,d) + \mu_1(c,d)^2]$$

$$= E(X^{2c}Y^{2d}) - 2E(X^cY^d)\mu_1(c,d) + \mu_1(c,d)^2$$

$$= E(X^{2c}Y^{2d}) - 2\mu_1(c,d)^2 + \mu_1(c,d)^2$$

since $E(X^cY^d) = \mu_1(c,d)$

$$\therefore E[X^cY^d - \mu_1(c,d)]^2 = E(X^{2c}Y^{2d}) - \mu_1(c,d)^2$$

$$= \mu_2(c,d) - \mu_1(c,d)^2$$

$$E[X^cY^d - \mu_1(c,d)]^3 = E\left(X^cY^d - \mu_1(c,d)\right)[X^cY^d - \mu_1(c,d)]^2$$

$$= E\{[X^cY^d - \mu_1(c,d)][X^{2c}Y^{2d} - 2X^cY^d\mu_1(c,d) + \mu_1(c,d)^2]\}$$

$$= E[X^{3c}Y^{3d} - 2X^{2c}Y^{2d}\mu_1(c,d) + X^cY^d\mu_1(c,d)^2 - X^{2c}Y^{2d}\mu_1(c,d) + 2X^cY^d\mu_1(c,d)^2 - \mu_1(c,d)^3]$$

$$= E(X^{3c}Y^{3d}) - 2E(X^{2c}Y^{2d})\mu_1(c,d) + E(X^cY^d)\mu_1(c,d)^2 - E(X^{2c}Y^{2d})\mu_1(c,d) + 2E(X^cY^d)\mu_1(c,d)^2 - \mu_1(c,d)^3$$

$$= \mu_3(c,d) - 2\mu_2(c,d)\mu_1(c,d) + \mu_1(c,d)\mu_1(c,d)^2 - \mu_2(c,d)\mu_1(c,d) + 2\mu_1(c,d)\mu_1(c,d)^2 - \mu_1(c,d)^3$$

$$= \mu_3(c,d) - 2\mu_2(c,d)\mu_1(c,d) + \mu_1(c,d)^3 - \mu_2(c,d)\mu_1(c,d) + 2\mu_1(c,d)^3 - \mu_1(c,d)^3$$

$$= \mu_3(c,d) - 3\mu_2(c,d)\mu_1(c,d) + 2\mu_1(c,d)^3$$

$$sk(X^cY^d) = \frac{E\left(X^cY^d - \mu_1(c,d)\right)^3}{\left[E\left(X^cY^d - \mu_1(c,d)\right)^2\right]^{\frac{3}{2}}}$$

$$\therefore sk(X^cY^d) = \frac{\mu_3(c,d) - 3\mu_2(c,d)\mu_1(c,d) + 2\mu_1(c,d)^3}{[\mu_2(c,d) - \mu_1(c,d)^2]^{\frac{3}{2}}}$$

The kurtosis (ku) is

$$ku = \frac{\mu_4(c,d) - 4\mu_3(c,d)\mu_1(c,d) + 6\mu_2(c,d)\mu_1(c,d)^2 - 3\mu_1(c,d)^4}{[\mu_2(c,d) - \mu_1(c,d)^2]^2} \qquad 3.32$$

Obtained as follows:

$$E[X^cY^d - \mu_1(c,d)]^4 = E\left(X^cY^d - \mu_1(c,d)\right)[X^cY^d - \mu_1(c,d)]^3$$

$$= E\left(X^cY^d - \mu_1(c,d)\right)(X^{3c}Y^{3d} - 2X^{2c}Y^{2d}\mu_1(c,d) + X^cY^d\mu_1(c,d)^2 - X^{2c}Y^{2d}\mu_1(c,d) + 2X^cY^d\mu_1(c,d)^2 - \mu_1(c,d)^3)$$

$$= E[X^{4c}Y^{4d} - 2X^{3c}Y^{3c}\mu_1(c,d) + X^{2c}Y^{2d}\mu_1(c,d)^2 - X^{3c}Y^{3d}\mu_1(c,d) + 2X^{2c}Y^{2d}\mu_1(c,d)^2 - X^cY^d\mu_1(c,d)^3 - X^{3c}Y^{3d}\mu_1(c,d) + 2X^{2c}Y^{2d}\mu_1(c,d)^2 - X^cY^d\mu_1(c,d)^3 + X^{2c}Y^{2d}\mu_1(c,d)^2 - 2X^cY^d\mu_1(c,d)^3 + \mu_1(c,d)^4]$$

$$= \mu_4(c,d) - 2\mu_3(c,d)\mu_1(c,d) + \mu_2(c,d)\mu_1(c,d)^2 - \mu_3(c,d)\mu_1(c,d) + 2\mu_2(c,d)\mu_1(c,d)^2 - \mu_1(c,d)\mu_1(c,d)^3 - \mu_3(c,d)\mu_1(c,d) + 2\mu_2(c,d)\mu_1(c,d)^2 - \mu_1(c,d)\mu_1(c,d)^3 + \mu_2(c,d)\mu_1(c,d)^2 - 2\mu_1(c,d)\mu_1(c,d)^3 + \mu_1(c,d)^4$$

$$= \mu_4(c,d) - 4\mu_3(c,d)\mu_1(c,d) + 6\mu_2(c,d)\mu_1(c,d)^2 - 3\mu_1(c,d)^4$$

$$\therefore E[X^cY^d - \mu_1(c,d)]^4 = \mu_4(c,d) - 4\mu_3(c,d)\mu_1(c,d) + 6\mu_2(c,d)\mu_1(c,d)^2 - 3\mu_1(c,d)^4$$

From Equation 3.12

$$ku = \frac{E[X^cY^d - \mu_1(c,d)]^4}{\left[E\left(X^cY^d - \mu_1(c,d)\right)\right]^2}$$

$$ku = \frac{\mu_4(c,d) - 4\mu_3(c,d)\mu_1(c,d) + 6\mu_2(c,d)\mu_1(c,d)^2 - 3\mu_1(c,d)^4}{[\mu_2(c,d) - \mu_1(c,d)^2]^2} \qquad 3.33$$

Suppose in Equation 3.28 we set $d = 0$, we obtain the marginal distribution of X^c,

$$\mu_{xn}(c) = \mu_n(c,0) \qquad 3.34$$

that is,

$$\mu_n(c, d = 0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^{cn}\, y^{n,d=0} f(x,y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^{cn}\, y^{0} f(x)f(y)dxdy$$

$$= \int_{-\infty}^{\infty} x^{nc}\, f(x)dx \int_{-\infty}^{\infty} f(y)\, dy$$

But

$$\int_{-\infty}^{\infty} f(y)\, dy = 1$$

$$\therefore \mu_n(c) = \mu_n(c, d = 0) = \int_{-\infty}^{\infty} x^{nc}\, f(x)dx$$

Similarly, we can obtain the n^{th} moment of Y^d about zero by setting $c = 0$ in Equation 3.28; that is,

$$\mu_{yn}(d) = \mu_n(0,d) \qquad 3.35$$

Hence, the skewness (sk) and kurtosis (ku) of the marginal distribution of X^c and Y^d may also be obtained as

$$sk(X^c) = \frac{\mu_{x3}(c) - 3\mu_{x2}(c)\mu_{x1}(c) + 2\mu_{x1}(c)^3}{[\mu_{x2}(c) - \mu_{x1}(c)^2]^{\frac{3}{2}}} \qquad 3.36$$

$$ku(X^c) = \frac{\mu_{x4}(c) - 4\mu_{x3}(c)\mu_{x1}(c) + 6\mu_{x2}(c)\mu_{x1}(c)^2 - 3\mu_{x1}(c)^4}{[\mu_{x2}(c) - \mu_{x1}(c)^2]^{\frac{3}{2}}} \qquad 3.37$$

$$sk(Y^d) = \frac{\mu_{y3}(d) - 3\mu_{y2}(d)\mu_{y1}(d) + 2\mu_{y1}(d)^3}{\left[\mu_{y2}(d) - \mu_{y1}(d)^2\right]^{\frac{3}{2}}} \qquad 3.38$$

$$ku(Y^d) = \frac{\mu_{y4}(d) - 4\mu_{y3}(d)\mu_{y1}(d) + 6\mu_{y2}(d)\mu_{y1}(d)^2 - 3\mu_{y1}(d)^4}{\left[\mu_{y2}(d) - \mu_{y1}(d)^2\right]^2} \qquad 3.39$$

Example:

Suppose two continuous random variables X and Y have the joint pdf

$$f(x,y) = \frac{2}{\beta^2} xye^{-\frac{y}{\beta}}; 0 < x < 1, y > 0 \qquad 3.40$$

then we have from Equation 3.28 that

$$\mu_n(c,d) = E(X^c Y^d)^n = E(X^{cn} Y^{dn}) = \int_0^1 \int_0^\infty x^{cn} y^{dn} f(x,y)\, dxdy$$

$$= \frac{2}{\beta^2} \int_0^1 \int_0^\infty x^{cn} y^{dn}\, xye^{-\frac{y}{\beta}} dxdy$$

$$= \frac{2}{\beta^2} \int_0^1 x^{cn+1}\, dx \int_0^\infty y^{dn+1}\, e^{-\frac{y}{\beta}} dy$$

$$= \frac{2}{\beta^2} \frac{x^{cn+2}}{cn+2}\Bigg]_0^1 \int_0^\infty y^{dn+1}\, e^{-\frac{y}{\beta}} dy$$

Now, let

$$w = \frac{y}{\beta}$$

$$\Rightarrow y = \beta w$$

$$\therefore \frac{dy}{dw} = \beta$$

$$\Rightarrow dy = \beta dw$$

$$\therefore \frac{2}{\beta^2}\frac{x^{cn+2}}{cn+2}\Big]_0^1 \int_0^\infty y^{dn+1}\, e^{-\frac{y}{\beta}} dy = \frac{2}{\beta^2}\frac{x^{cn+2}}{cn+2}\Big]_0^1 \int_0^\infty (\beta w)^{dn+1}\, e^{-w}\beta dw$$

$$= \frac{2}{\beta^2}\frac{x^{cn+2}}{cn+2}\Big]_0^1 \beta^{(dn+2)} \int_0^\infty w^{(dn+2)-1}\, e^{-w} dw$$

$$= \frac{2}{(cn+2)\beta^2}\beta^{(dn+2)}\Gamma(dn+2)$$

$$\mu_n(c,d) = \frac{2}{(cn+2)}\beta^{dn}\Gamma(dn+2) \qquad 3.41$$

$$\Rightarrow \mu_0(c,d) = \frac{2}{(0+2)}\beta^0\Gamma(0+2)$$

$$= \frac{2}{2}\Gamma(2) = 1$$

$$\therefore \mu_0(c,d) = 1$$

Also, if $n = 1$ we have that

$$\mu_1(c,d) = \frac{2}{(c+2)}\beta^d\Gamma(d+2) = \frac{2\beta^d\Gamma(d+2)}{(c+2)}$$

As stated earlier, c and d need not be integers. Thus, if $c = \frac{1}{3}$ and $d = \frac{1}{2}$ then, the mean of the joint distribution of $X^{\frac{1}{3}}$ and $Y^{\frac{1}{2}}$ is,

$$\mu\left(X^{\frac{1}{3}}, Y^{\frac{1}{2}}\right) = \mu_1\left(\frac{1}{3}, \frac{1}{2}\right) = \frac{2\beta^{\frac{1}{2}}\Gamma\left(\frac{1}{2}+2\right)}{\left(\frac{1}{3}+2\right)}$$

$$\mu\left(X^{\frac{1}{3}}, Y^{\frac{1}{2}}\right) = \frac{2\beta^{\frac{1}{2}}\Gamma\left(\frac{5}{2}\right)}{\frac{7}{3}} = \frac{2\beta^{\frac{1}{2}}\left(\frac{3}{2}\times\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\frac{7}{3}} = 2\beta^{\frac{1}{2}}\left(\frac{3}{4}\times\frac{3}{7}\right)\sqrt{\pi} = \frac{9}{14}\sqrt{\beta\pi}$$

$$\therefore \mu_1\left(\frac{1}{3}, \frac{1}{2}\right) \approx 1.1394\sqrt{\beta} \approx 1.14\sqrt{\beta}$$

Applying Equation 3.30 in Equation 3.41 we have that the joint variation of X^cY^d is

$$Var(X^cY^d) = \mu_2(c,d) - \mu_1(c,d)^2$$

$$= \frac{2\beta^{dn}\Gamma(2d+2)}{(2c+2)} - \left(\frac{2\beta^{d}\Gamma(d+2)}{(c+2)}\right)^2$$

$$= \frac{2\beta^{\left(2*\frac{1}{2}\right)}\Gamma(1+2)}{\frac{2}{3}+2} - \left(\frac{2\beta^{\frac{1}{2}}\Gamma\left(\frac{1}{2}+2\right)}{\frac{1}{3}+2}\right)^2$$

$$= \frac{2\beta\Gamma(3)}{\frac{8}{3}} - \left(\frac{2\beta^{\frac{1}{2}}\Gamma\left(\frac{5}{2}\right)}{\frac{7}{3}}\right)^2 = \frac{12\beta}{8} - \left(\frac{18}{28}\beta^{\frac{1}{2}}\pi^{\frac{1}{2}}\right)^2$$

$$= \frac{3}{2}\beta - \left(\frac{9}{14}\sqrt{\beta\pi}\right)^2 = \frac{3}{2}\beta - \frac{81}{196}\beta\pi = \left(\frac{3}{2} - \frac{81}{196}\pi\right)\beta$$

$$= (1.5 - 1.29831125)\beta \approx 0.20\beta$$

Equation 3.41 would have also been obtained using the joint distribution of X^cY^d given the joint distribution of X and Y. That is,

$$f(x,y) = \frac{2}{\beta^2} xye^{-\frac{y}{\beta}};\ 0 < x < 1, y > 0$$

$$E(X^c Y^d)^n = \frac{2}{\beta^2} \int_0^1 \int_0^\infty (x^c y^d)^n \, xye^{-\frac{y}{\beta}} dxdy$$

$$= \frac{2}{\beta^2} \int_0^1 \int_0^\infty x^{cn} \, y^{dn} xye^{-\frac{y}{\beta}} dxdy$$

Let $x^c = u$ and $y^d = v$

$$\Rightarrow x = u^{\frac{1}{c}};\ \frac{dx}{du} = \frac{u^{\frac{1}{c}-1}}{c}$$

$$\therefore dx = \frac{u^{\frac{1}{c}-1}}{c} du$$

Also,

$$y = v^{\frac{1}{d}};\ \frac{dy}{dv} = \frac{v^{\frac{1}{d}-1}}{d}$$

$$\therefore dy = \frac{v^{\frac{1}{d}-1}}{d} dv$$

$$\therefore f(u,v) = \frac{2}{\beta^2} u^{\frac{1}{c}} v^{\frac{1}{d}} e^{-\frac{v^{\frac{1}{d}}}{\beta}}$$

$$\therefore E(uv)^n = E(u^n v^n) = \frac{2}{\beta^2} \int_0^1 \int_0^\infty u^n * v^n * u^{\frac{1}{c}} * v^{\frac{1}{d}} * e^{-\frac{v^{\frac{1}{d}}}{\beta}} * \frac{u^{\frac{1}{c}-1}}{c} du * \frac{V^{\frac{1}{d}}}{d} dv$$

$$E(uv)^n = \frac{2}{cd\beta^2} \int_0^1 u^{\frac{nc+2}{c}-1} \, du \int_0^\infty v^{\frac{nd+2}{d}-1} \, e^{-\frac{v^{\frac{1}{d}}}{\beta}} dv$$

$$= \frac{2}{cd\beta^2}\left[\frac{u^{\frac{nc+2}{c}}}{\frac{nc+2}{c}}\right]_0^1 \int_0^{\infty} v^{\frac{nd+2}{d}-1} e^{-\left(\frac{v^{\frac{1}{d}}}{\beta}\right)} dv$$

Let $z = \frac{v^{\frac{1}{d}}}{\beta}$; $v^{\frac{1}{d}} = z\beta$

$$\therefore v = (z\beta)^d = z^d\beta^d; \ \frac{dv}{dz} = d\beta^d z^{d-1}$$

$$\therefore dv = d\beta^d z^{d-1} dz$$

$$\therefore E(uv)^n = \frac{2}{cd\beta^2}\left[\frac{u^{\frac{nc+2}{c}}}{\frac{nc+2}{c}}\right]_0^1 \int_0^{\infty} (z^d\beta^d)^{\frac{nd+2}{d}-1} e^{-z} d\beta^d z^{d-1} dz$$

$$= \frac{2\beta^{nd+2-d}}{cd\beta^2} d\beta^d \frac{c}{nc+2}\left[u^{\frac{nc+2}{c}}\right]_0^1 \Gamma(nd+2)$$

$$= \frac{2\beta^{nd+2}}{cd\beta^2} * \frac{cd}{nc+2} * \Gamma(nd+2) = \frac{2\beta^{nd}}{nc+2}\Gamma(nd+2)$$

$$\therefore \mu_n(c,d) = E(u^n v^n) = \frac{2\beta^{nd}\Gamma(dn+2)}{cn+2} \qquad 3.42$$

Equation 3.42 is the same result as Equation 3.41.

The n^{th} moment, $\mu_{xn}(c)$, of the marginal distribution of X^c for this illustrative example may be obtained by setting $d = 0$ in either Equation 3.41 or Equation 3.42 on the basis of the marginal distribution of X^c given in Equation 3.40 as

$$\mu_{xn}(c) = \mu_n(c,0) = \frac{2}{cn+2} \qquad 3.43$$

Similarly, $\mu_{yn}(d)$, the corresponding n^{th} moment of the marginal distribution of Y^d is

$$\mu_{yn}(d) = \mu_n(0, d) = \beta^{dn}\Gamma(dn + 2) \qquad 3.44$$

The first moment, $(n = 1)$, of the marginal distribution of $X^{\frac{1}{3}}$ about zero is obtained using Equation 3.43 as

$$\mu_1\left(\frac{1}{3}, 0\right) = \mu_{x1}\left(\frac{1}{3}\right) = \mu_x'\left(\frac{1}{3}\right) = \frac{2}{\frac{1}{3} + 2} = \frac{2 \times 3}{7}$$

$$\therefore \mu_x'\left(\frac{1}{3}\right) = \frac{6}{7}$$

and

$$\mu_{x2}'\left(\frac{1}{3}, 0\right) = \frac{2}{\frac{1}{3} * 2 + 2} = \frac{3}{4}$$

while the corresponding variance is obtained from Equation 3.34 and 3.43 for $c = \frac{1}{3}$ as,

$$var\left(X^{\frac{1}{3}}\right) = \mu_{x2}\left(\frac{1}{3}\right) - \mu_{x1}\left(\frac{1}{3}\right)^2$$

$$= \frac{3}{4} - \left(\frac{6}{7}\right)^2$$

$$\therefore var\left(X^{\frac{1}{3}}\right) = 0.01530612245 \approx 0.02$$

The skewness of the marginal distribution of $X^{\frac{1}{3}}$ from Equation 3.36 is,

$$sk\left(X^{\frac{1}{3}}\right) = \frac{\mu_{x3}\left(\frac{1}{3}\right) - 3\mu_{x2}\left(\frac{1}{3}\right)\mu_{x1}\left(\frac{1}{3}\right) + 2\mu_{x1}\left(\frac{1}{3}\right)^3}{\left(\mu_{x2}\left(\frac{1}{3}\right) - \mu_{x1}\left(\frac{1}{3}\right)^2\right)^{\frac{3}{2}}}$$

$$\mu_{x3}\left(\frac{1}{3}\right) = \frac{2}{(3)\frac{1}{3} + 2} = \frac{2}{3}$$

$$\mu_{x2} = \frac{2}{2\left(\frac{1}{3}\right) + 2} = \frac{3}{4}$$

$$\therefore sk\left(X^{\frac{1}{3}}\right) = \frac{\frac{2}{3} - 3\left(\frac{3}{4}\right)\left(\frac{6}{7}\right) + 2\left(\frac{6}{7}\right)^3}{\left(\frac{3}{4} - \frac{36}{49}\right)^{\frac{3}{2}}}$$

$$= \frac{\frac{2}{3} - \frac{54}{28} + \frac{432}{343}}{\left(\frac{3}{4} - \frac{36}{49}\right)^{\frac{3}{2}}} = -\frac{\frac{5}{2058}}{\left(\frac{3}{196}\right)^{\frac{3}{2}}} = -\frac{0.00242954324}{0.00189364155} = -1.283000597$$

$sk < 0$ implies that the distribution is negatively skewed (Arua *et al* 1997).

The marginal distribution of X^c is negatively skewed.

The kurtosis can be obtained from Equation 3.37 as follows:

$$ku\left(X^{\frac{1}{3}}\right) = \frac{\mu_{x4}\left(\frac{1}{3}\right) - 4\mu_{x3}\left(\frac{1}{3}\right)\mu_{x1}\left(\frac{1}{3}\right) + 6\mu_{x2}\left(\frac{1}{3}\right)\mu_{x1}\left(\frac{1}{3}\right)^2 - 3\mu_{x1}\left(\frac{1}{3}\right)^4}{\left(\mu_{x2}\left(\frac{1}{3}\right) - \mu_{x1}\left(\frac{1}{3}\right)^2\right)^2}$$

$$\mu_{x4}\left(\frac{1}{3}\right) = \frac{2}{4\left(\frac{1}{3}\right) + 2} = \frac{6}{10} = \frac{3}{5}$$

$$\therefore ku\left(X^{\frac{1}{3}}\right) = \frac{\frac{3}{5} - 4\left(\frac{2}{3}\right)\left(\frac{6}{7}\right) + 6\left(\frac{3}{4}\right)\left(\frac{6}{7}\right)^2 - 3\left(\frac{6}{7}\right)^4}{\left(\frac{3}{196}\right)^2} = \frac{0.00108288213}{0.00000005488}$$

$$\therefore ku\left(x^{\frac{1}{3}}\right) = 19729.69872$$

A value of $ku(c) > 3$ implies leptokurtic distribution (Arua et al 1997). Thus, the distribution of $X^{\frac{1}{3}}$ is leptokurtic.

Also, the skewness and kurtosis of the distribution of $Y^{\frac{1}{2}}$ can be obtained from Equations 3.11 and 3.12 as follows:

Using Equation 3.44

Substituting $n = 1, d = \frac{1}{2}$ yields

$$\mu_{y1}\left(\frac{1}{2}\right) = \frac{1}{2}\beta^{\frac{1}{2}}\sqrt{\pi} \approx 0.8862269255\beta^{\frac{1}{2}}$$

Substituting $n = 2, d = \frac{1}{2}$ yields

$$\mu_{y2}\left(\frac{1}{2}\right) = 2\beta$$

Substituting $n = 3, d = \frac{1}{2}$ yields

$$\mu_{y3}\left(\frac{1}{2}\right) = \frac{15}{8}\beta^{\frac{3}{3}}\sqrt{\pi}$$

Substituting $n = 4, d = \frac{1}{2}$ yields

$$\mu_{y4}\left(\frac{1}{2}\right) = 6\beta^2$$

Therefore,

$$sk\left(y^{\frac{1}{2}}\right) = \frac{\mu_{y3}\left(\frac{1}{2}\right) - 3\mu_{y2}\left(\frac{1}{2}\right)\mu_{y1}\left(\frac{1}{2}\right) + 2\mu_{y1}\left(\frac{1}{2}\right)^2}{\left(\mu_{y2}\left(\frac{1}{2}\right) - \mu_{y1}\left(\frac{1}{2}\right)^2\right)^{\frac{3}{2}}}$$

$$= \frac{\frac{15}{8}\beta^{\frac{3}{2}}\sqrt{\pi} - 3(2\beta)\left(\frac{1}{2}\beta^{\frac{1}{2}}\right)\sqrt{\pi} + 2\left(\frac{1}{4}\beta\right)\pi}{\left(2\beta - \frac{1}{4}\beta\pi\right)^{\frac{3}{2}}}$$

$$= \frac{\frac{15}{8}\beta^{\frac{3}{2}}\sqrt{\pi} - 3\beta^{\frac{3}{2}}\sqrt{\pi} + \frac{1}{2}\beta\pi}{\left(2\beta - \frac{1}{4}\beta\pi\right)^{\frac{3}{2}}}$$

$$= \frac{\beta\left(\frac{15}{8}\beta^{\frac{1}{2}}\sqrt{\pi} - 3\beta^{\frac{1}{2}}\sqrt{\pi} + \frac{1}{2}\pi\right)}{\beta^{\frac{3}{2}}\left(2 - \frac{1}{4}\pi\right)^{\frac{3}{2}}}$$

$$sk\left(y^{\frac{1}{2}}\right) = \frac{\left(\frac{15}{8}\sqrt{\beta\pi} - 3\sqrt{\beta\pi} + \frac{1}{2}\pi\right)}{\sqrt{\beta}\left(2 - \frac{1}{4}\pi\right)^{\frac{3}{2}}} \qquad 3.45$$

$$= \frac{3\sqrt{\beta\pi}\left(\frac{5}{8} - 1\right) + \frac{1}{2}\pi}{\sqrt{\beta}\left(2 - \frac{1}{4}\pi\right)^{\frac{3}{2}}}$$

$$sk\left(y^{\frac{1}{2}}\right) = \frac{\frac{-9}{8}\sqrt{\beta\pi} + \frac{1}{2}\pi}{\sqrt{\beta}\left(2 - \frac{1}{4}\pi\right)^{\frac{3}{2}}}$$

$$= \frac{\sqrt{\pi}\left(\frac{-9}{8}\sqrt{\beta} + \frac{1}{2}\sqrt{\pi}\right)}{\sqrt{\beta}\left(2 - \frac{1}{4}\pi\right)^{\frac{3}{2}}}$$

$$\therefore sk\left(y^{\frac{1}{2}}\right) = \sqrt{\frac{\pi}{\beta}}\left(\frac{\frac{1}{2}\sqrt{\pi} - \frac{9}{8}\sqrt{\beta}}{\left(2 - \frac{1}{4}\pi\right)^{\frac{3}{2}}}\right)$$

The distribution of $y^{\frac{1}{2}}$ is negatively skewed for $\beta > 0$

Also, the kurtosis of the distribution of $y^{\frac{1}{2}}$ can be obtained as follows:

$$ku\left(y^{\frac{1}{2}}\right) = \frac{\mu_{y4}\left(\frac{1}{2}\right) - 4\mu_{y3}\left(\frac{1}{2}\right)\mu_{y1}\left(\frac{1}{2}\right) + 6\mu_{y2}\left(\frac{1}{2}\right)\mu_{y1}\left(\frac{1}{2}\right)^2 - 3\mu_{y1}\left(\frac{1}{2}\right)^4}{\left(\mu_{y2}\left(\frac{1}{2}\right) - \mu_{y1}\left(\frac{1}{2}\right)^2\right)^2}$$

$$\therefore ku\left(y^{\frac{1}{2}}\right) = \frac{6\beta^2 - 4\left(\frac{15}{8}\beta^{\frac{3}{2}}\sqrt{\pi}\right)\left(\frac{1}{2}\beta^{\frac{1}{2}}\sqrt{\pi}\right) + 6(2\beta)\left(\frac{1}{2}\beta^{\frac{1}{2}}\sqrt{\pi}\right)^2 - 3\left(\frac{1}{2}\beta^{\frac{1}{2}}\sqrt{\pi}\right)^4}{\left(2\beta - \left(\frac{1}{2}\beta^{\frac{1}{2}}\sqrt{\pi}\right)^2\right)^2}$$

$$= \frac{6\beta^2 - 4\beta^2\left(\frac{15}{16}\right)\pi + \frac{6}{2}\beta^2\pi - \frac{3}{16}(\beta\pi)^2}{\left(2\beta - \frac{1}{4}\beta\pi\right)^2}$$

$$ku\left(y^{\frac{1}{2}}\right) = \frac{\beta^2\left(6 - \frac{15}{4}\pi + \frac{6}{2}\pi - \frac{3}{16}\pi^2\right)}{\beta^2\left(4 + \pi\left(\frac{\pi - 8}{16}\right)\right)} = \frac{6 + \pi\left(\frac{6}{2} - \frac{15}{4}\right) - \frac{3}{16}\pi^2}{4 + \pi\left(\frac{\pi - 8}{16}\right)}$$

$$\therefore ku\left(y^{\frac{1}{2}}\right) = \frac{6 - \frac{3}{4}\pi - \frac{3}{16}\pi^2}{4 + \pi\left(\frac{\pi - 8}{16}\right)} = -0.0291$$

This value of $ku\left(y^{\frac{1}{2}}\right)$ is negative and independent of β. The magnitude of β does not determine whether it is leptokurtic, mesokurtic or platykurtic as values of $ku > 3$ implies leptokurtic, $ku = 0$ implies mesokurtic while values of $ku < 3$ implies platykurtic (Arua *et al* 1997). Thus kurtosis and skewness do not depend on the parameter of the distribution of $Y^{\frac{1}{2}}$, β.

Suppose the random variables X and Y have the joint probability density function (pdf):

$$f(x,y) = (x+y)\frac{e^{-\left(\frac{x}{\beta_1}+\frac{y}{\beta_2}\right)}}{\beta_1\beta_2(\beta_1+\beta_2)};\ x>0, y>0 \qquad 3.46$$

Then

$$\mu_n(c,d) = E(X^cY^d)^n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^{cn}y^{dn}f(x,y)dxdy$$

$$\mu_n(c,d) = E(X^cY^d)^n = \frac{1}{\beta_1\beta_2(\beta_1+\beta_2)}\int_0^{\infty}\int_0^{\infty}(x+y)\,.\,e^{-\left(\frac{x}{\beta_1}+\frac{y}{\beta_2}\right)}dxdy$$

$$= \frac{1}{\beta_1\beta_2(\beta_1+\beta_2)}\int_0^{\infty}\int_0^{\infty} x^{cn}y^{dn}(x+y)e^{-\left(\frac{x}{\beta_1}+\frac{y}{\beta_2}\right)}dxdy$$

$$= \frac{1}{\beta_1\beta_2(\beta_1+\beta_2)}\int_0^{\infty}\int_0^{\infty}(x^{cn+1}y^{dn}+x^{cn}y^{dn+1})\,e^{-\left(\frac{x}{\beta_1}+\frac{y}{\beta_2}\right)}dxdy$$

$$= \frac{1}{\beta_1\beta_2(\beta_1+\beta_2)}\left[\int_0^{\infty}\int_0^{\infty} x^{cn+1}y^{dn}e^{-\left(\frac{x}{\beta_1}+\frac{y}{\beta_2}\right)}dxdy + \int_0^{\infty}\int_0^{\infty} x^{cn}y^{dn+1}\,e^{-\left(\frac{x}{\beta_1}+\frac{y}{\beta_2}\right)}dxdy\right]$$

Let $\frac{x}{\beta_1} = u; x = \beta_1 u;\ \frac{dx}{du} = \beta_1;\ \therefore dx = \beta_1 du$ and $\frac{y}{\beta_2} = v; y = \beta_2 v;\ \frac{dy}{dv} = \beta_2;\ \therefore$ $dy = \beta_2 dv$

$$\Rightarrow E(X^{cn}Y^{dn})$$

$$= \frac{1}{\beta_1\beta_2(\beta_1+\beta_2)}\left[\int_0^\infty\int_0^\infty (\beta_1 u)^{cn+1}(\beta_2 v)^{dn} e^{-(u+v)}\beta_1\beta_2 du dv + \int_0^\infty\int_0^\infty (\beta_1 u)^{cn}(\beta_2 v)^{dn+1}\, e^{-(u+v)}\,\beta_1\beta_2 du dv\right]$$

$$E(X^cY^d)^n = \frac{1}{\beta_1\beta_2(\beta_1+\beta_2)}\left[\int_0^\infty\int_0^\infty (\beta_1 u)^{cn+1}e^{-u}(\beta_2 v)^{dn}e^{-v}\beta_1\beta_2 du dv + \int_0^\infty\int_0^\infty (\beta_1 u)^{cn}e^{-u}(\beta_2 v)^{dn+1}e^{-v}\beta_1\beta_2\, du dv\right]$$

$$= \frac{\beta_1\beta_2}{\beta_1\beta_2(\beta_1+\beta_2)}\left[\int_0^\infty \beta_1^{cn+1}u^{cn+1}e^{-u}du\int_0^\infty \beta_2^{dn}v^{dn}e^{-v}dv + \int_0^\infty \beta_1^{cn}u^{cn}e^{-u}du\int_0^\infty \beta_2^{dn+1}v^{dn+1}e^{-v}dv\right]$$

$$= \frac{\beta_1^{cn+1}\Gamma(cn+2)\beta_2^{dn}\Gamma(dn+1) + \beta_1^{cn}\Gamma(cn+1)\beta_2^{dn+1}\Gamma(dn+2)}{\beta_1+\beta_2}$$

$$\therefore \mu_n(c,d) = \frac{\beta_1^{cn}\beta_2^{dn}[\beta_1\Gamma(cn+2)\Gamma(dn+1) + \beta_2\Gamma(cn+1)\Gamma(dn+2)]}{\beta_1+\beta_2} \qquad 3.47$$

The mean of the joint distribution of X and Y can be obtained by setting $c = d = 1$ and $n = 1$ in Equation 3.47. That is;

$$\mu_1(1,1) = \frac{\beta_1\beta_2\big(\beta_1\Gamma(1+2)\Gamma(1+1) + \beta_2\Gamma(1+2)\Gamma(1+1)\big)}{\beta_1+\beta_2}$$

$$= \frac{\beta_1\beta_2(2\beta_1+2\beta_2)}{\beta_1+\beta_2} = \frac{2\beta_1\beta_2(\beta_1+\beta_2)}{(\beta_1+\beta_2)}$$

$$\therefore \mu_1(1,1) = 2\beta_1\beta_2$$

The variance of the joint distribution of X and Y is obtained using Equation 3.30 in Equation 3.47 as follows:

$$var(x,y) = \mu_2(1,1) - \mu_1(1,1)^2$$

$$\mu_2(1,1) = \frac{\beta_1^2\beta_2^2[\beta_1\Gamma(2+2)\Gamma(2+1) + \beta_2\Gamma(2+1)\Gamma(2+2)]}{\beta_1+\beta_2}$$

$$= \frac{\beta_1^2\beta_2^2[\beta_1\Gamma 4\Gamma 3 + \beta_2\Gamma 3\Gamma 4]}{\beta_1+\beta_2} = \frac{\beta_1^2\beta_2^2(12\beta_1 + 12\beta_2)}{\beta_1+\beta_2}$$

$$= \frac{12\beta_1^2\beta_1^2(\beta_1+\beta_1)}{(\beta_1+\beta_2)} = 12\beta_1^2\beta_2^2$$

$$\therefore var(X,Y) = 12\beta_1^2\beta_2^2 - (2\beta_1\beta_2)^2 = 12\beta_1^2\beta_2^2 - 4\beta_1^2\beta_2^2$$

$$\Rightarrow var(X,Y) = 8\beta_1^2\beta_2^2$$

The skewness of the joint distribution of X and Y can be obtained by applying Equation 3.47 in Equation 3.31. That is,

$$sk(X^cY^d) = \frac{\mu_3(c,d) - 3\mu_2(c,d)\mu_1(c,d) + 2\mu_1(c,d)^3}{(\mu_2(c,d) - \mu_1(c,d)^2)^{\frac{3}{2}}}$$

$$\mu_3(1,1) = \frac{\beta_1^3\beta_2^3[\beta_1\Gamma(3+2)\Gamma(3+1) + \beta_2\Gamma(3+1)\Gamma(3+2)]}{(\beta_1+\beta_2)}$$

$$= \frac{\beta_1^3\beta_2^3[144\beta_1 + 144\beta_2]}{(\beta_1+\beta_2)} = \frac{144\beta_1^3\beta_2^3(\beta_1+\beta_2)}{(\beta_1+\beta_2)}$$

$$\therefore \mu_3(1,1) = 144\beta_1^3\beta_2^3$$

$$\therefore sk(X,Y) = \frac{144\beta_1^3\beta_2^3 - 3(12\beta_1^2\beta_2^2)(2\beta_1\beta_2) + 2(\beta_1\beta_2)^3}{(12\beta_1^2\beta_2^2 - (2\beta_1\beta_2)^2)^{\frac{3}{2}}}$$

$$= \frac{144\beta_1^3\beta_2^3 - 72\beta_1^3\beta_2^3 + 16\beta_1^3\beta_2^3}{(8\beta_1^2\beta_2^2)^{\frac{3}{2}}} = \frac{72\beta_1^3\beta_2^3 + 16\beta_1^3\beta_2^3}{(8\beta_1^2\beta_2^2)^{\frac{3}{2}}}$$

$$= \frac{88\beta_1^3\beta_2^3}{8^{\frac{3}{2}}\beta_1^3\beta_2^3}$$

$$\therefore sk(X,Y) = \frac{88}{8^{\frac{3}{2}}} = \frac{88}{\sqrt{512}}$$

Thus, the distribution of X^c and Y^d is positively skewed for $c = d = 1$.

The kurtosis can also be obtained by applying Equation 3.32 as follows:

$$ku(X,Y) = \frac{\mu_4(c,d) - 4\mu_3(c,d)\mu_1(c,d) + 6\mu_2(c,d)\mu_1(c,d)^2 - 3\mu_1(c,d)^4}{(\mu_2(c,d) - \mu_1(c,d)^2)^2}$$

From Equation 3.47

$$\mu_4(1,1) = \frac{\beta_1^4\beta_2^4[\beta_1\Gamma(6)\Gamma(2) + \beta_2\Gamma(2)\Gamma(6)]}{(\beta_1 + \beta_2)}$$

$$= \frac{\beta_1^4\beta_2^4[120\beta_1 + 120\beta_2]}{(\beta_1 + \beta_2)} = \frac{120\beta_1^4\beta_2^4(\beta_1 + \beta_2)}{(\beta_1 + \beta_2)}$$

$$\therefore \mu_4(1,1) = 120\beta_1^4\beta_2^4$$

$$\Rightarrow ku(X,Y)$$
$$= \frac{120\beta_1^4\beta_2^4 - 4(144\beta_1^3\beta_2^3)(2\beta_1\beta_2) + 6(12\beta_1^2\beta_2^2)(2\beta_1\beta_2)^2 - 3(2\beta_1\beta_2)^4}{(8\beta_1^2\beta_2^2)^2}$$

$$= \frac{120\beta_1^4\beta_2^4 - 1152(\beta_1^4\beta_2^4) + 288\beta_1^4\beta_2^4 - 48\beta_1^4\beta_2^4}{64\beta_1^4\beta_2^4} = \frac{-792\beta_1^4\beta_2^4}{64\beta_1^4\beta_2^4}$$

$$\therefore ku(X,Y) = -12.375$$

The marginal distribution of X^c is obtained with Equation 3.34. Hence,

$$\mu_{xn}(c,0) = \frac{\beta_1^{cn}[\beta_1\Gamma(cn+2) + \beta_2\Gamma(cn+1)]}{(\beta_1+\beta_2)} \;\ldots\ldots(3.48)$$

Thus,

$$\mu_{x1}(1,0) = \frac{\beta_1[\beta_1\Gamma(3) + \beta_2\Gamma(2)]}{(\beta_1+\beta_2)} = \frac{\beta_1[2\beta_1+\beta_2]}{(\beta_1+\beta_2)} \ldots\ldots\ldots\ldots\ldots(3.49)$$

$$\mu_{x2}(1,0) = \frac{\beta_1^2[\beta_1\Gamma(4) + \beta_2\Gamma(3)]}{(\beta_1+\beta_2)} = \frac{\beta_1^2[6\beta_1+2\beta_2]}{(\beta_1+\beta_2)}$$

$$\therefore \mu_{x2}(1,0) = \frac{2\beta_1^2(3\beta_1+\beta_2)}{(\beta_1+\beta_2)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.50)$$

$$\mu_{x3}(1,0) = \frac{\beta_1^3\left(\beta_1\Gamma(5) + \beta_2\Gamma(4)\right)}{(\beta_1+\beta_2)} = \frac{\beta_1^3(24\beta_1+6\beta_2)}{(\beta_1+\beta_2)}$$

$$\therefore \mu_{x3}(1,0) = \frac{6\beta_1^3(4\beta_1+\beta_2)}{(\beta_1+\beta_2)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.51)$$

$$\mu_{x4} = (1,0) = \frac{\beta_1^4[\beta_1\Gamma(6) + \beta_2\Gamma(5)]}{(\beta_1+\beta_2)} = \frac{\Gamma(5)\beta_1^4(5\beta_1+\beta_2)}{(\beta_1+\beta_2)}$$

$$\therefore \mu_4(1,0) = \frac{24\beta_1^4(5\beta_1+\beta_2)}{(\beta_1+\beta_2)} \;\ldots\ldots\ldots\ldots\ldots(3.51)$$

Thus,

$$\mu_1(1,0) = \frac{\beta_1(2\beta_1+\beta_2)}{(\beta_1+\beta_2)}$$

From Equation 3.34,

$$var(X) = \mu_2(1,0) - \mu_1(1,0)^2$$

$$= \frac{2\beta_1^2(3\beta_1+\beta_2)}{(\beta_1+\beta_2)} - \left(\frac{\beta_1[2\beta_1+\beta_2]}{(\beta_1+\beta_2)}\right)^2 = \frac{2\beta_1^2(3\beta_1+\beta_2)}{(\beta_1+\beta_2)} - \frac{4\beta_1^4+\beta_1^2\beta_2^2+4\beta_1^3\beta_2}{(\beta_1+\beta_2)^2}$$

$$= \frac{(\beta_1 + \beta_2)(6\beta_1^3 + 2\beta_1^2\beta_2) - (4\beta_1^4 + \beta_1^2\beta_2^2 + 4\beta_1^3\beta_2)}{(\beta_1 + \beta_2)^2}$$

$$= \frac{6\beta_1^4 + 2\beta_1^3\beta_2 + 6\beta_1^3\beta_2 + 2\beta_1\beta_2^3 - 4\beta_1^4 + \beta_1^2\beta_2^2 + 4\beta_1^3\beta_2}{(\beta_1 + \beta_2)^2}$$

$$= \frac{2\beta_1^4 + 12\beta_1^3\beta_2 + \beta_1^2\beta_2^2 + 2\beta_1\beta_2^2}{(\beta_1 + \beta_2)^2}$$

$$\therefore var(X) = \frac{2\beta_1^3(\beta_1 + 6\beta_2) + \beta_1\beta_2^2(\beta_1 + 2)}{(\beta_1 + \beta_2)^2} \quad \ldots\ldots\ldots\ldots\ldots(3.52)$$

Using Equation 3.36, we can obtain the skewness of the marginal distribution of X as follows:

$$sk(X) = \frac{\mu_{x3}(1) - 3\mu_{x2}(1)\mu_{x1}(1) + 2\mu_{x1}(1)^3}{(\mu_{x2}(1) - \mu_{x1}(1)^2)^{\frac{3}{2}}}$$

$$= \frac{\frac{6\beta_1^3(4\beta_1 + \beta_2)}{(\beta_1 + \beta_2)} - 3\left[\frac{2\beta_1^2(3\beta_1 + \beta_2)}{(\beta_1 + \beta_2)}\right]\left[\frac{\beta_1(2\beta_1 + \beta_2)}{(\beta_1 + \beta_2)}\right] + 2\left[\frac{\beta_1(2\beta_1 + \beta_2)}{\beta_1 + \beta_2}\right]^3}{\left\{\frac{2\beta_1^2(3\beta_1 + \beta_2)}{(\beta_1 + \beta_2)} - \left[\frac{\beta_1(2\beta_1 + \beta_2)}{(\beta_1 + \beta_2)}\right]^2\right\}^{\frac{3}{2}}}$$

$$= \frac{\frac{6\beta_1^3(4\beta_1 + \beta_2)}{(\beta_1 + \beta_2)} - 3\left[\frac{2\beta_1^2(3\beta_1 + \beta_2)\beta_1(2\beta_1 + \beta_2)}{(\beta_1 + \beta_2)^2}\right] + 2\left[\frac{\beta_1(2\beta_1 + \beta_2)}{(\beta_1 + \beta_2)}\right]^3}{\left[\frac{6\beta_1^3 + 2\beta_1^2\beta_2}{\beta_1 + \beta_2} - \frac{(2\beta_1^2 + \beta_1\beta_2)^2}{(\beta_1 + \beta_2)^2}\right]^{\frac{3}{2}}}$$

Considering the numerator,

$$\frac{6\beta_1^3(4\beta_1 + \beta_2)}{\beta_1 + \beta_2} - \frac{3[6\beta_1^3 + 2\beta_1^2\beta_2][2\beta_1^2 + \beta_1\beta_2]}{(\beta_1 + \beta_2)^2} + \frac{2[2\beta_1^2 + \beta_1\beta_2]^3}{(\beta_1 + \beta_2)^2}$$

$$= \frac{24\beta_1^4 + 6\beta_1^3\beta_2}{\beta_1 + \beta_2} - \frac{3\left(12\beta_1^5 + 10\beta_1^4\beta_2 + 2\beta_1^3\beta_2^2\right)}{(\beta_1 + \beta_2)^2}$$

$$+ \frac{2[8\beta_1^6 + 6\beta_1^4\beta_2^2 + 12\beta_1^5\beta_2 + \beta_1^3\beta_2^3]}{(\beta_1 + \beta_2)^3}$$

$$= \frac{4\beta_1^6 + 12\beta_1^5\beta_2 + 12\beta_1^4\beta_2^2 + 2\beta_1^3\beta_2^3}{(\beta_1 + \beta_2)^3}$$

Now considering the denominator,

$$\left[\frac{(\beta_1 + \beta_2)(6\beta_1^3 + 2\beta_1^2\beta_2) - (2\beta_1^2 + \beta_1\beta_2)^2}{(\beta_1 + \beta_2)^2}\right]^{\frac{3}{2}}$$

$$= \left[\frac{6\beta_1^4 + 2\beta_1^3\beta_2 + 6\beta_1^3\beta_2 + 2(\beta_1\beta_2)^2 - 4\beta_1^4 - (\beta_1\beta_2)^2 - 4\beta_1^3\beta_2}{(\beta_1 + \beta_2)^2}\right]^{\frac{3}{2}}$$

$$= \left[\frac{2\beta_1^4 + 4\beta_1^3\beta_2 + (\beta_1\beta_2)^2}{(\beta_1 + \beta_2)^2}\right]^{\frac{3}{2}}$$

$$\therefore sk = \frac{4\beta_1^6 + 12\beta_1^5\beta_2 + 12\beta_1^4\beta_2^2 + 2\beta_1^3\beta_2^3}{(\beta_1 + \beta_2)^3} * \frac{(\beta_1 + \beta_2)^{\frac{2*3}{2}}}{(2\beta_1^4 + 4\beta_1^3\beta_2 + (\beta_1\beta_2)^2)^{\frac{2*3}{2}}}$$

$$= \frac{4\beta_1^6 + 12\beta_1^5\beta_2 + 12\beta_1^4\beta_2^2 + 2\beta_1^3\beta_2^3}{(2\beta_1^4 + 4\beta_1^3\beta_2 + (\beta_1\beta_2)^2)^3}$$

$$(4\beta_1^3 + 12\beta_1^2\beta_2 + 12\beta_1\beta_2^2 + 2\beta_2^3)$$
$$\div \Big(\beta_1^3\big(8\beta_1^6 + 48\beta_1^5\beta_2 + 88\beta_1^4\beta_2^2 + 8\beta_1^3\beta_2^4 + 41\beta_1^3\beta_2^2 + 4\beta_1^3\beta_2^2$$
$$+ 16\beta_1^2\beta_2^5 + 36\beta_1^2\beta_2^4 + 8\beta_1^2\beta_2^3 + 4\beta_1\beta_2^6 + 8\beta_1\beta_2^5 + 2\beta_1\beta_2^4 + \beta_2^6\big)\Big)$$

Suppose $\beta_1 = \beta_2 = 1$

$$sk = \frac{66}{272} = 0.24265 \ \ldots\ldots\ldots.(3.53)$$

This value is close to symmetry with an infinitesimal sign of positive skewness. The larger the values of $\beta_1$and β_2, the more positively skewed the distribution of X becomes.

The kurtosis of the marginal distribution of X is from Equation 3.37

$$Ku(X^c) = \left\{ \frac{24\beta_1^4[5\beta_1 + \beta_2]}{\beta_1 + \beta_2} - 4\left[\frac{6\beta_1^3(4\beta_1 + \beta_2)}{\beta_1 + \beta_2}\right]\left[\frac{\beta_1(2\beta_1 + \beta_2)}{\beta_1 + \beta_2}\right] + 6\left[\frac{2\beta_1^2(3\beta_1 + \beta_2)}{\beta_1 + \beta_2}\right]\left[\frac{\beta_1(2\beta_1 + \beta_2)}{\beta_1 + \beta_2}\right]^2 - 3\left[\frac{\beta_1(2\beta_1 + \beta_2)}{\beta_1 + \beta_2}\right]^4 \right\} \div \left\{ \left[\frac{2\beta_1^2(3\beta_1 + \beta_2)}{\beta_1 + \beta_2}\right] - \left[\frac{2\beta_1^2(3\beta_1 + \beta_2)}{\beta_1 + \beta_2}\right]^2 \right\}$$

$$= \frac{3\beta_1^4[8\beta_1^7(3\beta_1 + 8\beta_2) + 4\beta_1^5\beta_2^2(17\beta_1 + 4\beta_2) + 4\beta_1^4(\beta_2^4 + 4) - 8\beta_1^2\beta_2^2(4\beta_1 + 3) - \beta_2^3(8\beta_1 + \beta_2)]}{\{(\beta_1 + \beta_2)^2[2\beta_1^3(\beta_1 + 6\beta_2) + \beta_1\beta_2^2(\beta_1 + 2)]\}}$$

$------(3.54)$

For values of β_1 and β_2 greater than zero the marginal kurtosis of the distribution of X is positive. Suppose $\beta_1 = \beta_2 = 1$, Equation 3.54 equals 6.4576271. This means that the marginal distribution of X is leptokurtic (Arua *et al* 1997).

The marginal distribution of Y can be obtained using Equation 3.47. Thus,

$$\mu_{yn}(0, d) = \frac{\beta_1^0\beta_2^{dn}[\beta_1\Gamma(0 + 2)\Gamma(dn + 1) + \beta_2\Gamma(0 + 1)\Gamma(dn + 2)]}{\beta_1 + \beta_2}$$

$$\therefore \mu_{yn}(0, d) = \frac{\beta_2^{dn}[\beta_1\Gamma(dn + 1) + \beta_2\Gamma(dn + 2)]}{\beta_1 + \beta_2} \quad \ldots\ldots\ldots\ldots..(3.55)$$

$$\Rightarrow \mu_{y1}(0,1) = \frac{\beta_2[\beta_1\Gamma(1 + 1) + \beta_2\Gamma(1 + 2)]}{\beta_1 + \beta_2}$$

$$\therefore \mu_{y1}(0,1) = \frac{\beta_2[\beta_1 + 2\beta_2]}{\beta_1 + \beta_2} \quad \ldots\ldots\ldots.(3.56)$$

$$\mu_{y2}(0,1) = \frac{\beta_2^2[\beta_1\Gamma(2+1) + \beta_2\Gamma(2+2)]}{\beta_1 + \beta_2}$$

$$\therefore \mu_{y2}(0,1) = \frac{\beta_2^2[2\beta_1 + 6\beta_2]}{\beta_1 + \beta_2} \quad \ldots\ldots\ldots\ldots..(3.57)$$

$$\mu_{y3}(0,1) = \frac{\beta_2^3[\beta_1\Gamma(3+1) + \beta_2\Gamma(3+2)]}{\beta_1 + \beta_2}$$

$$= \frac{\beta_2^3[6\beta_1 + 24\beta_2]}{\beta_1 + \beta_2} \ldots\ldots\ldots\ldots.(3.58)$$

$$\mu_{y4}(0,1) = \frac{\beta_2^4[\beta_1\Gamma(4+1) + \beta_2\Gamma(4+2)]}{\beta_1 + \beta_2}$$

$$= \frac{\beta_2^4[24\beta_1 + 120\beta_2]}{\beta_1 + \beta_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots.(3.59)$$

Using Equation 3.35 we have that

$$Var(Y^d) = \mu_{y2}(d) - \mu_{y1}(d)^2$$

Thus the variance of the marginal distribution of Y^1 can be obtained as,

$$Var(Y) = \mu_{y2}(d) - \mu_{y1}(d)^2$$

$$= \frac{\beta_2^2[2\beta_1 + 6\beta]}{\beta_1 + \beta_2} - \left[\frac{\beta_2[\beta_1 + 2\beta_2]}{\beta_1 + \beta_2}\right]^2$$

$$Var(Y) = \frac{\beta_2^2(2\beta_1 + 6\beta)}{\beta_1 + \beta_2} - \frac{\beta_1^2\beta_2^2 + 4\beta_1\beta_2^3 + 4\beta_2^4}{(\beta_1 + \beta_2)^2}$$

$$= \frac{\beta_2^2(2\beta_1 + 6\beta)(\beta_1 + \beta_2) - (\beta_1^2\beta_2^2 + 4\beta_1\beta_2^2 + 4\beta_2^4)}{(\beta_1 + \beta_2)^2}$$

$$= \frac{(2\beta_1^2\beta_2^2 + 6\beta_1\beta_2^3 + 2\beta_2^3\beta_1 + 6\beta_2^4) - (\beta_1^2\beta_2^2 + 4\beta_1\beta_2^3 - 4\beta_2^4)}{(\beta_1 + \beta_2)^2}$$

$$= \frac{\beta_1^2\beta_2^2 - 2\beta_1\beta_2^3 + \beta_2^4(\beta_1 - 4) + 6\beta_2^5}{(\beta_1 + \beta_2)^2}$$

$$= \frac{2\beta_1^2\beta_2^2 + 6\beta_1\beta_2^3 + 2\beta_2^3\beta_1 + 6\beta_2^4 - \beta_1^2\beta_2^2 - 4\beta_1\beta_2^3 - 4\beta_2^4}{(\beta_1 + \beta_2)^2}$$

$$= \frac{\beta_1^2\beta_2^2 + 6\beta_1\beta_2^3 + 2\beta_2^3\beta_1 - 4\beta_1\beta_2^3 + 2\beta_2^4}{(\beta_1 + \beta_2)^2}$$

Also,

$$Var(Y) = \frac{\beta_2^2(\beta_1^2 + 4\beta_1\beta_2 + 4\beta_1 + 2\beta_2^2)}{\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2} \quad \ldots\ldots\ldots..3.60$$

Using Equation 3.38, the skewness of the marginal distribution of Y is obtained as follows:

$$sk(Y) = \left\{\frac{\beta_2^3[\beta_1\Gamma 4 + \beta_2\Gamma 5]}{\beta_1 + \beta_2} - 3\left[\frac{\beta_2^2(2\beta_1 + 6\beta_2)}{\beta_1 + \beta_2}\right]\left[\frac{\beta_2(\beta_1 + 2\beta_2)}{\beta_1 + \beta_2}\right] + \frac{2\beta_2^2(\beta_1 + 2\beta_2)^2}{\beta_1 + \beta_2}\right\} \div \left\{\frac{\beta_2^2[2\beta_1^2 - \beta_1\beta_2 + \beta_2^2(\beta_1 - 4) + 6\beta_2^3]}{\beta_1 + \beta_2}\right\}^{\frac{3}{2}}$$

Thus,

$$sk(Y) = \frac{8\beta_1^2\beta_2^3 + 2\beta_1^2\beta_2^2 + 8\beta_2^4 - 12\beta_2^5}{(\beta_1 + \beta_2)^2} * \frac{(\beta_1 + \beta_2)^2(\beta_1 + \beta_2)}{\beta_2^2(2\beta_1^2 - \beta_1\beta_2 + \beta_2^2(\beta_1 - 4) + 6\beta_2^3)^{\frac{3}{2}}}$$

$$= \frac{(\beta_1 + \beta_2)\left(8\beta_1^2\beta_2^3 + 2\beta_1^2\beta_2^2 + 8\beta_2^4 - 12\beta_2^5\right)}{(2\beta_1^2\beta_2^2 - \beta_1\beta_2^3 + \beta_2^4(\beta_1 - 4) + 6\beta_2^5)^{\frac{3}{2}}}$$

Now, suppose $\beta_1 = \beta_2 = 1$, we have that

$$sk(Y) = \frac{2(8 + 2 + 8 - 12)}{(2 - 1 - 3 + 6)^{\frac{3}{2}}} = \frac{12}{8} = 1.5$$

Then, the marginal distribution of Y is positively skewed where $\beta_1 = \beta_2 = 1$.

Using Equation 3.39, the kurtosis of the marginal distribution of Y is obtained as follows:

$$ku(Y) = \left\{\frac{\beta_2^4(24\beta_1 + 6\beta_2)}{\beta_1 + \beta_2} - 4\left[\frac{\beta_2^3(6\beta_1 + 24\beta_2)}{\beta_1 + \beta_2}\right]\left[\frac{\beta_2(\beta_1 + 2\beta_2)}{\beta_1 + \beta_2}\right] + 6\left[\frac{\beta_2^2(2\beta_1 + 6\beta_2)}{\beta_1 + \beta_2}\right]\left[\frac{\beta_2(\beta_1 + 2\beta_2)}{\beta_1 + \beta_2}\right]^2 - 3\left[\frac{\beta_2(\beta_1 + 2\beta_2)}{\beta_1 + \beta_2}\right]^4\right\} \div \left\{\frac{\beta_2^2(2\beta_1 + 6\beta_2)}{\beta_1 + \beta_2} - \left(\frac{\beta_2(\beta_1 + 2\beta_2)}{\beta_1 + \beta_2}\right)^2\right\}^2$$

Consider the numerator.

$$\frac{\beta_2^4(24\beta_1 + 120\beta_2)}{\beta_1 + \beta_2} - 4\left[\frac{\beta_2^3(6\beta_1 + 24\beta_2)}{\beta_1 + \beta_2}\right]\left[\frac{\beta_2(\beta_1 + 2\beta_2)}{\beta_1 + \beta_2}\right] + 6\left[\frac{\beta_2^2(2\beta_1 + 6\beta_2)}{\beta_1 + \beta_2}\right]\left[\frac{\beta_2(\beta_1 + 2\beta_2)}{\beta_1 + \beta_2}\right]^2 - 3\left[\frac{\beta_2(\beta_1 + 2\beta_2)}{\beta_1 + \beta_2}\right]^4$$

$$= \frac{9\beta_1^4\beta_2^4 + 72\beta_1^3\beta_2^5 + 132\beta_1^2\beta_2^6 + 96\beta_1\beta_2^7 + 8\beta_2^8}{(\beta_1 + \beta_2)^4}$$

Considering the denominator,

$$\left\{\frac{\beta_2^2(2\beta_1 + 6\beta_2)}{\beta_1 + \beta_2} - \left[\frac{\beta_2(\beta_1 + 2\beta_2)}{\beta_1 + \beta_2}\right]^2\right\}^2 = \frac{(\beta_1^2\beta_2^2 + 4\beta_1\beta_2^3 + 2\beta_2^4)^2}{(\beta_1 + \beta_2)^4}$$

$$= \frac{\beta_1^4\beta_2^4 + 8\beta_1^3\beta_2^5 + 20\beta_1^2\beta_2^6 + 16\beta_1\beta_2^7 + 4\beta_2^8}{(\beta_1 + \beta_2)^4}$$

$$\therefore ku(Y) = \frac{9\beta_1^4\beta_2^4 + 72\beta_1^3\beta_2^5 + 132\beta_1^2\beta_2^6 + 96\beta_1\beta_2^7 + 8\beta_2^8}{(\beta_1 + \beta_2)^4} \times \frac{(\beta_1 + \beta_2)^4}{\beta_1^4\beta_2^4 + 8\beta_1^3\beta_2^5 + 20\beta_1^2\beta_2^6 + 16\beta_1\beta_2^7 + 4\beta_2^8}$$

$$= \frac{9\beta_1^4\beta_2^4 + 72\beta_1^3\beta_2^5 + 132\beta_1^2\beta_2^6 + 96\beta_1\beta_2^7 + 48\beta_2^8}{\beta_1^4\beta_2^4 + 8\beta_1^3\beta_2^5 + 20\beta_1^2\beta_2^6 + 16\beta_1\beta_2^7 + 4\beta_2^8} \quad \ldots\ldots\ldots (3.61)$$

Observation here is that all coefficients in the numerator are higher than their counterparts in the denominator.

Suppose $\beta_1 = \beta_2 = 1$, we have that;

$$ku(Y) = \frac{357}{46} = 7.761$$

Thus, the kurtosis is greater than 3 which implies leptokurtic (Arua *et al* 1997).

Suppose in Equation 3.47 we let $c = \frac{3}{2}$ and $d = \frac{1}{2}$, the n^{th} mgoment of the joint distribution of $X^{\frac{3}{2}}$ and $Y^{\frac{1}{2}}$ about zero can be obtained as

$$\mu_1\left(\frac{3}{2}, \frac{1}{2}\right) = E\left(X^{\frac{3}{2}}, Y^{\frac{1}{2}}\right)$$

$$= \frac{\beta_1^{\frac{3}{2}}\beta_2^{\frac{1}{2}}\left[\beta_1\Gamma\left(\frac{3}{2}+2\right)\Gamma\left(\frac{1}{2}+1\right) + \beta_2\Gamma\left(\frac{3}{2}+1\right)\Gamma\left(\frac{1}{2}+2\right)\right]}{\beta_1 + \beta_2}$$

$$= \frac{\beta_1^{\frac{3}{2}}\beta_2^{\frac{1}{2}}\left[\beta_1\Gamma\frac{7}{2}\Gamma\frac{3}{2} + \beta_2\Gamma\frac{5}{2}\Gamma\frac{5}{2}\right]}{\beta_1 + \beta_2}$$

$$= \frac{\beta_1^{\frac{3}{2}}\beta_2^{\frac{1}{2}}\frac{3}{16}\Gamma\left(\frac{1}{2}\right)^2(5\beta_1 + 3\beta_2)}{\beta_1 + \beta_2}$$

But $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ in Equation 3.25b. $\therefore \left(\Gamma\frac{1}{2}\right)^2 = \pi$

Hence,

$$\mu_1\left(\frac{3}{2},\frac{1}{2}\right)=\frac{\beta_1^{\frac{3}{2}}\beta_2^{\frac{1}{2}}\frac{3\pi}{16}(5\beta_1+3\beta_2)}{\beta_1+\beta_2}\quad \ldots\ldots\ldots\ldots\ldots(3.63)$$

Using Equation 3.48 and substituting for $c=\frac{3}{2}$, the n^{th} moment of the marginal distribution of $X^{\frac{3}{2}}$ about zero is

$$\mu_n\left(\frac{3}{2},0\right)=\frac{\beta_1^{\frac{3n}{2}}\left[\beta_1\Gamma\left(\frac{3n}{2}+2\right)+\beta_2\Gamma\left(\frac{3n}{2}+1\right)\right]}{\beta_1+\beta_2}\quad \ldots\ldots\ (3.64)$$

Letting $n=1$, the first moment of the marginal distribution of $X^{\frac{3}{2}}$ becomes

$$\mu_1\left(\frac{3}{2},0\right)=\mu\left(\frac{3}{2},0\right)=\frac{\beta_1^{\frac{3}{2}}\left[\beta_1\Gamma\left(\frac{3}{2}+2\right)+\beta_2\Gamma\left(\frac{3}{2}+1\right)\right]}{\beta_1+\beta_2}$$

$$=\frac{\beta_1^{\frac{3}{2}}\left[\beta_1\Gamma\left(\frac{7}{2}\right)+\beta_2\Gamma\left(\frac{5}{2}\right)\right]}{\beta_1+\beta_2}=\frac{\frac{3}{4}\Gamma\left(\frac{1}{2}\right)\beta_1^{\frac{3}{2}}\left[\frac{5}{2}\beta_1+\beta_2\right]}{\beta_1+\beta_2}$$

$$\therefore\mu\left(\frac{3}{2},0\right)=\frac{3\sqrt{\pi}\beta_1^{\frac{3}{2}}\left[\frac{5}{2}\beta_1+\beta_2\right]}{4(\beta_1+\beta_2)}\quad \ldots\ldots\ldots\ldots(3.65)$$

Suppose $c=\frac{1}{3}$ and $n=1$ in Equation 3.48, we have that

$$\mu_1\left(\frac{1}{3},0\right)=\frac{\beta_1^{\frac{1}{3}}\left[\beta_1\Gamma\left(\frac{7}{3}\right)+\beta_2\Gamma\left(\frac{4}{3}\right)\right]}{\beta_1+\beta_2}=\frac{\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\beta_1^{\frac{1}{3}}\left(\frac{4}{3}\beta_1+\beta_2\right)}{\beta_1+\beta_2}$$

$$=\frac{\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\beta_1^{\frac{1}{3}}\left(\frac{4\beta_1+3\beta_2}{3}\right)}{\beta_1+\beta_2}=\frac{\frac{1}{9}\Gamma\left(\frac{1}{3}\right)\beta_1^{\frac{1}{3}}(4\beta_1+3\beta_2)}{\beta_1+\beta_2}$$

$$\therefore \mu_1\left(\frac{1}{3},0\right)=\frac{\Gamma\left(\frac{1}{3}\right)\beta_1^{\frac{1}{3}}(4\beta_1+3\beta_2)}{9(\beta_1+\beta_2)} \quad \ldots\ldots\ldots\ldots\ldots.(3.66)$$

To illustrate the needed modification the probability distribution and moment generating function of the random variable, X, note from Equation 3.46 that

$$f(x,y)=(x+y)\frac{e^{-(x/\beta_1+y/\beta_2)}}{\beta_1\beta_2(\beta_1+\beta_2)} \quad x>0,y>0;\ \beta_1,\beta_2>0$$

We obtain the marginal distribution of X as

$$f(x)=\int_0^\infty f(x,y)dy$$

$$=k\left[\int_0^\infty xe^{-x/\beta_1}\,e^{-y/\beta_2}dy+\int_0^\infty ye^{-x/\beta_1}.\,e^{-y/\beta_2}dy\right]$$

where $k=\frac{1}{\beta_1\beta_2(\beta_1+\beta_2)}$

$$=k\left[xe^{-x/\beta_1}\int_0^\infty e^{-y/\beta_2}dy+e^{-x/\beta_1}\int_0^\infty ye^{-y/\beta_2}dy\right]$$

Integrating with transformation to gamma function gives

$$=k\left[xe^{-x/\beta_1}(\beta_2)+\beta_2^2e^{x/\beta_1}\right]$$

Substituting for k and evaluating, we have

$$f(x)=\frac{xe^{-x/\beta_1}+\beta_2e^{-x/\beta_1}}{\beta_1(\beta_1+\beta_2)} \quad ----------(3.67)$$

therefore, the pdf of $Y=X^c;\ c\geq 0$ is, using the theorem,

$$f(y)=f\left(g^{-1}(y)\right).\frac{d[g^{-1}(y)]}{dy}$$

where $y = g(x)$ and $x = g^{-1}(y)$

(William and Richard, 1973 and Chukwu and Amuji, 2012).

Now,

$$y = g(x) = X^c$$

$$\Rightarrow x = g^{-1}(y) = y^{\frac{1}{c}}$$

$$\therefore \frac{dx}{dy} = \frac{1}{c} y^{\frac{1}{c}-1}$$

$$\Rightarrow dx = \frac{1}{c} y^{\frac{1}{c}-1} dy$$

$$\therefore f(y) = \left[\frac{y^{\frac{1}{c}} e^{-y^{\frac{1}{c}}/\beta_1} + \beta_2 e^{-y^{\frac{1}{c}}/\beta_1}}{\beta_1(\beta_1 + \beta_2)}\right] \frac{1}{c} y^{\frac{1}{c}-1}$$

$$\therefore f(y) = \frac{1}{c}\left[\frac{y^{\frac{2}{c}-1} e^{-y^{\frac{1}{c}}/\beta_1} + \beta_2 y^{\frac{1}{c}-1} e^{-y^{\frac{1}{c}}/\beta_1}}{\beta_1(\beta_1 + \beta_2)}\right]$$

$$\therefore M_y(t) = M_{X^c}(t) = E\left(e^{tx^c}\right) = E\left(e^{ty}\right)$$

$$= \frac{1}{c[\beta_1(\beta_1 + \beta_2)]}\left\{\int_0^\infty y^{\frac{2}{c}-1} e^{-\frac{y^{\frac{1}{c}}}{\beta_1}} . e^{ty} . dy + \beta_2 \int_0^\infty y^{\frac{1}{c}-1} e^{-\frac{y^{\frac{1}{c}}}{\beta_1}} . e^{ty} . dy\right\}$$

$$M_y(t) = k\left(\int_0^\infty y^{\frac{2}{c}-1} e^{-\frac{y^{\frac{1}{c}}}{\beta_1}} . e^{ty} . dy + \int_0^\infty y^{\frac{1}{c}-1} e^{-\frac{y^{\frac{1}{c}}}{\beta_1}} . e^{ty} . dy\right)$$

where $k = \frac{1}{c[\beta_1(\beta_1+\beta_2)]}$

Let

$$\frac{y^{\frac{1}{c}}}{\beta_1} = v \ \Rightarrow y = (\beta_1 v)^c; dy = c\beta_1^c v^{c-1} dv$$

$$\therefore M_Y(t) = k\int_0^\infty (\beta_1^c v^c)^{\frac{2}{c}-1} e^{-v} e^{t(\beta_1^c v^c)} c\beta_1^c v^{c-1} dv$$

$$+ k\beta_2 \int_0^\infty (\beta_1^c v^c)^{\frac{1}{c}-1} e^{-v}.e^{t(\beta_1 v)^c}.c\beta_1^c v^{c-1} dv$$

Considering the first term,

$$kc\int_0^\infty (\beta_1^c v^c)^{\frac{2}{c}-1}.\beta_1^c v^{c-1} e^{-v} e^{t(\beta_1^c v^c)} dv$$

$$= kc\int_0^\infty \left[1 + \frac{t(\beta_1^c v^c)}{1!} + \cdots + \frac{t^r(\beta_1^c v^c)^r}{r!} + \cdots\right] e^{-v}\beta_1^2 v$$

Substituting for k and evaluating gives

$$\frac{\beta_1}{(\beta_1+\beta_2)}\int_0^\infty v\left(1 + \frac{t}{1!}(\beta_1^c v^c) + \cdots + \frac{t^r}{r!}(\beta_1^c v^c)^r + \cdots\right) e^{-v} dv$$

Now, considering the second term;

$$k\beta_2\int_0^\infty (\beta_1^c v^c)^{\frac{1}{c}-1} e^{-v} e^{t(\beta_1 v)^c}.c.\beta_1^c v^{c-1} dv = kc\beta_2\beta_1 \int_0^\infty e^{t(\beta_1 v)^c} e^{-v} dv$$

$$= kc\beta_1\beta_2 \int_0^\infty e^{t(\beta_1 v)^c} e^{-v} dv$$

$$= kc\beta_2\beta_1 \int_0^\infty \left[1 + \frac{t}{1!}(\beta_1^c v^c) + \cdots + \frac{t^r}{r!}(\beta_1^c v^c)^r + \cdots\right] e^{-v} dv$$

Substituting for k and evaluating we have,

$$\frac{\beta_2}{\beta_1+\beta_2}\int_0^\infty \left(1+\frac{t}{1!}(\beta_1^c v^c)+\cdots+\frac{t^r}{r!}(\beta_1^c v^c)^r+\cdots\right)e^{-v}dv$$

Thus,

$$M_Y(t)=\frac{\beta_1}{(\beta_1+\beta_2)}\int_0^\infty v\left[1+\frac{t}{1!}(\beta_1^c v^c)+\cdots+\frac{t^r}{r!}(\beta_1^c v^c)^r+\cdots\right]e^{-v}dv$$

$$+\frac{\beta_2}{(\beta_1+\beta_2)}\int_0^\infty \left[1+\frac{t}{1!}(\beta_1^c v^c)+\cdots+\frac{t^r}{r!}(\beta_1^c v^c)^r+\cdots\right]e^{-v}dv$$

$$=\frac{\beta_1}{(\beta_1+\beta_2)}\int_0^\infty \left[ve^{-v}dv+\frac{t}{1!}\beta_1^c v^{c+1}e^{-v}dv+\cdots+\frac{t^r}{r!}\beta_1^{cr}v^{cr+1}e^{-v}dv+\cdots\right]$$

$$+\frac{\beta_2}{(\beta_1+\beta_2)}\int_0^\infty [e^{-v}dv+\frac{t}{1!}\beta_1 ve^{-v}dv+\cdots+\frac{t^r}{r!}\beta_1^{cr}v^{cr}e^{-v}dv+\cdots]$$

$$M_Y(t)=\frac{1}{(\beta_1+\beta_2)}\left(\beta_1\sum_{r=0}^{\infty}\frac{t^r}{r!}\beta_1^{cr}\Gamma(cr+2)+\beta_2\sum_{r=0}^{\infty}\frac{t^r}{r!}\beta^{cr}\,\Gamma(cr+1)\right)$$

$$\therefore M_Y(t)=\frac{\beta_1\sum_{r=0}^{\infty}\frac{t^r}{r!}\beta_1^{cr}\Gamma(cr+2)+\beta_2\sum_{r=0}^{\infty}\frac{t^r}{r!}\beta_1^{cr}\Gamma(cr+1)}{\beta_1+\beta_2} \quad ------(3.68)$$

The n^{th} moment of the distribution of $Y=X^c$ about zero is taken as the co-efficient of $\sum_r^\infty \frac{t^r}{r!}$ or the n^{th} derivative of $M_Y(t)$ with respect to t, evaluated at $t=0$, in the expansion of Equation 3.68.

That is,

$$M_Y^{(n)}(o)=M_{X^c}^{(n)}(0)=\frac{\beta_1^{cn}[\beta_1\Gamma(cn+2)+\beta_2\Gamma(cn+1)]}{\beta_1+\beta_2}$$

$$\therefore M_Y^{(n)}(0) = \frac{cn\Gamma(cn)\beta_1^{cn}[\beta_1(cn+1)+\beta_2]}{\beta_1+\beta_2} \qquad (3.69)$$

The modification yields the same result as $\mu_{xn}(c)$ in Equation 3.48.

3.6 GENERALIZED MULTIVARIATE MOMENT GENERATING FUNCTION (GMMGF)

This section develops the generalized moment generating function for multivariate random variable $Y = X^c$ about a constant vector or matrix, λ. This is called the Multivariate Generalized Moment Generating Function, $\boldsymbol{G}_n(c; \boldsymbol{\lambda})$.

Suppose $t \in \mathbb{R}$ is a $(p \times p)$ square matrix or a $(p \times 1)$ column vector, $Y = X^c$ and $c \in \mathbb{R}$.

Therefore,

$$M_{(Y;\lambda)}(t) = M_{(X^c;\lambda)}(t) = E\left(e^{t'(X^c+\lambda)}\right) \quad ---------(3.70)$$

Equation 3.70 may be read as the moment generating function of X^c about λ.

Equation 3.70 may be evaluated with the Maclaurin's series expansion as

$$E\left(e^{t'(X^c+\lambda)}\right) = E\left(\sum_{n=0}^{\infty} \frac{[t'(X^c+\lambda)]^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{(t')^n}{n!} E(X^c+\lambda)^n$$

$$\therefore M_{(X^c;\lambda)}(t) = E(X^c+\lambda)^n \sum_{n=0}^{\infty} \frac{(t')^n}{n!} \quad ---------(3.71)$$

The coefficient of $\sum_{n=0}^{\infty} \frac{(t')^n}{n!}$ in Equation 3.71 yields the n^{th} moment of the random variable $Y = X^c$ and may be termed the Multivariate Generalized Moment Generating Function, $\boldsymbol{G}_n(c; \boldsymbol{\lambda})$. It generates all conceivable moments of $\boldsymbol{X}^c$ about $\boldsymbol{\lambda}$. Obviously, if $c = 1$, $\lambda = 0$ and $n = 1$, Equation 3.71 yields the first moment of $\boldsymbol{X}$ about zero also called the mean of the distribution of $\boldsymbol{X}$.

If $c = 1$, $\lambda = -\mu$, and $n = 2$, we have from Equation 3.71 that $G_n(1; -\mu) = Var(X)$. That is;

$$Var(X) = E(X - \mu)^2$$

Higher moments of the distribution of X are similarly obtained by varying the value of n accordingly.

Equation 3.71 may be evaluated as

$$\boldsymbol{G}_n(c; \boldsymbol{\lambda}) = E(\boldsymbol{X}^c + \boldsymbol{\lambda})^n = E\left(\sum_{r=0}^{n} \binom{n}{r} \boldsymbol{\lambda}^{n-r} \boldsymbol{X}^{cr}\right)$$

$$= \sum_{r=0}^{n} \binom{n}{r} \boldsymbol{\lambda}^{n-r} E(\boldsymbol{X}^{cr}) \quad ----(3.72)$$

(Riordan, 1958)

The Generalized Multivariate Moment Generating Functions, $G_n(c; \lambda)$ will be developed for the Multivariate Gamma, Normal and the Dirrichlet Distributions.

3.7 GENERALIZED MULTIVARIATE MOMENT GENERATING FUNCTION (GMMGF) FOR THE MULTIVARIATE GAMMA DISTRIBUTION

Let X be a positive-definite real $p \times p$ matrix distributed as Multivariate Gamma with shape parameter, α, scale parameter, β, and scale, Σ (a positive-definite real $p \times p$ matrix). Then, the probability density function, PDF, of X is given as,

$$f(X) = \frac{|\Sigma|^{-\alpha}}{\beta^{p\alpha}\Gamma_p(\alpha)} |X|^{\alpha-(p+1)/2} exp\left(tr\left(-\frac{1}{\beta}\Sigma^{-1}X\right)\right) \quad -------- \ (3.73a)$$

where Γ_p is the multivariate gamma function. (Gupta and Nagar, 1999 and Royen, 2006)

Where the shape parameter, $\alpha = \frac{\eta}{2}$, and the scale parameter, $\beta = 2$, the Multivariate Gamma Distribution reduces to the Wishart Distribution.

$$f(\boldsymbol{X}) = \frac{|v|^{-\frac{\eta}{2}}}{2^{\frac{\eta p}{2}}\Gamma_p\left(\frac{\eta}{2}\right)} |X|^{\frac{\eta-p-1}{2}} e^{-\frac{1}{2}tr(v^{-1}\boldsymbol{X})} \ ---------(3.73b)$$

(Wishart 1928).

Equation 3.73b gives the wishart distribution for $\beta = 2$ and $\alpha = \frac{\eta}{2}$ where η is the sample size.

Γ_p is defined in two forms. In the first definition,

$$\Gamma_p(\alpha) = \int_{s>0} exp\big(-tr(s)\big)\,|s|^{\alpha-(p+1)/2} ds \ ----------(3.74)$$

where $s > 0$ means s is positive-definite.

The other definition, more useful in practice, is

$$\Gamma_p(\alpha) = \pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma[\alpha + (1-j)/2] -----------(3.75)$$

(James, 1964 and Royen, 2006).

From Equation 3.75 we have the recursive relationship:

$$\Gamma_p(\alpha) = \pi^{(p-1)/2}\Gamma(\alpha)\Gamma_{p-1}\left(\alpha - \frac{1}{2}\right) = \pi^{(p-1)/2}\Gamma_{p-1}(\alpha).\Gamma[\alpha + (1-p)/2] \quad (3.76)$$

Thus,

$$\left.\begin{aligned} \Gamma_1(\alpha) &= \Gamma(\alpha) \\ \Gamma_2(\alpha) &= \pi^{1/2}\Gamma(\alpha)\Gamma\left(\alpha-\frac{1}{2}\right) \\ \Gamma_3(\alpha) &= \pi^{3/2}\Gamma(\alpha)\Gamma\left(\alpha-\frac{1}{2}\right)\Gamma(\alpha-1) \end{aligned}\right\} ---------(3.77)$$

and so on. (James 1964)

Now, applying Equation 3.72, the generalized moment generating function about $\lambda_{p\times p}$ is developed as

$$\boldsymbol{G}_n(c;\boldsymbol{\lambda}) = E(\boldsymbol{X}^c+\boldsymbol{\lambda})^n = E\left(\sum_{r=0}^{n}\boldsymbol{\lambda}^{n-r}.\boldsymbol{X}^{cr}.\binom{n}{r}\right) = \sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}E(\boldsymbol{X}^{cr})$$

where $E(X^{cr}) = \int_{-\infty}^{\infty} x^{cr}\, f(x)dx$, for p —variate gamma distribution from Equation 3.73 is

$$\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\int_{X>0}\frac{|\Sigma|^{-\alpha}}{\beta^{p\alpha}\Gamma_p(\alpha)}|X|^{cr+\alpha-(p+1)/2}exp\left(tr\left(-\frac{1}{\beta}\boldsymbol{\Sigma}^{-1}X\right)\right)d\boldsymbol{X}$$

$$=\sum_{r=0}^{n}\binom{n}{r}\lambda^{n-r}\frac{|\Sigma|^{-\alpha}}{\beta^{p\alpha}\Gamma_p(\alpha)}\int_{X>0}|X|^{cr+\alpha-(p+1)/2}exp\left(tr\left(-\frac{1}{\beta}\boldsymbol{\Sigma}^{-1}X\right)\right)d\boldsymbol{X}$$

$$=\sum_{r=0}^{n}\binom{n}{r}\boldsymbol{\lambda}^{n-r}\frac{|\boldsymbol{\Sigma}|^{\alpha+cr}\beta^{p(\alpha+cr)}\Gamma_p(\alpha+cr)}{\boldsymbol{\Sigma}^{\alpha}\beta^{p\alpha}\Gamma_p(\alpha)}$$

$$\therefore \boldsymbol{G}_n(c;\boldsymbol{\lambda}) = E(\boldsymbol{X}^c+\boldsymbol{\lambda})^n = \sum_{r=0}^{n}\binom{n}{r}\boldsymbol{\lambda}^{n-r}\frac{|\boldsymbol{\Sigma}|^{cr}\beta^{pcr}\Gamma_p(\alpha+cr)}{\Gamma_p(\alpha)} ------(3.78)$$

Substituting $\beta = 2$ and $\alpha = \frac{\eta}{2}$ in Equation 3.78 gives the multivariate generalized moment generating function of the wishart distribution as

$$\boldsymbol{G}_n(c;\boldsymbol{\lambda}) = \sum_{r=0}^{n} \binom{n}{r} \boldsymbol{\lambda}^{n-r} \frac{|\boldsymbol{\Sigma}|^{cr} 2^{pcr} \Gamma_p\left(\frac{\eta}{2} + cr\right)}{\Gamma_p\left(\frac{\eta}{2}\right)} \quad --------- (3.79)$$

Equation 3.79 depends on p, the number of variables, and η, the number of observations (sample size), which shows that it is the Multivariate Generalized Moment Generating Function, $\boldsymbol{G}_n(c;\boldsymbol{\lambda})$, of a p −variate extension of the chi-square random variable.

The practical application of this function is better appreciated where $n = 1$. That is,

$$n = 1, r = 0, 1 \; and \; c = 1$$

$$\boldsymbol{G}_1(1;\ \boldsymbol{\lambda}) = \sum_{r=0}^{1} \binom{1}{0} \boldsymbol{\lambda}^{1-r} \frac{\boldsymbol{\Sigma}^r \beta^{pr} \Gamma_p(\alpha + r)}{\Gamma_p(\alpha)} \quad --------- (3.79a)$$

$$= \binom{1}{0} \boldsymbol{\lambda} + \binom{1}{1} \frac{\boldsymbol{\Sigma} \beta^p \Gamma_p(\alpha + 1)}{\Gamma_p(\alpha)} = \boldsymbol{\lambda} + \alpha\beta^p \Sigma \quad ------(3.79\text{b})$$

Now substituting λ with $-\lambda$ in order to get the first central moment, it implies that

$$-\boldsymbol{\mu} + \alpha\beta^p \boldsymbol{\Sigma} = \big((0)\big)_{\boldsymbol{p}\times\boldsymbol{p}}$$

$$\therefore \boldsymbol{\lambda} = \beta^p \alpha \boldsymbol{\Sigma} = \beta^{p-1}(\alpha\beta\boldsymbol{\Sigma}) ----- \quad (3.79c)$$

The coefficient of β^{p-1} in Equation 3.79c is the mean of the distribution while β^{p-1} indicates that the dimension (number of variables) of the distribution is p.

Where $\beta = 2, \alpha = \frac{\eta}{2}$, we have for the Wishart distribution;

$$\boldsymbol{\lambda} = 2^{p-1}\left(2\frac{n}{2}\boldsymbol{\Sigma}\right) = 2^{p-1}(n\boldsymbol{\Sigma}) --------- \quad (3.79d)$$

This may be interpreted as a p −variate Wishart distribution with mean, $n\boldsymbol{\Sigma}$, which is the coefficient of 2^{p-1} in Equation 3.79d.

To further illustrate the use of the Generalized Multivariate Moment Generating Function, suppose $c = 1, n = 2, r = 0, 1, 2$ and $p = 3$, we have from Equation 3.78

$$\boldsymbol{G}_2(c; \boldsymbol{\lambda}) = E(\boldsymbol{X}^c + \boldsymbol{\lambda})^2 = \sum_{r=0}^{2} \binom{2}{r} \boldsymbol{\lambda}^{2-r} \frac{\Gamma_p(r+\alpha)}{\Gamma_p(\alpha)} \boldsymbol{\Sigma}^r \beta^{3r}$$

$$= \boldsymbol{\lambda}'\boldsymbol{\lambda} + \frac{2\Gamma_2(\alpha+1)\Gamma(\alpha)\boldsymbol{\lambda}'\boldsymbol{\Sigma}\beta^3}{\Gamma_2(\alpha)\Gamma(\alpha-1)} + \frac{\Gamma_2(\alpha+2)\Gamma(\alpha+1)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\beta^6}{\Gamma_2(\alpha)\Gamma(\alpha-1)}$$

Evaluating further using equation of 3.77 yields

$$\boldsymbol{G}_2(1; \boldsymbol{\lambda}) = \boldsymbol{\lambda}'\boldsymbol{\lambda} + \frac{2\pi^{\frac{1}{2}}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\alpha)\boldsymbol{\lambda}'\boldsymbol{\Sigma}\beta^3}{\pi^{\frac{1}{2}}.\Gamma(\alpha).\Gamma\left(\alpha-\frac{1}{2}\right)\Gamma(\alpha-1)}$$

$$+ \frac{\pi^{\frac{1}{2}}\Gamma(\alpha+2)\Gamma\left(\alpha-\frac{3}{2}\right)\Gamma(\alpha+1)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\beta^6}{\pi^{\frac{1}{2}}\Gamma(\alpha)\Gamma\left(\alpha-\frac{1}{2}\right).\Gamma(\alpha-1)}$$

$$\therefore \boldsymbol{G}_2(1; \boldsymbol{\lambda}) = \boldsymbol{\lambda}'\boldsymbol{\lambda} + \frac{2\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)\boldsymbol{\lambda}'\boldsymbol{\Sigma}\beta^3}{\Gamma\left(\alpha-\frac{1}{2}\right)\Gamma(\alpha-1)}$$

$$+ \frac{\Gamma(\alpha+2).\Gamma\left(\alpha-\frac{3}{2}\right)\Gamma(\alpha+1)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}\beta^6}{\Gamma(\alpha)\Gamma\left(\alpha-\frac{1}{2}\right)\Gamma(\alpha-1)} \quad ----(3.80)$$

Thus, for the wishart distribution, where $\beta = 2$ and $\alpha = \frac{\eta}{2}$, we have

$$\boldsymbol{G}_2(1; \boldsymbol{\lambda}) = \boldsymbol{\lambda}'\boldsymbol{\lambda} + \frac{2\Gamma\left(\frac{\eta}{2}+1\right)\Gamma\left(\frac{\eta}{2}+\frac{1}{2}\right)\boldsymbol{\lambda}'\boldsymbol{\Sigma}2^3}{\Gamma\left(\frac{\eta}{2}-\frac{1}{2}\right)\Gamma\left(\frac{\eta}{2}-1\right)}$$

$$+ \frac{\Gamma\left(\frac{\eta}{2}+2\right).\Gamma\left(\frac{\eta}{2}-\frac{3}{2}\right)\Gamma\left(\frac{\eta}{2}+1\right)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}2^6}{\Gamma\left(\frac{\eta}{2}\right)\Gamma\left(\frac{\eta}{2}-\frac{1}{2}\right)\Gamma\left(\frac{\eta}{2}-1\right)}$$

$$\therefore \boldsymbol{G}_2(1;\lambda) = \boldsymbol{\lambda}'\boldsymbol{\lambda} + \frac{16\Gamma\left(\frac{\eta+2}{2}\right)\Gamma\left(\frac{\eta+1}{2}\right)\boldsymbol{\lambda}'\boldsymbol{\Sigma}}{\Gamma\left(\frac{\eta-1}{2}\right)\Gamma\left(\frac{\eta-2}{2}\right)} + \frac{64\Gamma\left(\frac{\eta+4}{2}\right).\Gamma\left(\frac{\eta-3}{2}\right)\Gamma\left(\frac{\eta+2}{2}\right)\boldsymbol{\Sigma}'\boldsymbol{\Sigma}}{\Gamma\left(\frac{\eta}{2}\right)\Gamma\left(\frac{\eta-1}{2}\right)\Gamma\left(\frac{\eta-2}{2}\right)} \quad ------- (3.81)$$

This is the second moment of a $p = 3$ variate Wishart distribution about $\boldsymbol{\lambda}$. It is a function of η, the number of observations (sample size).

3.8 CASE OF INDEPENDENCE OF GAMMA RANDOM VARIABLES

Suppose $x_1, x_2, \dots, x_i, \dots, x_p$ are independent gamma random variables each with probability density function

$$f(x_i) = \frac{1}{\beta_i^{\alpha_i}\Gamma(\alpha_i)} x_i^{\alpha_i - 1} e^{-x_i/\beta_i} \quad --------- (3.82)$$

then, the joint density function of the random variable is

$$f(x_1, x_2, \dots, x_i, \dots, x_p) = \prod_{i=1}^{p} \frac{x_i^{\alpha_i-1} e^{-x_i/\beta_i}}{\beta_i^{\alpha_i}\Gamma(\alpha_i)} \quad ------- (3.83)$$

(Furman, 2008)

Hence, the generalized moment generating function about λ of the joint distribution of the p independent gamma random variables is obtained as follows:

$$\boldsymbol{G}_n(c;\lambda) = \left(\sum_{r=0}^{n} \binom{n}{r} \boldsymbol{\lambda}^{n-r}\right)^p E(\boldsymbol{X}^{cr})$$

where

$$E(X^{cr}) = \prod_{i=1}^{p}\int_{0}^{\infty}\frac{x_i^{\alpha_i+cr-1}e^{-x_i/\beta_i}dx_i}{\beta_i^{\alpha_i}\Gamma(\alpha_i)}$$

Let

$$\frac{x_i}{\beta_i} = y_i;\ \ x_i = \beta_i y_i;\ \ \frac{dx_i}{dy_i};\ \Rightarrow dx_i = \beta_i dy_i$$

$$\therefore E(X^{cr}) = \prod_{i=1}^{p}\int_{0}^{\infty}\frac{(\beta_i y_i)^{(\alpha_i+cr)-1}e^{-y_i}\beta_i dy_i}{\beta_i^{\alpha_i}\Gamma(\alpha_i)} = \prod_{i=1}^{p}\frac{\beta_i^{cr}}{\Gamma(\alpha_i)}\int_{0}^{\infty} y_i^{(\alpha_i+cr)-1}e^{-y_i}\,dy_i$$

$$= \prod_{i=1}^{p}\frac{\beta_i^{cr}\Gamma(\alpha_i+cr)}{\Gamma(\alpha_i)}$$

$$\therefore \boldsymbol{G}_n(c;\boldsymbol{\lambda}) = \left(\sum_{r=0}^{n}\binom{n}{r}\boldsymbol{\lambda}^{n-r}\right)^{p}\prod_{i=1}^{p}\beta_i^{cr}\frac{\Gamma(\alpha_i+cr)}{\Gamma(\alpha_i)} \quad ----(3.84)$$

Suppose c, λ, n and r vary amongst the random variables, we have that

$$\boldsymbol{G}_{n_i}(c_i;\ \lambda_i) = \prod_{i=1}^{p}\sum_{r_i=0}^{n_i}\binom{n_i}{r_i}\lambda_i^{n_i-r_i}\beta_i^{c_i r_i}\frac{\Gamma(\alpha_i+c_i r_i)}{\Gamma(\alpha_i)} \quad ----(3.85)$$

However, if all the parameters ($c_i, \lambda_i, \beta_i, \alpha_i, n_i$ and r_i) are constant for all the random variables, we have

$$\boldsymbol{G}_{n_i}(c_i;\ \lambda_i) = \boldsymbol{G}_n(c;\lambda) = \left[\sum_{r=0}^{n}\binom{n}{r}\boldsymbol{\lambda}^{n-r}\frac{\beta^{cr}\Gamma(\alpha+cr)}{\Gamma(\alpha)}\right]^{p} \quad ----(3.86)$$

Example: Suppose all parameters are constant for a p-independent-variate gamma distribution; $n = 2, c = 1, \lambda = -\mu = -\alpha\beta$ then,

$$G_2(1;(-\alpha\beta)) = \left[\lambda^2 + 2\alpha\beta\lambda\frac{\Gamma(\alpha)}{\Gamma(\alpha)} + \beta^2(\alpha+1)\alpha\frac{\Gamma(\alpha)}{\Gamma(\alpha)}\right]^{p}$$

$$= [(-\alpha\beta)^2 + 2(-\alpha\beta)(\alpha\beta) + \alpha(\alpha+1)\beta^2]^p = [-(\alpha\beta)^2 + (\alpha\beta)^2 + \alpha\beta^2]^p$$

$$\therefore \boldsymbol{G}_2[1;\ (-\alpha\beta)] = (\alpha\beta^2)^p \quad ---- \quad (3.87)$$

Thus, the second moment about the mean of p-dimensional independent multivariate gamma random variable is the p^{th} power of the variance of their univariate equivalents.

3.9 GENERALIZED MULTIVARIATE MOMENT GENERATING FUNCTION (GMMGF) OF THE NORMAL DISTRIBUTION

A random variable, X, is said to have a univariate normal density if its density function is of the form:

$$f(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ---- \ (3.88)$$

The joint density of independent normal variates, $x_1, x_2, \dots, x_i, \dots, x_p$ is

$$f\left(x_1, x_2, \dots, x_i, \dots x_p\right) = \frac{1}{(2\pi)^{\frac{p}{2}}\sigma_1 \dots \sigma_p} e^{-\frac{1}{2}\sum_{i=1}^{p}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2} \quad ---- \ (3.89)$$

Let

$$\boldsymbol{X}_{(p\times1)} = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_p \end{pmatrix}, \boldsymbol{\mu}_{(p\times1)} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ . \\ . \\ . \\ \mu_p \end{pmatrix} \text{ and } \boldsymbol{\Sigma}_{(p\times p)} = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \sigma_p^2 \end{pmatrix}$$

Then,

$$f(\boldsymbol{X}) = \frac{1}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} exp\left[-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})\right] \quad ---- \ (3.90)$$

$$-\infty \leq \boldsymbol{X} \leq \infty, |\boldsymbol{\Sigma}| > 0$$

The covariance matrix of the random vector, $\boldsymbol{X}$, with correlated random variables is given as

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \dots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} \dots & \sigma_{2p} \\ \vdots & \vdots & \vdots \\ \sigma_{p1} & \sigma_{p2} \dots & \sigma_{pp} \end{pmatrix}$$

By substituting this in Equation 3.90, it becomes the multivariate density function of the random vector of p —correlated random variables. (Ogum 2002; Onyeagu 2003; Johnson and Wichern 1992)

Thus $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

From Equation 3.72,

$$\boldsymbol{G}_n(c; \boldsymbol{\lambda}) = E(\boldsymbol{X}^c + \boldsymbol{\lambda})^n = \sum_{r=0}^{n} \binom{n}{r} \boldsymbol{\lambda}^{n-r} E(\boldsymbol{X}^{cr})$$

where

$$E(\boldsymbol{X}^{cr}) = \int_{-\infty}^{\infty} \boldsymbol{X}^{cr} f(\boldsymbol{X}) d\boldsymbol{X}_p$$

$$\therefore E(\boldsymbol{X}^{cr}) = \int_{-\infty}^{\infty} \frac{\boldsymbol{X}^{cr}}{(2\pi)^{\frac{p}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}}} exp\left[-\frac{1}{2}(\boldsymbol{X} - \boldsymbol{\mu})' \Sigma^{-1} (\boldsymbol{X} - \boldsymbol{\mu})\right] d\boldsymbol{X}_p$$

$$E(\boldsymbol{X}^{cr}) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\boldsymbol{X}^{cr}}{(2\pi)^{\frac{p}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}}} e^{-\left[\frac{1}{2}\sum_{i=1}^{p}\left(\frac{x_i - \mu_i}{\sigma_i}\right)^2\right]} dx_1 \dots dx_p$$

Let

$$\left(\frac{x_i - \mu_i}{\sigma_i \sqrt{2}}\right)^2 = v_i; \Rightarrow v_i^{\frac{1}{2}} = \frac{x_i - \mu_i}{\sigma_i \sqrt{2}}; \; x_i = \mu_i + \sigma_i \sqrt{2} v_i^{\frac{1}{2}}; \; \frac{dx_i}{dv_i} = \sqrt{2} \sigma_i \frac{1}{2} v_i^{\frac{1}{2}}$$

$$\therefore dx_i = \frac{\sigma_i v_i^{-\frac{1}{2}}}{\sqrt{2}} dv_i$$

Hence,

$$E(\boldsymbol{X}^{cr}) = \frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left(\mu_i + \sigma_i\sqrt{2}v_i^{\frac{1}{2}}\right)^{cr} . e^{-\frac{1}{2}\sum_{i=1}^{p}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2} dx_1 \dots dx_p$$

$$= \frac{2^p}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \int_{0}^{\infty} \dots \int_{0}^{\infty} \left(\mu_i + \sigma_i\sqrt{2}v_i^{\frac{1}{2}}\right)^{cr} e^{-\frac{1}{2}\sum_{i=1}^{p}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2} dx_1 \dots . dx_p$$

But,

$$\left(\mu_i + \sigma_i\sqrt{2}v_i^{\frac{1}{2}}\right)^{cr} = \sum_{t=0}^{cr} \binom{n}{t} \mu_i^{cr-t} \left(2\sigma_i^2 v_i\right)^{\frac{t}{2}}$$

Now,

$$E(\boldsymbol{X}^{cr}) = \frac{\sum_{t=0}^{cr}\binom{cr}{t} \mu_i^{cr-t} 2^p \left(2^p\sigma_i^2\right)^{\frac{t}{2}}}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \int_{0}^{\infty} \dots \int_{0}^{\infty} v_i^{\frac{t}{2}} e^{-\sum_{i=1}^{p} v_i} . \frac{\sigma_i v_i^{-\frac{1}{2}}}{\sqrt{2}} dv_1 \dots dv_p$$

$$E(\boldsymbol{X}^{cr}) = \frac{\sum_{t=0}^{cr}\binom{cr}{t} \mu_i^{cr-t} 2^p \left(2^p\sigma_i^2\right)^{\frac{t}{2}}}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} . \frac{\sigma_i}{2^{\frac{p}{2}}} \int_{0}^{\infty} \dots \int_{0}^{\infty} v_i^{\left(\frac{t}{2}+\frac{1}{2}\right)-1} e^{-\sum_{i=1}^{p} v_i} dv_1 \dots dv_p$$

$$= \frac{\sum_{t=0}^{cr}\binom{cr}{t} \boldsymbol{\mu}^{cr-t} 2^p (2^p\boldsymbol{\Sigma})^{\frac{t}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}}}{(2\pi)^{\frac{p}{2}} . 2^{\frac{p}{2}} |\boldsymbol{\Sigma}|^{\frac{1}{2}}} \left[\Gamma\left(\frac{t}{2} + \frac{1}{2}\right)\right]^p$$

$$\therefore E(\boldsymbol{X}^{cr}) = \frac{\sum_{t=0}^{cr}\binom{cr}{t} \boldsymbol{\mu}^{cr-t} (2^p\boldsymbol{\Sigma})^{\frac{t}{2}}}{\pi^{\frac{p}{2}}} \left[\Gamma\left(\frac{t}{2} + \frac{1}{2}\right)\right]^p$$

Hence,

$$\therefore \boldsymbol{G}_n(c;\boldsymbol{\lambda}) = \frac{\sum_{r=0}^{n}\binom{n}{r}\boldsymbol{\lambda}^{n-r}\sum_{t=0}^{cr}\binom{cr}{t}\boldsymbol{\mu}^{cr-t}(2^p\boldsymbol{\Sigma})^{\frac{t}{2}}\left[\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}} ----(3.91)$$

As with Equation 3.24, Equation 3.91 is evaluated at even number values of t. That is, where $t = 0, 2, 4, \ldots$

Examples:

For $c = 1, n = 1, r = 0,1$ and $t = 0$

$$\boldsymbol{G}_1(1;\boldsymbol{\lambda}) = \sum_{r=0}^{1}\binom{1}{0}\boldsymbol{\lambda}^{1-0}\frac{\left[\sum_{t=0}^{r}\binom{r}{t}\boldsymbol{\mu}^{r-t}(2^p\boldsymbol{\Sigma})^{\frac{t}{2}}\left[\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)\right]^p\right]}{\pi^{\frac{p}{2}}}$$

$$\therefore \boldsymbol{G}_1(1;\boldsymbol{\lambda}) = \boldsymbol{\lambda} + \boldsymbol{\mu} \quad ---- (3.92)$$

Suppose $\boldsymbol{\lambda} = -\boldsymbol{\mu}$, the first moment of $\boldsymbol{X}^1 = \boldsymbol{X}$ about $\boldsymbol{\mu}$ is obtained as

$$\boldsymbol{G}_1(1;-\mu) = -\boldsymbol{\mu} + \boldsymbol{\mu} = 0$$

as expected.

For $c = 1, n = 2, r = 0, 1, 2$ and $t = 0$,2; then

$$\boldsymbol{G}_2(1;\boldsymbol{\lambda}) = \sum_{r=0}^{2}\binom{2}{r}\boldsymbol{\lambda}^{2-r}\left[\frac{\sum_{t=0}^{r}\binom{r}{t}\boldsymbol{\mu}^{r-t}(2^p\boldsymbol{\Sigma})^{\frac{t}{2}}\left[\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}}\right]$$

If $r = 0, t = 0$

$$\binom{2}{0}\boldsymbol{\lambda}^{2-0} = \boldsymbol{\lambda}^2$$

$$r = 1, t = 0$$

$$\binom{2}{1}\boldsymbol{\lambda}^{2-1}\left[\binom{1}{0}\boldsymbol{\mu}^{1-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}\frac{\left(\Gamma\left(\frac{0}{2}+\frac{1}{2}\right)\right)^p}{\pi^{\frac{p}{2}}}\right] = 2\boldsymbol{\lambda}\boldsymbol{\mu}$$

For $r = 2;\ t = 0, 2$

$$\binom{2}{2}\boldsymbol{\lambda}^{2-2}\sum_{t=0}^{2}\binom{2}{t}\boldsymbol{\mu}^{2-t}(2^p\boldsymbol{\Sigma})^{\frac{t}{2}}\frac{\left[\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)\right]^p}{\pi^p}$$

$$= \binom{2}{0}\boldsymbol{\mu}^2(2^p\boldsymbol{\Sigma})^0\frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}} + \binom{2}{2}\boldsymbol{\mu}^0(2^p\Sigma)^{\frac{2}{2}}\frac{\left[\Gamma\left(\frac{3}{2}\right)\right]^p}{\pi^{\frac{p}{2}}} = \boldsymbol{\mu}^2 + (2^p\boldsymbol{\Sigma})\left(\frac{1}{2}\right)^p$$

$$= \boldsymbol{\mu}'\boldsymbol{\mu} + \boldsymbol{\Sigma}$$

$$\therefore \boldsymbol{G}_2(1;\boldsymbol{\lambda}) = \boldsymbol{\lambda}^2 + 2\boldsymbol{\lambda}\boldsymbol{\mu} + \boldsymbol{\mu}^2 + \boldsymbol{\Sigma} = \boldsymbol{\lambda}'\boldsymbol{\lambda} + 2\boldsymbol{\lambda}'\mu + \boldsymbol{\lambda}'\boldsymbol{\lambda} + \boldsymbol{\Sigma} \quad --- \quad (3.93)$$

Now, let $\boldsymbol{\lambda} = -\boldsymbol{\mu}$; hence the second moment of $\boldsymbol{X}^1 = \boldsymbol{X}$ about the mean, μ, is

$$_2(1;-\boldsymbol{\mu}) = (-\boldsymbol{\mu})^2 + 2(-\boldsymbol{\mu})(\boldsymbol{\mu}) + \boldsymbol{\mu}^2 + \boldsymbol{\Sigma} = 2\boldsymbol{\mu}^2 - 2\boldsymbol{\mu}^2 + \boldsymbol{\Sigma}$$

$$\therefore \boldsymbol{G}_2(1;-\boldsymbol{\mu}) = \boldsymbol{\Sigma} \quad ----- \quad (3.94)$$

That is the variance-covariance matrix as expected.

Also, $\boldsymbol{G}_3(1;\boldsymbol{\lambda})$ is obtained from Equation 3.91 as follows:

$$n = 3;\ r = 0, 1, 2, 3; t = 0, 2$$

Now, where $r = 0$

$$\binom{3}{0}\boldsymbol{\lambda}^{3-0}\left[\binom{0}{0}\boldsymbol{\mu}^{0-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}\frac{\left[\Gamma\left(\frac{0}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}}\right] = \boldsymbol{\lambda}^3$$

where $r = 1; t = 0$

$$\binom{3}{1}\boldsymbol{\lambda}^2\left[\binom{1}{0}\boldsymbol{\mu}^{1-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}\left(\frac{\pi^{\frac{p}{2}}}{\pi^{\frac{p}{2}}}\right)\right] = 3\boldsymbol{\lambda}^2\boldsymbol{\mu}$$

where $r = 2; t = 0, 2$

$$\binom{3}{2}\boldsymbol{\lambda}^{3-2}\left[\binom{2}{0}\boldsymbol{\mu}^{2-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}\frac{\pi^{\frac{p}{2}}}{\pi^{\frac{p}{2}}} + \binom{2}{2}\boldsymbol{\mu}^{2-2}(2^p\boldsymbol{\Sigma})^{\frac{2}{2}}\frac{\left[\Gamma\left(\frac{2}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}}\right] = 3\boldsymbol{\lambda}\boldsymbol{\mu}^{\mathbf{2}} + 3\boldsymbol{\lambda}\boldsymbol{\Sigma}$$

where $r = 3;\ t = 0, 2$

$$\binom{3}{3}\boldsymbol{\lambda}^0\left[\binom{3}{0}\boldsymbol{\mu}^{3-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}\frac{\left[\Gamma\left(\frac{0}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}} + \binom{3}{2}\boldsymbol{\mu}^{3-2}(2^p\boldsymbol{\Sigma})^{\frac{2}{2}}\frac{\left[\Gamma\left(\frac{2}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}}\right]$$

$$= \boldsymbol{\mu}^3 + 3\boldsymbol{\mu}(2^p\boldsymbol{\Sigma})\frac{\left[\Gamma\left(\frac{3}{2}\right)\right]^p}{\pi^{\frac{p}{2}}} = \boldsymbol{\mu}^3 + 3\boldsymbol{\mu}\boldsymbol{\Sigma}$$

$$\therefore \boldsymbol{G}_3(1;\boldsymbol{\lambda}) = \boldsymbol{\lambda}^3 + 3\boldsymbol{\lambda}^2\boldsymbol{\mu} + 3\boldsymbol{\lambda}\boldsymbol{\mu}^2 + 3\boldsymbol{\lambda}\boldsymbol{\Sigma} + \boldsymbol{\mu}^3 + 3\boldsymbol{\mu}\boldsymbol{\Sigma} ---- (3.95)$$

Now, let $\lambda = -\mu$. Thus,

$$\boldsymbol{G}_3(1;-\boldsymbol{\mu}) = (-\boldsymbol{\mu})^3 + 3(-\boldsymbol{\mu})^2\boldsymbol{\mu} + 3(-\boldsymbol{\mu})\boldsymbol{\mu}^{\mathbf{2}} + 3(-\boldsymbol{\mu})\boldsymbol{\Sigma} + \boldsymbol{\mu}^3 + 3\boldsymbol{\mu}\boldsymbol{\Sigma}$$

$$\therefore \boldsymbol{G}_3(1;-\boldsymbol{\mu}) = -(\boldsymbol{\mu}'\boldsymbol{\mu})\boldsymbol{\mu} + 3(\boldsymbol{\mu}'\boldsymbol{\mu})\boldsymbol{\mu} - 3(\boldsymbol{\mu}'\boldsymbol{\mu})\boldsymbol{\mu} - \boldsymbol{\Sigma}\boldsymbol{\mu} + (\boldsymbol{\mu}'\boldsymbol{\mu})\boldsymbol{\mu} + 3\boldsymbol{\Sigma}\boldsymbol{\mu} = \big((0)\big)_{p\times 1} -- (3.96)$$

In the same reasoning, $\boldsymbol{G}_4(1;\boldsymbol{\lambda})$ is obtained as follows:

$$n = 4; r = 0, 1, 2, 3, 4; t = 0, 2, 4$$

where $r = 0; t = 0$

$$\binom{4}{0}\boldsymbol{\lambda}^{4-0}\left[\binom{0}{0}\boldsymbol{\mu}^{0-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}\frac{\left[\Gamma\left(\frac{0}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}}\right]=\boldsymbol{\lambda}^4$$

where $r=1; t=0$

$$\binom{4}{1}\boldsymbol{\lambda}^{4-1}\left[\binom{1}{0}\boldsymbol{\mu}^{1-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}\frac{\left[\Gamma\left(\frac{0}{2}+\frac{1}{2}\right)\right]^p}{\pi^{\frac{p}{2}}}\right]=4\boldsymbol{\lambda}^3\boldsymbol{\mu}$$

where $r=2; t=0,2$

$$\binom{4}{3}\boldsymbol{\lambda}^{4-3}\left[\binom{3}{0}\boldsymbol{\mu}^{3-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}+\binom{3}{2}\boldsymbol{\mu}^{3-2}(2^p\boldsymbol{\Sigma})^{\frac{2}{2}}\left(\frac{1}{2}\right)^p\frac{\pi^{\frac{p}{2}}}{\pi^{\frac{p}{2}}}\right]=4\boldsymbol{\lambda}[\boldsymbol{\mu}^3+3\boldsymbol{\mu}\boldsymbol{\Sigma}]$$

$$=4\boldsymbol{\lambda}\boldsymbol{\mu}^{\mathbf{3}}+12\boldsymbol{\lambda}\boldsymbol{\mu}\boldsymbol{\Sigma}$$

where $r=4;\ t=0,2,4$

$$\binom{4}{4}\boldsymbol{\lambda}^{4-4}\left[\binom{4}{0}\boldsymbol{\mu}^{4-0}(2^p\boldsymbol{\Sigma})^{\frac{0}{2}}+\binom{4}{2}\boldsymbol{\mu}^{4-2}(2^p\boldsymbol{\Sigma})^{\frac{2}{2}}\left(\frac{1}{2}\right)^p\frac{\pi^{\frac{p}{2}}}{\pi^{\frac{p}{2}}}\right.$$

$$\left.+\binom{4}{4}\boldsymbol{\mu}^{4-4}(2^p\boldsymbol{\Sigma})^{\frac{4}{2}}\frac{\left[\Gamma\left(\frac{5}{2}\right)\right]^p}{\pi^{\frac{p}{2}}}\right]$$

$$=(\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}')+6\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma}+(2^p\boldsymbol{\Sigma})^2\left(\frac{3}{2}.\frac{1}{2}\right)^p\frac{\pi^{\frac{p}{2}}}{\pi^{\frac{p}{2}}}$$

$$=(\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}')+6\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma}+\frac{2^{2p}}{2^{2p}}.3^p\boldsymbol{\Sigma}'\boldsymbol{\Sigma}$$

$$=(\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}')+6\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma}+3^p\boldsymbol{\Sigma}'\boldsymbol{\Sigma}$$

$$\boldsymbol{G}_4(1;\boldsymbol{\lambda}) = \boldsymbol{\lambda}^4 + 4\boldsymbol{\lambda}^3\boldsymbol{\mu} + 6\boldsymbol{\lambda}^2\boldsymbol{\mu}^2 + 6\boldsymbol{\lambda}^2\boldsymbol{\Sigma} + 4\boldsymbol{\lambda}\mu^3 + 12\boldsymbol{\lambda\mu\Sigma} + \boldsymbol{\mu}^4 + 6\boldsymbol{\mu^2\Sigma} + \boldsymbol{3}^p\boldsymbol{\Sigma}^2 \\ ------- (3.97)$$

Now, let $\lambda = -\mu$; therefore, the fourth moment of $X^1 = X$ about its mean, μ, is obtained as

$$G_4(1;-\mu) = (\boldsymbol{\mu\mu}')(\boldsymbol{\mu\mu}') - 4(\boldsymbol{\mu\mu}')(\boldsymbol{\mu\mu}') + 6(\boldsymbol{\mu\mu}')(\boldsymbol{\mu\mu}') + 6\boldsymbol{\mu\mu}'\boldsymbol{\Sigma} - 4(\boldsymbol{\mu\mu}')(\boldsymbol{\mu\mu}') \\ - 12\boldsymbol{\mu\mu}'\boldsymbol{\Sigma} + (\boldsymbol{\mu\mu}')(\boldsymbol{\mu\mu}') + 6\boldsymbol{\mu\mu}'\boldsymbol{\Sigma} + 3^p\boldsymbol{\Sigma}'\boldsymbol{\Sigma}$$

$$= 12\boldsymbol{\mu\mu}'\Sigma - 12\boldsymbol{\mu\mu}'\boldsymbol{\Sigma} + 3^p\boldsymbol{\Sigma}'\boldsymbol{\Sigma}$$

$$\therefore \boldsymbol{G}_4(1;-\boldsymbol{\mu}) = 3^p\boldsymbol{\Sigma}'\boldsymbol{\Sigma} \quad ---- (3.98)$$

Using Equation 3.96 and Equation 3.94 in Equation 3.11, the skewness of the distribution of $X^1 = X$ is obtained as

$$\boldsymbol{SK}(1) = \frac{\boldsymbol{G}_3(1;-\boldsymbol{\mu})}{[\boldsymbol{G}_2(1;-\boldsymbol{\mu})]^{\frac{3}{2}}} = \big((0)\big)_{p\times 1} \quad ---- \ (3.99)$$

Thus, it is observed that the ratio of the third moment of $\boldsymbol{X}^1 = \boldsymbol{X}$ about its mean, $\boldsymbol{\mu}$, to the $\left(\frac{3}{2}\right)^{th}$ power of its second moment is its skewness which equals zero, 0.

This shows that the multivariate normal distribution is symmetric.

Using Equation 3.98 and Equation 3.94 in Equation 3.12, the kurtosis of the distribution of $X^1 = X$ is obtained as

$$\boldsymbol{K_U}(1) = \frac{\boldsymbol{G}_4(1;-\boldsymbol{\mu})}{[\boldsymbol{G}_2(1;-\boldsymbol{\mu})]^2} = \frac{3^p\Sigma'\Sigma}{\Sigma'\Sigma} = 3^p \quad ---- \ (3.100)$$

This may be interpreted as mesokurtic because the base is equal to 3 or because the p^{th} root of $\boldsymbol{K_u}(1)$ is equal to 3. It may equally be termed a p −variate mesokurtic distribution.

All conceivable moments of the Multivariate Normal Distribution including situations where c is real can be obtained with the multivariate generalized moment generating function (Equation 3.91).

The traditional moment generating function, which demands on more tedious derivations, yields a skewness of zero and a kurtosis value of 3 as can be seen below

3.10 THE TRADITIONAL MULTIVARIATE NORMAL MOMENT GENERATING FUNCTION

This section tends to review the notion of multivariate moment generating function and specifically shows the derivation of the moment generating function of a multivariate normal distribution.

As a prelude to the discussion of multivariate moment generating function, Bulmer (1979) stated more generally that, where $X = (X_1, \dots, X_n)'$, an n-dimensional random vector, one uses $t.x = t'X$ instead of $t.X$

$$\therefore M_X(t) = E\left(e^{t'X}\right) \quad ------ \quad (3.101)$$

If $X \sim N(\mu, \sigma^2)$, then the moment generating function for the univariate case is

$$M_X(t) = e^{ut+\frac{1}{2}\sigma^2 t^2}$$

For the multivariate case, let

$X_{p\times 1} \sim N_p(\mu, \Sigma)$ and $M_X(t)$ denote the moment generating function of X.

$$M_X(t) = E\left(e^{t'X}\right) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{e^{t'X} e^{-\frac{1}{2}\{(X-\mu)'\Sigma^{-1}(X-\mu)\}}}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} \, dX_1 \dots dX_p \quad (3.102)$$

(Onyeagu, 2003).

Considering the numerator,

$$e^{t'X}e^{-\frac{1}{2}\{(X-\mu)'\Sigma^{-1}(X-\mu)\}} = e^{-\frac{1}{2}\{(X-\mu)'\Sigma^{-1}(X-\mu)-2t'X\}}$$

Now, considering the exponent,

$$(X-\mu)'\Sigma^{-1}(X-\mu) - 2t'X$$

$$= X'\Sigma^{-1}X - X'\Sigma^{-1}\mu - \mu'\Sigma^{-1}X + \mu'\Sigma^{-1}\mu - 2t'X$$

$$= X'\Sigma^{-1}X - 2(\mu+\Sigma t)'\Sigma^{-1}X + \mu'\Sigma^{-1}\mu$$

$$= X'\Sigma^{-1}X - 2(\mu+\Sigma t)'\Sigma^{-1}X + (\mu+\Sigma t)'\Sigma^{-1}(\mu+\Sigma t) + \mu'\Sigma^{-1}\mu - (\mu+\Sigma \text{t})'\Sigma^{-1}(\mu+\Sigma t)$$

$$= (X-\mu-\Sigma t)'\Sigma^{-1}(X-\mu-\Sigma t) - (2t'\mu + t'\Sigma t)$$

$$\therefore e^{-\frac{1}{2}\{(X-\mu)'\Sigma^{-1}(X-\mu)-2t'X\}} = e^{t'\mu+\frac{1}{2}t'\Sigma t} . e^{-\frac{1}{2}\{(X-\mu-\Sigma t)'\Sigma^{-1}(X-\mu-\Sigma t)\}}$$

$$\therefore M_X(t) = e^{t'\mu+\frac{1}{2}t'\Sigma t} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\{(X-\mu-\Sigma t)'\Sigma^{-1}(X-\mu-\Sigma t)\}}}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}} dX_1 \dots dX_p$$

$$\Rightarrow M_X(t) = e^{t'\mu+\frac{1}{2}t'\Sigma t} \quad ---- \quad (3.103)$$

The n^{th} moment of the multivariate normal random variable, X, is obtained by taking the n^{th} derivative of $M_X(t)$, equation 3.103, and evaluating at $t = 0$ (Onyeagu, 2003).

Now,

$$\frac{dM_X(t)}{dt} = (\mu + t'\Sigma)e^{t'\mu+\frac{1}{2}t'\Sigma t}$$

Thus, the first moment about zero is obtained as

$$\left.\frac{dM_X(t)}{dt}\right|_{t=0} = E(X) = \mu \quad ---- (104)$$

Also,

$$\frac{d^2M_x(t)}{dt^2} = \Sigma e^{t'\mu+\frac{1}{2}t'\Sigma t} + (\mu + \Sigma t)'(\mu + \Sigma \mathrm{t})e^{t'\mu+\frac{1}{2}t'\Sigma t}$$

Thus, the second moment about zero can be obtained as

$$\left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0} = \mu'\mu + \Sigma \quad ---- \;(105)$$

Hence the variance of the multivariate normal random variable and its probability density function is obtained using Equations 105 and 104 as follows:

$$Var(X) = E(X^2) - [E(X)]^2 = \mu'\mu + \Sigma - \mu'\mu$$

$$\therefore Var(X) = \Sigma$$

$$\frac{d^3M_X(t)}{dt^3} = \frac{d}{dt}(\Sigma e^{t'\mu+\frac{1}{2}t'\Sigma t} + (\mu + \Sigma \mathrm{t})'(\mu + \Sigma t)e^{t'\mu+\frac{1}{2}t'\Sigma \mathrm{t}})$$

$$= (\mu + \Sigma t)\Sigma e^{t'\mu+\frac{1}{2}t'\Sigma t} + (2\mu'\Sigma + 2\Sigma'\Sigma t)e^{t'\mu+\frac{1}{2}t'\Sigma \mathrm{t}}$$

$$+ (\mu'\mu + 2\mu'\Sigma t + (\Sigma \mathrm{t})'\Sigma t)(\mu + t'\Sigma)e^{t'\mu+\frac{1}{2}t'\Sigma t}$$

$$= [\Sigma\mu + \Sigma'\Sigma t + 2\mu'\Sigma + 2(\Sigma \mathrm{t})'\Sigma + (\mu'\mu)\mu + \mu'\mu\Sigma t + 2\mu'\Sigma t\mu + 2\mu'(\Sigma t)'\Sigma t$$

$$+ (\Sigma t)'\Sigma t + (\Sigma t)'\Sigma t(\Sigma \mathrm{t})]e^{t'\mu+\frac{1}{2}t'\Sigma \mathrm{t}}$$

Thus, the third moment about is

$$\left.\frac{d^3M_X(t)}{dt^3}\right|_{t=0} = 2\Sigma\mu + (\mu\mu')\mu + \Sigma\mu = 3\Sigma\mu + (\mu\mu')\mu \quad ---- (106)$$

Hence the skewness of the distribution can be obtained using a combination of Equations 104, 105 and 106 as

$$sk = \frac{3\Sigma\mu + (\mu\mu')\mu - 3(\mu\mu' + \Sigma)\mu + 2(\mu\mu')\mu}{(\mu_2 - \mu'\mu)^{\frac{3}{2}}}$$

$$= \frac{3\Sigma'\mu + (\mu\mu')\mu - 3(\mu\mu')\mu - 3\Sigma'\mu + 2(\mu\mu')\mu}{(\mu_2 - \mu'\mu)^{\frac{3}{2}}}$$

$$= \frac{0}{(\mu_2 - \mu'\mu)^{\frac{3}{2}}}$$

$$\therefore sk = 0 \;---- (107)$$

This implies that the Multivariate Distribution is symmetric.

Also,

$$\frac{d^4 M_X(t)}{dt^4} = \frac{d}{dt}[\Sigma\mu + \Sigma'\Sigma t + 2\mu'\Sigma + 2(\Sigma \text{t})'\Sigma + (\mu'\mu)\mu + \mu'\mu\Sigma t + 2\mu'\Sigma t\mu + 2\mu'(\Sigma t)'\Sigma t + (\Sigma t)'\Sigma t + (\Sigma t)'\Sigma t(\Sigma \text{t})]e^{t'\mu + \frac{1}{2}t'\Sigma t}$$

$$= (\Sigma'\Sigma)e^{t'\mu + \frac{1}{2}t'\Sigma t} + (\Sigma'\mu + \Sigma' t\Sigma)(\mu + \Sigma t) + (2\Sigma'\Sigma)e^{t'\mu + \frac{1}{2}t'\Sigma t} + (2\Sigma'\mu + 2\Sigma'\Sigma t)(\mu + \Sigma' t) + (2\mu'\Sigma\mu + 2\Sigma' t\Sigma + \mu'\Sigma\mu + 4\mu'\Sigma t\Sigma + 3\Sigma t'\Sigma t\Sigma)e^{t'\mu + \frac{1}{2}t'\Sigma t} +$$

$$\left.\frac{d^4 M_X(t)}{dt^4}\right|_{t=0} = 3\Sigma'\Sigma + 6\mu\mu'\Sigma + (\mu\mu')(\mu\mu') \;---- (3.108)$$

Hence the kurtosis of the distribution can be obtained using a combination of Equations (3.104), (3.105), (3.106), (3.107) and (3.108) as

$$ku = \frac{3\Sigma'\Sigma + 6\mu\mu'\Sigma + (\mu\mu')(\mu\mu') - 4(3\Sigma'\mu + (\mu\mu')\mu)\mu' + 6(\mu\mu'\mu + \mu\Sigma)\mu' - 3(\mu\mu')(\mu\mu')}{(\mu_2 + \mu'\mu)^2}$$

$$= \frac{3\Sigma'\Sigma + 6\mu\mu'\Sigma + (\mu\mu')(\mu\mu') - 12\mu\mu'\Sigma - 4(\mu\mu')(\mu\mu') + 6(\mu\mu')(\mu\mu') + 6\mu\mu'\Sigma - 3(\mu\mu')(\mu\mu')}{(\mu_2 + \mu'\mu)^2}$$

$$= \frac{3\Sigma'\Sigma}{\Sigma'\Sigma}$$

$$\therefore ku = 3 - - - - (3.109)$$

This also implies that the Multivariate Normal Distribution is leptokurtic but this method does not give any account of the number of variables in the analysis.

Higher moments and other parameters of the multivariate normal distribution that depend on them can be obtained with this method. However, evaluating moments about the mean (central moments) using the relationships between moments about zero (crude moments) and taking n^{th} derivative become more difficult as n gets higher.

3.11 GENERALIZED MULTIVARIATE MOMENT GENERATING FUNCTION (GMMGF) FOR DIRRICHLET (MULTIVARIATE BETA) DISTRIBUTION

The Dirrichlet distribution, often denoted $Dir(\alpha)$, is a family of continuous multivariate probability distributions parameterized by a vector α of positive reals. It is a multivariate generalization of the beta distribution (kotz *et al*; 2000).

The density function of the Dirichlet distribution is given as

$$f(x_1, \dots, x_k) = \frac{1}{\beta(\alpha)} \prod_{i=1}^{k} x_i^{\alpha_i - 1} \qquad (3.110)$$

where

$$\beta(\alpha) = \frac{\prod_{i=1}^{k} \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)} \; and \; \alpha = (\alpha_1, \dots, \alpha_k)$$

For $k \geq 2$ number of categories (integers), $\alpha_1, \dots, \alpha_k$ concentration parameters, where $\alpha_i > 0$ with support variables: $x_1, \dots, x_k \in (0, 1)$ and $\sum_{i=1}^{k} x_i = 1$.

$$\therefore f(x_i, \dots x_k) = \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \prod_{i=1}^{k} x_i^{\alpha_i - 1} \tag{3.111}$$

Using equation (3.111) in (3.72)

$$E(X_i^{cr}) = \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \int_0^1 x_i^{cr} \prod_{i=1}^{k} x_i^{\alpha_i - 1}\, dx_i$$

$$= \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \int_0^1 \prod_{i=1}^{k} x_i^{\alpha_i + cr - 1}\, dx_i$$

$$\therefore E(X_i^{cr}) = \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i + cr)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i + cr\right)} \tag{3.112}$$

Hence,

$$\boldsymbol{G}_n(c; \boldsymbol{\lambda}) = \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i + cr)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i + cr\right)} \tag{3.113}$$

Now, let $n = 1, c = 1\ and\ r = 0, 1$

$$\boldsymbol{G}_1(1; \lambda) = \lambda^1 + \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i + cr)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i + cr\right)} = \lambda + \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i}$$

Suppose $\lambda = -\mu$

$$\boldsymbol{G}_{\boldsymbol{1}}(1; -\mu) = -\mu + \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i} = 0$$

(First moment about the mean)

Hence,

$$E(X_i) = \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i} \qquad (3.114)$$

Suppose $n = 2; r = 0, 1, 2; c = 1$ and $\lambda = \frac{-\alpha_i}{\sum_{i=1}^{k} \alpha_i}$ from Equation (3.113)

$$\boldsymbol{G}_2(1; \lambda) = \sum_{i=1}^{2} \binom{2}{r} \lambda^{2-r} \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i + r)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i + r\right)}$$

$r = 0;$

$$\binom{2}{0} \lambda^{2-0} \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)} = \lambda^2$$

$r = 1;$

$$\binom{2}{1} \lambda^{2-1} \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i + 1)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i + 1\right)} = 2\lambda \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \frac{\alpha_i \prod_{i=1}^{k} \Gamma(\alpha_i)}{\sum_{i=1}^{k} \alpha_i \, \Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}$$

$$= 2\lambda \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i}$$

$r = 2$

$$\binom{2}{2} \lambda^{0} \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}{\prod_{i=1}^{k} \alpha_i} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i + 2)}{\Gamma\left(\sum_{i=1}^{k} \alpha_i + 2\right)}$$

$$= \frac{\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)(\alpha_i + 1)(\alpha_i) \prod_{i=1}^{k} \Gamma(\alpha_i)}{\prod_{i=1}^{k} \Gamma(\alpha_i) \left(\sum_{i=1}^{k} \alpha_i + 1\right)\left(\sum_{i=1}^{k} \alpha_i\right)\Gamma\left(\sum_{i=1}^{k} \alpha_i\right)}$$

$$= \frac{(\alpha_i + 1)(\alpha_i) \prod_{i=1}^{k} \Gamma(\alpha_i)}{\prod_{i=1}^{k} \Gamma(\alpha_i) \left(\sum_{i=1}^{k} \alpha_i + 1\right)\left(\sum_{i=1}^{k} \alpha_i\right)} = \frac{(\alpha_i + 1)(\alpha_i)}{\left(\sum_{i=1}^{k} \alpha_i + 1\right)\left(\sum_{i=1}^{k} \alpha_i\right)}$$

$$\therefore G_2(1;\ \lambda) = \lambda^2 + 2\lambda \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i} + \frac{(\alpha_i + 1)(\alpha_i)}{\left(\sum_{i=1}^{k} \alpha_i + 1\right)\left(\sum_{i=1}^{k} \alpha_i\right)} \qquad (3.115)$$

Thus the second central moment; that is, $\lambda = \frac{-\alpha_i}{\sum_{i=1}^{k}\alpha_i}$ becomes

$$\boldsymbol{G}_2(1;\lambda) = \frac{\alpha_i^2}{\left(\sum_{i=1}^{k}\alpha_i\right)^2} - 2\frac{\alpha_i^2}{\left(\sum_{i=1}^{k}\alpha_i\right)^2} + \frac{\alpha_i(\alpha_i+1)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)}$$

$$= \frac{\alpha_i(\alpha_i+1)\left(\sum_{i=1}^{k}\alpha_i\right) - \alpha_i^2\left(\sum_{i=1}^{k}\alpha_i+1\right)}{\left(\sum_{i=1}^{k}\alpha_i\right)^2\left(\sum_{i=1}^{k}\alpha_i+1\right)}$$

$$= \frac{\alpha_i\left[\left(\sum_{i=1}^{k}\alpha_i\right)(\alpha_i+1) - \alpha_i\left(\sum_{i=1}^{k}\alpha_i+1\right)\right]}{\left(\sum_{i=1}^{k}\alpha_i\right)^2\left(\sum_{i=1}^{k}\alpha_i+1\right)}$$

$$= \frac{\alpha_i}{\left(\sum_{i=1}^{k}\alpha_i\right)^2}\frac{\left[\alpha_i\sum_{i=1}^{k}\alpha_i + \sum_{i=1}^{k}\alpha_i - \alpha_i\sum_{i=1}^{k}\alpha_i - \alpha_i\right]}{\left(\sum_{i=1}^{k}\alpha_i+1\right)}$$

$$\boldsymbol{G}_2(1;\boldsymbol{\lambda}) = Var(X_i) = \frac{\alpha_i}{\left(\sum_{i=1}^{k}\alpha_i\right)^2}\frac{\left(\sum_{i=1}^{k}\alpha_i - \alpha_i\right)}{\left(\sum_{i=1}^{k}\alpha_i+1\right)}$$

Let $\sum_{i=1}^{k}\alpha_i = \alpha_0$. Thus,

$$Var(X_i) = \frac{\alpha_i}{\alpha_0^2}\frac{(\alpha_0-\alpha_i)}{(\alpha_0+1)} \qquad (3.116)$$

Now, for $n = 3;\ c = 1; r = 0,1,2,3;$ we have

$r = 0$

$$\binom{3}{0}\lambda^{3-0}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)} = \lambda^3$$

$r = 1$

$$\binom{3}{1}\lambda^{3-1}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i+1)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i+1\right)} = 3\lambda^2\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\alpha_i}\frac{\alpha_i\prod_{i=1}^{k}\Gamma(\alpha_i)}{\sum_{i=1}^{k}\alpha_i\,\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$$= 3\lambda^2\frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i}$$

$r = 2$

$$\binom{3}{2}\lambda^{3-2}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i+2)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i+2\right)}$$

$$= 3\lambda\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\alpha_i(\alpha_i+1)\prod_{i=1}^{k}\Gamma(\alpha_i)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$r = 3$

$$\binom{3}{3}\lambda^{3-3}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i+3)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i+3\right)}$$

$$= \frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\alpha_i(\alpha_i+1)(\alpha_i+2)\prod_{i=1}^{k}\Gamma(\alpha_i)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$$= \frac{\alpha_i(\alpha_i+1)(\alpha_i+2)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)}$$

$$\therefore G_3(1;\lambda) = \lambda^3 + 3\lambda^2\frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i} + 3\lambda\frac{\alpha_i(\alpha_i+1)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)} + \frac{\alpha_i(\alpha_i+1)(\alpha_i+2)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)} \qquad (3.117)$$

Now, let $\lambda = \frac{-\alpha_i}{\sum_{i=1}^{k}\alpha_i}$. Hence to obtain the third central moment of the Dirrichlet distribution we have;

$$G_3(1;\lambda) = \frac{-\alpha_i^3}{\left(\sum_{i=1}^{k}\alpha_i\right)^3} + 3\frac{\alpha_i^2}{\left(\sum_{i=1}^{k}\alpha_i\right)^2}\frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i} - 3\frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i}\frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i}\frac{(\alpha_i+1)}{\left(\sum_{i=1}^{k}\alpha_i+1\right)} + \frac{\alpha_i(\alpha_i+1)(\alpha_i+2)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)}$$

$$= \frac{-\alpha_i^3}{\left(\sum_{i=1}^k \alpha_i\right)^3} + 3\frac{\alpha_i^3}{\left(\sum_{i=1}^k \alpha_i\right)^3} - 3\frac{\alpha_i^2}{\left(\sum_{i=1}^k \alpha_i\right)^2}\frac{(\alpha_i + 1)}{\left(\sum_{i=1}^k \alpha_i + 1\right)}$$

$$+ \frac{\alpha_i}{\sum_{i=1}^k \alpha_i}\frac{\left(\alpha_i^2 + 3\alpha_i + 2\right)}{\left(\sum_{i=1}^k \alpha_i + 1\right)\left(\sum_{i=1}^k \alpha_i + 2\right)}$$

$$= \frac{2\alpha_i^3\left(\sum_{i=1}^k \alpha_i + 1\right)\left(\sum_{i=1}^k \alpha_i + 2\right) - 3\alpha_i\left(\alpha_i^3 - \alpha_i^2\right) + \left(\sum_{i=1}^k \alpha_i\right)^2\left(\alpha_i^3 + 3\alpha_i^2 + 2\alpha_i\right)}{\left(\sum_{i==1}^k \alpha_i\right)^3\left(\sum_{i=1}^k \alpha_i + 1\right)\left(\sum_{i=1}^k \alpha_i + 2\right)}$$

$$= \frac{\left(2\alpha_i^3\sum_{i=1}^k \alpha_i + 2\alpha_i^3\right)\left(\sum_{i=1}^k \alpha_i + 2\right) - 3\alpha_i^3\sum_{i=1}^k \alpha_i - 3\alpha_i^2\sum_{i=1}^k \alpha_i + \left(\sum_{i=1}^k \alpha_i\right)^2(\alpha_i^3 + 3\alpha_i^2 + 2\alpha_i)}{\left(\sum_{i=1}^k \alpha_i\right)^3\left(\sum_{i=1}^k \alpha_i + 1\right)\left(\sum_{i=1}^k \alpha_i + 2\right)}$$

$$= \frac{3\alpha_i^3\sum_{i=1}^k \alpha_i\left(\sum_{i=1}^k \alpha_i + 1\right) + \alpha_i^3\left[\left(\sum_{i=1}^k \alpha_i\right)^2 + 4\right] - 3\alpha_i^2\sum_{i=1}^k \alpha_i\left(1 - \sum_{i=1}^k \alpha_i\right) + 2\alpha_i\left(\sum_{i=1}^k \alpha_i\right)^2}{\left(\sum_{i=1}^k \alpha_i\right)^3\left(\sum_{i=1}^k \alpha_i + 1\right)\left(\sum_{i=1}^k \alpha_i + 2\right)}$$

$$\therefore G_3\left(1; \frac{-\alpha_i}{\sum_{i=1}^k \alpha_i}\right)$$

$$= \frac{3\alpha_i^3\alpha_0(\alpha_0 + 1) + \alpha_i^3(\alpha_0 + 4) - 3\alpha_i^2\alpha_0(1 - \alpha_0) + 2\alpha_i\alpha_0^2}{\alpha_0^3(\alpha_0 + 1)(\alpha_0 + 2)} \qquad (3.118)$$

Hence skewness can be determined as

$$Sk(1) = \frac{3\alpha_i^3\alpha_0(\alpha_0 + 1) + \alpha_i^3(\alpha_0 + 4) - 3\alpha_i^2\alpha_0(1 - \alpha_0) + 2\alpha_i\alpha_0^2}{\alpha_0^3(\alpha_0 + 1)(\alpha_0 + 2)} * \frac{\alpha_0^3(\alpha_0 + 1)^{\frac{3}{2}}}{\alpha_i^3(\alpha_0 - \alpha_i)^{\frac{3}{2}}}$$

$$\therefore Sk(1)$$

$$= \frac{3\alpha_i^3\alpha_0(\alpha_0 + 1) + \alpha_i^3(\alpha_0 + 4) - 3\alpha_i^2\alpha_0(1 - \alpha_i) + 2\alpha_i\alpha_0^2}{\left(\alpha_i^3\alpha_0^2 + 3\alpha_i^3\alpha_0 + 2\alpha_i^3\right)}\frac{(\alpha_0 + 1)^{\frac{3}{2}}}{(\alpha_0 - \alpha_i)^{\frac{3}{2}}} \qquad (3.119)$$

Hence the Dirrichlet distribution is positively skewed since $\alpha_i > 0;\ \forall\ i$.

Now, let $n = 4; c = 1; r = 0, 1, 2, 3, 4$

$r = 0$

$$\binom{4}{0}\lambda^{4-0}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}=\lambda^4$$

$r=1$

$$\binom{4}{1}\lambda^{4-1}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i+1)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i+1\right)}=4\lambda^3\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\alpha_i}{\left(\sum_{i=1}^{k}\alpha_i\right)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

Hence, for $r=1$ we have

$$\binom{4}{1}\lambda^{4-1}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}.\frac{\prod_{i=1}^{k}\Gamma(\alpha_i+1)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i+1\right)}=4\lambda^3\frac{\alpha_i}{\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$r=2$

$$\binom{4}{2}=\lambda^{4-2}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}.\frac{\prod_{i=1}^{k}\Gamma(\alpha_i+2)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i+2\right)}$$

$$=6\lambda^2\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma\,(\alpha_i)}\frac{\alpha_i(\alpha_i+1)\Gamma(\alpha_i)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$$=6\lambda^2\frac{\alpha_i}{\left(\sum_{i=1}^{k}\alpha_i\right)}\frac{(\alpha_i+1)}{\left(\sum_{i=1}^{k}\alpha_i+1\right)}$$

$r=3$

$$\binom{4}{3}\lambda^{4-3}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i+3)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i+3\right)}$$

$$=4\lambda\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma\,(\alpha_i)}\frac{\alpha_i(\alpha_i+1)(\alpha_i+2)\prod_{i=1}^{k}\Gamma(\alpha_i)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$$=4\lambda\frac{\alpha_i}{\left(\sum_{i=1}^{k}\alpha_i\right)}\frac{(\alpha_i+1)(\alpha_i+2)}{\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)}$$

$r=4$

$$\binom{4}{4}\lambda^{4-4}\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma(\alpha_i)}\frac{\prod_{i=1}^{k}\Gamma(\alpha_i+4)}{\Gamma\left(\sum_{i=1}^{k}\alpha_i+4\right)}$$

$$=\frac{\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}{\prod_{i=1}^{k}\Gamma\left(\alpha_i\right)}\frac{\alpha_i(\alpha_i+1)(\alpha_i+2)(\alpha_i+3)\prod_{i=1}^{k}\Gamma(\alpha_i)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)\left(\sum_{i=1}^{k}\alpha_i+3\right)\Gamma\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$$=\frac{\alpha_i(\alpha_i+1)(\alpha_i+2)(\alpha_i+3)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)\left(\sum_{i=1}^{k}\alpha_i+3\right)}$$

Hence,

$$G_4(1;\lambda)=\lambda^4+4\lambda^3\frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i}+6\lambda^2\frac{\alpha_i(\alpha_i+1)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)}$$

$$+4\lambda\frac{\alpha_i(\alpha_i+1)(\alpha_i+2)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)}$$

$$+\frac{\alpha_i(\alpha_i+1)(\alpha_i+2)(\alpha_i+3)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)\left(\sum_{i=1}^{k}\alpha_i+3\right)} \qquad (3.120)$$

Now, let $\lambda=\frac{-\alpha_i}{\sum_{i=1}^{k}\alpha_i}$. Hence the fourth central moment becomes

$$G_4(1;\lambda)=\frac{\alpha_i^4}{\left(\sum_{i=1}^{k}\alpha_i\right)^4}-4\frac{\alpha_i^3}{\left(\sum_{i=1}^{k}\alpha_i\right)^3}\frac{\alpha_i}{\left(\sum_{i=1}^{k}\alpha_i\right)}$$

$$+6\frac{\alpha_i^2}{\left(\sum_{i=1}^{k}\alpha_i\right)^2}\frac{\alpha_i(\alpha_i+1)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)}$$

$$-4\frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i}\frac{\alpha_i(\alpha_i+1)(\alpha_i+2)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)}$$

$$+\frac{\alpha_i(\alpha_i+1)(\alpha_i+2)(\alpha_i+3)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)\left(\sum_{i=1}^{k}\alpha_i+3\right)}$$

$$= 3\frac{\alpha_i^4}{\left(\sum_{i=1}^{k}\alpha_i\right)^4} + 6\frac{\alpha_i^3}{\left(\sum_{i=1}^{k}\alpha_i\right)^3}\frac{(\alpha_i+1)}{\left(\sum_{i=1}^{k}\alpha_i+1\right)}$$

$$- 4\frac{\alpha_i^2}{\left(\sum_{i=1}^{k}\alpha_i\right)^2}\frac{(\alpha_i+1)(\alpha_i+2)}{\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)}$$

$$+ \frac{\alpha_i(\alpha_i+1)(\alpha_i+2)(\alpha_i+3)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)\left(\sum_{i=1}^{k}\alpha_i+3\right)}$$

$$= \frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i}\left\{\frac{6\alpha_i^3\sum_{i=1}^{k}\alpha_i + 6\alpha_i^2\sum_{i=1}^{k}\alpha_i - 3\alpha_i^3\sum_{i=1}^{k}\alpha_i - 3\alpha_i^3}{\left(\sum_{i=1}^{k}\alpha_i\right)^3\left(\sum_{i=1}^{k}\alpha_i+1\right)}\right.$$

$$\left. + \frac{\left(\sum_{i=1}^{k}\alpha_i\right)\left(6-3\alpha_i^3-6\alpha_i^2-21\alpha_i\right)-12\alpha_i^2(\alpha_i-4)}{\left(\sum_{i=1}^{k}\alpha_i\right)\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)\left(\sum_{i=1}^{k}\alpha_i+3\right)}\right\}$$

$\therefore G_4(1;\lambda)$

$$= \frac{\alpha_i}{\sum_{i=1}^{k}\alpha_i}\left\{\frac{\left(\sum_{i=1}^{k}\alpha_i+2\right)\left(\sum_{i=1}^{k}\alpha_i+3\right)\left(3\alpha_i^3\sum_{i=1}^{k}\alpha_i + 6\alpha_i^2\sum_{i=1}^{k}\alpha_i - 3\alpha_i^3\right)}{\left(\sum_{i=1}^{k}\alpha_i\right)^3\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)\left(\sum_{i=1}^{k}\alpha_i+3\right)}\right.$$

$$\left. + \frac{(\sum_{i=1}^{k}\alpha_i)^2[\sum_{i=1}^{k}\alpha_i\,(6-3\alpha_i^3-6\alpha_i^2-21\alpha_i)]-12\alpha_i^2(\alpha_i+4)}{\left(\sum_{i=1}^{k}\alpha_i\right)^3\left(\sum_{i=1}^{k}\alpha_i+1\right)\left(\sum_{i=1}^{k}\alpha_i+2\right)\left(\sum_{i=1}^{k}\alpha_i+3\right)}\right\}$$

$\therefore G_4(1;\lambda)$

$$= \frac{\alpha_i}{\alpha_0}\left\{\frac{(\alpha_0+2)(\alpha_0+3)\left(3\alpha_i^3\alpha_0 + 6\alpha_i^2\alpha_0 - 3\alpha_i^3\right)}{\alpha_0^3(\alpha_0+1)(\alpha_0+2)(\alpha_0+3)}\right.$$

$$\left. + \frac{\alpha_0^2[\alpha_0(6-3\alpha_i^3-6\alpha_i^2-21\alpha_i)]-12\alpha_i^2(\alpha_i+4)}{\alpha_0^3(\alpha_0+1)(\alpha_0+2)(\alpha_0+3)}\right\} \qquad (3.121)$$

Hence the kurtosis of the distribution may be obtained as

$$ku(1) = \left\{\frac{(\alpha_0+2)(\alpha_0+3)\left(3\alpha_i^3\alpha_0+6\alpha_i^2\alpha_0-3\alpha_i^3\right)}{\alpha_0^3(\alpha_0+1)(\alpha_0+2)(\alpha_0+3)} + \frac{\alpha_0^2[\alpha_0(6-3\alpha_i^3-6\alpha_i^2-21\alpha_i)]-12\alpha_i^2(\alpha_i+4)}{(\alpha_0+1)(\alpha_0+2)(\alpha_0+3)}\right\} \times \frac{(\alpha_0+1)^2}{\alpha_i(\alpha_0-\alpha_0)^2}$$

$$\therefore ku(1) = \frac{6\alpha_i^2\alpha_0^3+3\alpha_i^3\alpha_0^2+18\alpha_i^3\alpha_0+39\alpha_i^2\alpha_0+24\alpha_0^3+15\alpha_0^4-66\alpha_i^2\alpha_0^2+3\alpha_0^5}{(\alpha_0^3+6\alpha_0^2+11\alpha_0+6)} \times \frac{(\alpha_0+1)^2}{\alpha_i(\alpha_0-\alpha_i)^2} \qquad (3.122)$$

The value of $ku(1)$ is positive and may be less than, equal to or greater than 3 depending on the values of $\alpha_i \forall\ i = 1,2,\dots,k$ where $\alpha_0 = \sum_{i=1}^{k}\alpha_i$.

To illustrate the practical application and consistency of the functions developed and presented in this chapter, they shall be used to analyze numerical data in the next chapter.

CHAPTER FOUR

PRESENTATION AND ANALYSIS OF DATA

4.1 PRESENTATION OF DATA

The Data for illustration of the application and consistency of the developed methods are presented as follows:

Table 4.1: P = 3 - Variate Multivariate Gamma Distribution with Shape Parameter = 2 and Scale Parameter = 3

Serial No	x_1	x_2	x_3
1	6.3848	13.4597	3.9665
2	15.8473	1.7828	9.373
3	1.5872	5.4326	2.1957
4	0.6203	4.4363	0.8513
5	2.632	10.9161	2.9904
6	7.1946	6.8832	7.0914
7	5.3607	15.3467	3.4509
8	3.4233	3.2901	3.225
9	6.775	10.6918	0.3861
10	3.727	1.6152	2.657
11	13.3058	5.0971	4.8853
12	7.9797	3.7563	6.8347
13	8.5766	8.8345	8.2838
14	6.5292	6.911	2.5925
15	10.461	0.3759	3.51
16	4.7776	2.1998	8.0186
17	8.8577	4.453	3.42
18	12.3543	3.0492	4.0527
19	12.0012	2.9735	17.6135
20	9.2999	6.9735	17.3728
21	11.3636	6.6637	9.1995
22	2.3759	8.7596	3.5722
23	5.0573	8.2106	6.324
24	4.3035	6.6792	2.6099
25	2.604	9.0652	3.8959
26	6.777	3.2669	3.8998
27	3.7905	22.3057	10.7298
28	4.1491	6.0776	5.2878
29	2.6876	3.7828	6.7316
30	1.7262	1.1808	9.5079
31	4.8571	8.5815	2.2269
32	2.5318	4.9717	5.3598
33	14.2059	2.4589	2.4659
34	4.7154	12.4749	10.3825
35	5.6329	10.7214	2.4159

TABLE 4.2: SCORES OF FIFTY-THREE DIPLOMA LAW STUDENTS OF THE DELTA STATE UNIVERSITY, OLEH CAMPUS IN THREE COURSES

Serial No	x_1	x_2	x_3		Serial No	x_1	x_2	x_3
1	60	63	71		**34**	82	72	80
2	60	52	62		**35**	80	82	74
3	55	62	40		**36**	92	80	96
4	44	41	63		**37**	30	25	20
5	46	62	44		**38**	37	92	94
6	45	64	56		**39**	38	90	20
7	63	61	63		**40**	94	95	93
8	54	50	40		**41**	10	20	26
9	52	53	52		**42**	50	64	70
10	56	55	46		**43**	72	60	54
11	75	64	74		**44**	70	62	56
12	54	52	51		**45**	75	72	52
13	72	55	42		**46**	55	50	54
14	44	43	40		**47**	60	53	71
15	43	54	51		**48**	60	52	52
16	42	62	42		**49**	55	62	40
17	53	52	50		**50**	44	41	63
18	55	52	50		**51**	56	62	44
19	50	52	64		**52**	45	64	56
20	60	55	65		**53**	63	61	63
21	55	50	54					
22	75	72	52					
23	70	62	56					
24	72	60	54					
25	50	64	60					
26	80	82	74					
27	82	72	80					
28	74	80	82					
29	62	50	72					
30	50	74	65					
31	50	74	65					
32	62	50	72					
33	74	80	82					

TABLE 4.3: PROPORTIONAL SCORES OF STUDENTS OF THE DELTA STATE UNIVERSITY, OLEH CAMPUS IN THREE COURSES

Serial No	x_1	x_2	x_3		Serial No	x_1	x_2	x_3
1	0.309278	0.324742	0.365979		**34**	0.350427	0.307692	0.34188
2	0.344828	0.298851	0.356322		**35**	0.338983	0.347458	0.313559
3	0.350318	0.394904	0.254777		**36**	0.343284	0.298507	0.358209
4	0.297297	0.277027	0.425676		**37**	0.4	0.333333	0.266667
5	0.302632	0.407895	0.289474		**38**	0.165919	0.412556	0.421525
6	0.272727	0.387879	0.339394		**39**	0.256757	0.608108	0.135135
7	0.336898	0.326203	0.336898		**40**	0.333333	0.336879	0.329787
8	0.375	0.347222	0.277778		**41**	0.178571	0.357143	0.464286
9	0.33121	0.33758	0.33121		**42**	0.271739	0.347826	0.380435
10	0.356688	0.350318	0.292994		**43**	0.387097	0.322581	0.290323
11	0.352113	0.300469	0.347418		**44**	0.37234	0.329787	0.297872
12	0.343949	0.33121	0.324841		**45**	0.376884	0.361809	0.261307
13	0.426036	0.325444	0.248521		**46**	0.345912	0.314465	0.339623
14	0.346457	0.338583	0.314961		**47**	0.326087	0.288043	0.38587
15	0.290541	0.364865	0.344595		**48**	0.365854	0.317073	0.317073
16	0.287671	0.424658	0.287671		**49**	0.350318	0.394904	0.254777
17	0.341935	0.335484	0.322581		**50**	0.297297	0.277027	0.425676
18	0.350318	0.33121	0.318471		**51**	0.345679	0.382716	0.271605
19	0.301205	0.313253	0.385542		**52**	0.272727	0.387879	0.339394
20	0.333333	0.305556	0.361111		**53**	0.336898	0.326203	0.336898
21	0.345912	0.314465	0.339623					
22	0.376884	0.361809	0.261307					
23	0.37234	0.329787	0.297872					
24	0.387097	0.322581	0.290323					
25	0.287356	0.367816	0.344828					
26	0.338983	0.347458	0.313559					
27	0.350427	0.307692	0.34188					
28	0.313559	0.338983	0.347458					
29	0.336957	0.271739	0.391304					
30	0.350427	0.391534	0.343915					
31	0.26455	0.391534	0.343915					
32	0.336957	0.271739	0.391304					
33	0.313559	0.338983	0.347458					

The data above follows the Dirrichlet Distribution and may be assigned the parameters, $\alpha_i = 2 \; \forall \; i = 1, 2, 3$; $\sum_{i=1}^{3} \alpha_i = 6$ and $\alpha_i = 1.2, 1.3, 1.4; i = 1, 2, 3$, respectively; $\sum_{i=1}^{3} \alpha_i = 3.9$.

4.2 DATA ANALYSIS

4.2.1 The Multivariate Gamma Family of Distributions

The shape parameter, α, scale parameter, β, and scale matrix, $\mathbf{\Sigma}$, of the data in Table 4.1 for the Multivariate Gamma distribution are respectively

$$\alpha = 2, \beta = 3 \ and \ \mathbf{\Sigma} = \begin{pmatrix} 15.4116 & -4.9758 & 4.7261 \\ -4.9758 & 20.9363 & 0.2525 \\ 4.7261 & 0.2525 & 16.3018 \end{pmatrix}$$

Using Equation 3.79a,

$$\boldsymbol{G}_1(1;\ \boldsymbol{\lambda}) = \boldsymbol{\lambda} + \alpha\beta^p\mathbf{\Sigma}$$

$$\therefore \boldsymbol{G}_1(1;\ \boldsymbol{\lambda}) = \boldsymbol{\lambda} + 2 * 3^p \begin{pmatrix} 15.4116 & -4.9758 & 4.7261 \\ -4.9758 & 20.9363 & 0.2525 \\ 4.7261 & 0.2525 & 16.3018 \end{pmatrix} \qquad (4.1)$$

Now, using Equation 3.79c where $p = 3$ yields

$$\boldsymbol{\lambda} = 2 * 3 \begin{pmatrix} 15.4116 & -4.9758 & 4.7261 \\ -4.9758 & 20.9363 & 0.2525 \\ 4.7261 & 0.2525 & 16.3018 \end{pmatrix} * (-3^2) \qquad (4.2)$$

Thus, the mean of the distribution is the coefficient of 3^2 therefore,

$$\boldsymbol{\mu} = 6 \begin{pmatrix} 15.4116 & -4.9758 & 4.7261 \\ -4.9758 & 20.9363 & 0.2525 \\ 4.7261 & 0.2525 & 16.3018 \end{pmatrix} \qquad (4.3)$$

In the same way, using Equation 3.79d for the Wishart distribution, that is, the distribution of $\mathbf{\Sigma}$

$$\boldsymbol{\lambda} = 2^2 * \eta * \mathbf{\Sigma} = 2^2 * 53 * \begin{pmatrix} 15.4116 & -4.9758 & 4.7261 \\ -4.9758 & 20.9363 & 0.2525 \\ 4.7261 & 0.2525 & 16.3018 \end{pmatrix} \qquad (4.4)$$

Hence, the mean of the distribution is the coefficient of 2^2 which is

$$\boldsymbol{\mu} = 53 * \begin{pmatrix} 15.4116 & -4.9758 & 4.7261 \\ -4.9758 & 20.9363 & 0.2525 \\ 4.7261 & 0.2525 & 16.3018 \end{pmatrix} \quad (4.5)$$

This shows that we have a $p = 3$ –variate Wishart distribution with mean given in Equation 4.5.

4.2.2 The Normal Distribution

The estimates of the mean vector, $\overline{\boldsymbol{X}}$, and the covariance matrix, $\boldsymbol{S}$, in Table 4.2 are

$$\overline{\boldsymbol{X}} = \begin{pmatrix} 58.62 \\ 61.00 \\ 58.72 \end{pmatrix} \; \boldsymbol{\Sigma} = \begin{pmatrix} 247.201 & 130.019 & 161.026 \\ 130.019 & 216.923 & 137.385 \\ 161.026 & 137.385 & 282.861 \end{pmatrix}$$

Now, we represent $\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$

Applying Equation 3.92,

$$\boldsymbol{G}_1(1;\ \boldsymbol{\lambda}) = \boldsymbol{\lambda} + \boldsymbol{\mu}$$

where $\lambda = -\mu = -\overline{\boldsymbol{X}} = -\begin{pmatrix} 58.62 \\ 61.00 \\ 58.72 \end{pmatrix}$

$$\therefore \boldsymbol{G_1}(1;\ \boldsymbol{\lambda}) = \begin{pmatrix} 58.62 \\ 61.00 \\ 58.72 \end{pmatrix} - \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$$

$$\boldsymbol{G}_1(1;\ -\boldsymbol{\mu}) = \begin{pmatrix} 58.62 \\ 61.00 \\ 58.72 \end{pmatrix} - \begin{pmatrix} 58.62 \\ 61.00 \\ 58.72 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \boldsymbol{G_1}(1;\ -\boldsymbol{\mu}) = 0 \quad (4.6)$$

As expected, the first moment of the $p = 3$-variate normal distribution about its mean equals zero.

Now, evaluating Equation 3.91 at $n = 2$ and $c = 1$ yields Equation 3.93. That is,

$$\boldsymbol{G}_2(1;\ \boldsymbol{\lambda}) = \boldsymbol{\lambda\lambda}' + \mathbf{2}\boldsymbol{\lambda\mu}' + \boldsymbol{\lambda\lambda}' + \boldsymbol{\Sigma}$$

where $\boldsymbol{\lambda} = -\boldsymbol{\mu}$ we have

$$\boldsymbol{G_2}(1;\ -\boldsymbol{\mu}) = (-\boldsymbol{\mu})(-\boldsymbol{\mu})' + 2(-\boldsymbol{\mu})\boldsymbol{\mu}' + (-\boldsymbol{\mu})(-\boldsymbol{\mu})' + \boldsymbol{\Sigma}$$

where $(-\boldsymbol{\mu})(-\boldsymbol{\mu})' = \begin{pmatrix} -58.62 \\ -61.00 \\ -58.72 \end{pmatrix} (-58.62 \quad -61.00 \quad -58.72)$

$$= \begin{pmatrix} 3438.3044 & 3575.82 & 3442.1664 \\ 3575.82 & 3721 & 3581.92 \\ 3442.1664 & 3581.92 & 3448.0384 \end{pmatrix}$$

$$\therefore \boldsymbol{G_2}(1;\ -\boldsymbol{\mu}) = 2\boldsymbol{\mu\mu}' - 2\boldsymbol{\mu\mu}' + \boldsymbol{\Sigma}$$

$$= 2\begin{pmatrix} 3438.3044 & 3575.82 & 3442.1664 \\ 3575.82 & 3721 & 3581.92 \\ 3442.1664 & 3581.92 & 3448.0384 \end{pmatrix}$$
$$- 2\begin{pmatrix} 3438.3044 & 3575.82 & 3442.1664 \\ 3575.82 & 3721 & 3581.92 \\ 3442.1664 & 3581.92 & 3448.0384 \end{pmatrix}$$
$$+ \begin{pmatrix} 247.201 & 130.019 & 161.026 \\ 130.019 & 216.923 & 137.385 \\ 161.026 & 137.385 & 282.861 \end{pmatrix}$$

Hence,

$$\boldsymbol{G}_2(1;\ -\boldsymbol{\mu}) = \begin{pmatrix} 247.201 & 130.019 & 161.026 \\ 130.019 & 216.923 & 137.385 \\ 161.026 & 137.385 & 282.861 \end{pmatrix} \qquad (4.7)$$

Therefore the second moment of the $p = 3$ −variate normal distribution is its covariance matrix as expected.

Also, evaluating Equation 3.91 at $n = 3$ and $c = 1$ yields 3.95. That is,

$$\boldsymbol{G}_3(1;\ \boldsymbol{\lambda}) = (\boldsymbol{\lambda\lambda}')\boldsymbol{\lambda} + 3(\boldsymbol{\lambda\mu}')\boldsymbol{\lambda} + 3(\boldsymbol{\mu\mu}')\boldsymbol{\lambda} + 3\boldsymbol{\Sigma\lambda} + (\boldsymbol{\mu\mu}')\boldsymbol{\mu} + 3\boldsymbol{\Sigma\lambda}$$

At $\boldsymbol{\lambda} = -\boldsymbol{\mu}$**,** Equation 3.95 becomes

$$\boldsymbol{G}_3(1;\ -\boldsymbol{\mu}) = [(-\boldsymbol{\mu})(-\boldsymbol{\mu}')](-\boldsymbol{\mu}) + 3[(-\boldsymbol{\mu})\boldsymbol{\mu}'](-\boldsymbol{\mu}) + 3(\boldsymbol{\mu}\boldsymbol{\mu}')(-\boldsymbol{\mu}) + 3\boldsymbol{\Sigma}(-\boldsymbol{\mu}) + (\boldsymbol{\mu}\boldsymbol{\mu}')\boldsymbol{\mu} + 3\boldsymbol{\Sigma}\boldsymbol{\mu}$$

$$= 4(\boldsymbol{\mu}\boldsymbol{\mu}')\boldsymbol{\mu} - 4(\boldsymbol{\mu}\boldsymbol{\mu}')\boldsymbol{\mu} + 3\boldsymbol{\Sigma}\boldsymbol{\mu} - 3\boldsymbol{\Sigma}\boldsymbol{\mu}$$

$$\therefore \boldsymbol{G}_3(1;\ -\boldsymbol{\mu})$$

$$= 4\begin{pmatrix} 3438.3044 & 3575.82 & 3442.1664 \\ 3575.82 & 3721 & 3581.92 \\ 3442.1664 & 3581.92 & 3448.0384 \end{pmatrix}\begin{pmatrix} 58.62 \\ 61.00 \\ 58.72 \end{pmatrix} - 4\begin{pmatrix} 3438.3044 & 3575.82 & 3442.1664 \\ 3575.82 & 3721 & 3581.92 \\ 3442.1664 & 3581.92 & 3448.0384 \end{pmatrix}\begin{pmatrix} 58.62 \\ 61.00 \\ 58.72 \end{pmatrix} + 3\begin{pmatrix} 247.201 & 130.019 & 161.026 \\ 130.019 & 216.923 & 137.385 \\ 161.026 & 137.385 & 282.861 \end{pmatrix}\begin{pmatrix} 58.62 \\ 61.00 \\ 58.72 \end{pmatrix} - 3\begin{pmatrix} 247.201 & 130.019 & 161.026 \\ 130.019 & 216.923 & 137.385 \\ 161.026 & 137.385 & 282.861 \end{pmatrix}\begin{pmatrix} 58.62 \\ 61.00 \\ 58.72 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (4.8)$$

Also, evaluating Equation 3.91 at $n = 4$ and $c = 1$ gives Equation 3.97. That is,

$$\boldsymbol{G}_4(1;\ \boldsymbol{\lambda}) = (\boldsymbol{\lambda}\boldsymbol{\lambda}')(\boldsymbol{\lambda}\boldsymbol{\lambda}') + 4(\boldsymbol{\lambda}\boldsymbol{\lambda}')(\boldsymbol{\lambda}\boldsymbol{\mu}') + 6(\boldsymbol{\lambda}\boldsymbol{\lambda}')(\boldsymbol{\mu}\boldsymbol{\mu}') + 6(\boldsymbol{\lambda}\boldsymbol{\lambda}')\boldsymbol{\Sigma} + 4(\boldsymbol{\lambda}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}') + 12(\boldsymbol{\lambda}\boldsymbol{\mu}')\boldsymbol{\Sigma} + (\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}') + 6(\boldsymbol{\mu}\boldsymbol{\mu}')\boldsymbol{\Sigma} + 3^{\boldsymbol{p}}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}$$

Let $\boldsymbol{\lambda} = -\boldsymbol{\mu}$ then Equation 3.97 becomes

$$G_4(1;\ -\mu) = (\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}') - 4(\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}') + 6(\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}') + 6(\boldsymbol{\mu}\boldsymbol{\mu}')\boldsymbol{\Sigma} - 4(\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}') - 12(\boldsymbol{\mu}\boldsymbol{\mu}')\boldsymbol{\Sigma} + (\boldsymbol{\mu}\boldsymbol{\mu}')(\boldsymbol{\mu}\boldsymbol{\mu}') + 6(\boldsymbol{\mu}\boldsymbol{\mu}')\boldsymbol{\Sigma} + 3^{p}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}$$

$$= 12\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma} - 12\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma} + 3^{p}\boldsymbol{\Sigma}'\boldsymbol{\Sigma}$$

$G_4(1;\ -\mu)$

$$= 12\begin{pmatrix} 3438.3044 & 3575.82 & 3442.1664 \\ 3575.82 & 3721 & 3581.92 \\ 3442.1664 & 3581.92 & 3448.0384 \end{pmatrix}$$

$$- 12\begin{pmatrix} 3438.3044 & 3575.82 & 3442.1664 \\ 3575.82 & 3721 & 3581.92 \\ 3442.1664 & 3581.92 & 3448.0384 \end{pmatrix}$$

$$+ 3^p\begin{pmatrix} 247.201 & 130.019 & 161.026 \\ 130.019 & 216.923 & 137.385 \\ 161.026 & 137.385 & 282.861 \end{pmatrix}\begin{pmatrix} 247.201 & 130.019 & 161.026 \\ 130.019 & 216.923 & 137.385 \\ 161.026 & 137.385 & 282.861 \end{pmatrix}$$

$$G_4(1;\ -\mu) = 3^p\begin{pmatrix} 103943 & 82467.5 & 103216 \\ 82467.5 & 82835.2 & 89599.0 \\ 103216 & 89599.0 & 124814 \end{pmatrix} \qquad (4.9)$$

Now using Equation 3.99, the skewness of the distribution can be obtained as

$$\boldsymbol{Sk}(1) = \frac{\boldsymbol{G_3}(1;\ -\boldsymbol{\mu})}{|\boldsymbol{G}_3(1;\ -\boldsymbol{\mu})|^{\frac{3}{2}}} = \frac{1}{1.414 \times 10^{10}}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \boldsymbol{Sk}(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \big((0)\big)_{3\times 1} \qquad (4.10)$$

The Multivariate Normal Distribution under consideration is thus a $p = 3$ −variate symmetric distribution.

Applying Equations 4.4 and 4.2 in Equation 3.100 yields

$$\boldsymbol{Ku}(1) = \frac{\boldsymbol{G}_4(1; -\boldsymbol{\mu})}{[\boldsymbol{G}_2(1; -\mu)]^{\mathbf{2}}} = \frac{\boldsymbol{3^p}\begin{pmatrix} 103943 & 82467.5 & 103216 \\ 82467.5 & 82835.2 & 89599.0 \\ 103216 & 89599.0 & 124814 \end{pmatrix}}{\begin{pmatrix} 103943 & 82467.5 & 103216 \\ 82467.5 & 82835.2 & 89599.0 \\ 103216 & 89599.0 & 124814 \end{pmatrix}}$$

$$\therefore \boldsymbol{Ku}(1) = 3^3 \qquad (4.11)$$

This implies that the distribution under consideration is a $p = 3$ −variate Mesokurtic distribution.

4.2.3 The Dirrichlet Distribution (Multivariate Extension of the Beta Family of Distributions)

The data in Table 4.3 represent a symmetric Dirrichlet Distribution with parameters $\alpha_i = 2, \forall\, i = 1, 2, 3;\ \alpha_0 = 6$

The mean of the i^{th} random variable is obtained from Equation 3.114 as

$$E(X_i) = \frac{\alpha_i}{\sum_{i=1}^{3} \alpha_i}$$

$$= \frac{2}{6}$$

$$\therefore E(X_i) = 0.3333 \qquad (4.12)$$

Hence the mean vector becomes

$$E(\boldsymbol{X}) = \overline{\boldsymbol{X}} = \begin{pmatrix} 0.3333 \\ 0.3333 \\ 0.3333 \end{pmatrix}$$

The Skewness of the distribution may be obtained with Equation 3.119 as

$$sk_i(1) = \frac{3\alpha_i^3\alpha_0(\alpha_0+1) + \alpha_i^3(\alpha_0+4) - 3\alpha_i^2\alpha_0(1-\alpha_i) + 2\alpha_i\alpha_0^2}{(\alpha_i^3\alpha_0^2 + 3\alpha_i^3\alpha_0 + 2\alpha_i^3)} \cdot \frac{(\alpha_0+1)^{\frac{3}{2}}}{(\alpha_0-\alpha_i)^{\frac{3}{2}}}$$

$$= \frac{3(2^3)6(6+1) + 2^3(6+4) - 3(2^2)6(1-2) + 2(2)6^2}{\left(2^3 * 6^2 + 3(2^3)6 + 2(2^3)\right)} * \frac{(6+1)^{\frac{3}{2}}}{(6-2)^{\frac{3}{2}}}$$

$$= \frac{3*8*6*7 + 8*10 + 3*4*6 + 4*36}{(8*32 + 3*8*6 + 2*8)} * \frac{7^{\frac{3}{2}}}{4^{\frac{3}{2}}}$$

$$= \frac{1008 + 80 + 72 + 144}{256 + 144 + 16} \times \frac{18.5208}{8} = \frac{1304}{416} \times \frac{18.5208}{8}$$

$$= \frac{163 \times 18.5208}{416} = \frac{3018.89}{416}$$

$$\therefore sk_i = 7.2569 \qquad (4.13)$$

Thus the skewness vector becomes

$$\boldsymbol{sk}(1) = \begin{pmatrix} 7.2569 \\ 7.2569 \\ 7.2569 \end{pmatrix}$$

Hence the symmetric Dirrichlet Distribution in Table 4.3 is $p = 3$ positively skewed distribution.

Kurtosis of the Dirrichlet Distribution in Table 4.3 may be obtained with Equation 3.122 as

$$ku_i(1) = \frac{6\alpha_i^2\alpha_0^3 + 3\alpha_i^3\alpha_0^2 + 18\alpha_i^3\alpha_0 + 39\alpha_i^2\alpha_0 + 24\alpha_0^3 + 15\alpha_0^4 - 66\alpha_i^2\alpha_0^2 + 3\alpha_0^5}{(\alpha_0^3 + 6\alpha_0^2 + 11\alpha_0 + 6)} \times \frac{(\alpha_0 + 1)^2}{\alpha_i(\alpha_0 - \alpha_i)^2}$$

$$= \frac{6.4.6^3 + 3.8.6^2 + 18.8.6 + 39.4.6 + 24.8 + 15.16 - 66.4.6^2 + 3.6^5}{(6^3 + 6.6^2 + 11.6 + 6)} \times \frac{(6+1)^2}{2(6-2)^2}$$

$$= \frac{864 + 144 + 144 + 156 + 32 + 40 - 1584 + 3888}{84} \times \frac{49}{32}$$

$$= \frac{3684}{84} \times \frac{49}{32} = \frac{180516}{2688}$$

$$\therefore ku_i(1) = 67.15625 \qquad (4.14)$$

Thus the kurtosis vector becomes

$$\boldsymbol{ku}(1) = \begin{pmatrix} 67.15625 \\ 67.15625 \\ 67.15625 \end{pmatrix}$$

Hence the $\alpha_i = 2, \forall\ i = 1, 2, 3$ symmetric Dirrichlet Distribution in Table 4.3 is a $p = 3$ platykurtic distribution.

The data on Table 4.4 represents $p = 3$-variate Dirrichlet Distribution with parameters, $\alpha_i = 1.2, 1.3, 1.4$ for $i = 1, 2, 3$ respectively and $\alpha_0 = 3.9$.

Applying Equation 3.114, the expected mean vector may be obtained as

$$E(X) = \bar{X} = \left(\left(\frac{\alpha_i}{\sum_{i=1}^{3} \alpha_i}\right)\right)$$

$$\therefore \overline{\boldsymbol{X}} = \begin{pmatrix} 1.2/3.9 \\ 1.3/3.9 \\ 1.4/3.9 \end{pmatrix} = \begin{pmatrix} 0.3077 \\ 0.3333 \\ 0.3590 \end{pmatrix}$$

Skewness of the non-symmetric distribution may be obtained using Equation 3.119 as

$$sk_i(1) = \frac{3\alpha_i^3\alpha_0(\alpha_0 + 1) + \alpha_i^3(\alpha_0 + 4) - 3\alpha_i^3\alpha_0(1 - \alpha_i) + 2\alpha_i\alpha_0^2}{\left(\alpha_i^3\alpha_0^2 + 3\alpha_i^3\alpha_0 + 2\alpha_i^3\right)} \times \frac{(\alpha_0 + 1)^{\frac{3}{2}}}{(\alpha_0 - \alpha_i)^{\frac{3}{2}}}$$

$$\therefore sk_1(1) = \frac{3(1.2^3)3.9(3.9 + 1) + 1.2^3(3.9 + 4) - 3(1.2^2)3.9(1 - 1.2) + 2(1.2)3.9^2}{\left(1.2^3(3.9^2) + 3(1.2^3)3.9 + 2(1.2^3)\right)} \times \frac{(3.9 + 1)^{\frac{3}{2}}}{(3.9 - 1.2)^{\frac{3}{2}}}$$

$$= \frac{99.06624 + 13.6512 + 3.3696 + 36.504}{26.28288 + 20.2176 + 3.456} \times \frac{10.84661237}{4.4366}$$

$$= \frac{152.59104}{49.95648} \times \frac{10.84661237}{4.4366} = \frac{1655.095862}{221.6369192}$$

$$\therefore sk_1(1) = 7.4676$$

$sk_2(1)$

$$= \frac{3(1.3^3)3.9(3.9+1) + 1.3^3(3.9+4) - 3(1.3^2)3.9(1-1.3) + 2(1.3)3.9^2}{\left(1.3^3(3.9^2) + 3(1.3^3)3.9 + 2(1.3^3)\right)}$$

$$\times \frac{(3.9+1)^{\frac{3}{2}}}{(3.9-1.3)^{\frac{3}{2}}}$$

$$= \frac{125.95401 + 17.3563 + 1.521 + 39.546}{33.41637 + 25.7049 + 4.394} \times \frac{10.84661237}{4.192374029}$$

$$= \frac{184.37731}{63.51527} \times \frac{10.84661237}{4.192374029} = \frac{1999.869211}{266.2797684}$$

$$\therefore sk_2(1) = 7.5104$$

$sk_3(1)$

$$= \frac{3(1.4)^3 3.9(3.9+1) + (1.4)^3(3.9+4) - 3(1.4)^2 3.9(1-1.4) + 2(1.4)(3.9)^2}{((1.4)^3(3.9)^2 + 3(1.4)^3 3.9 + 2(1.4)^3)}$$

$$\times \frac{(3.9+1)^{\frac{3}{2}}}{(3.9-1.4)^{\frac{3}{2}}}$$

$$= \frac{157.3152 + 21.6776 + 9.1728 + 42.588}{41.73624 + 32.1048 + 54.88} \times \frac{10.84661237}{3.952847075}$$

$$= \frac{2502.87663}{508.8145865}$$

$$\therefore sk_3(1) = 4.9190$$

Hence the skewness vector becomes

$$\boldsymbol{sk}(1) = \begin{pmatrix} 7.4676 \\ 7.5104 \\ 4.9190 \end{pmatrix}$$

Kurtosis of the non-symmetric distribution in Table 4.4 may be obtained using Equation 3.122 as

$$ku_i(1) = \frac{6\alpha_i^2\alpha_0^3 + 3\alpha_i^3\alpha_0^2 + 18\alpha_i^3\alpha_0 + 39\alpha_i^2\alpha_0 + 24\alpha_0^3 + 15\alpha_0^4 - 66\alpha_i^2\alpha_0^2 + 3\alpha_0^5}{(\alpha_0^3 + 6\alpha_0^2 + 11\alpha_0 + 6)} \times \frac{(\alpha_0 + 1)^2}{\alpha_i(\alpha_0 - \alpha_i)^2}$$

Thus,

$$ku_1(1) = \left(\frac{6(1.2)^2(3.9)^3 + 3(1.2)^3(3.9)^2 + 18(1.2)^3 3.9 + 39(1.2)^2 3.9 + 24(3.9)^3}{((3.9)^3 + 6(3.9)^2 + 11(3.9) + 6)} + \frac{15(3.9)^4 - 66(1.2)^2(3.9)^2 + 3(3.9)^5}{((3.9)^3 + 6(3.9)^2 + 11(3.9) + 6)}\right) \times \frac{(3.9 + 1)^2}{1.2(3.9 - 1.2)^2}$$

$$= \frac{170151.1741}{1745.042292}$$

$$\therefore ku_1(1) = 97.5055$$

Also,

$$ku_2(1) = \left(\frac{6(1.3)^2(3.9)^3 + 3(1.3)^3(3.9)^2 + 18(1.3)^3 3.9 + 39(1.3)^2 3.9 + 24(3.9)^3}{(3.9^3 + 6(3.9)^2 + 11 * 3.9 + 6)} + \frac{15(3.9)^4 - 66(1.3)^2(3.9)^2 + 3(3.9)^5}{(3.9^3 + 6(3.9)^2 + 11 * 3.9 + 6)}\right) \times \frac{(3.9 + 1)^2}{1.3(3.9 - 1.3)^2}$$

$$= \frac{7017.04224}{199.479} \times \frac{24.01}{8.788} = \frac{168479.1842}{1753.021452}$$

$$\therefore ku_2(1) = 95.1079$$

Now,

$$ku_3(1)$$
$$= \left(\frac{6(1.4)^2(3.9)^3 + 3(1.4)^3(3.9)^2 + 18(1.4)^3 3.9 + 39(1.4)^2 3.9 + 24(3.9)^3}{(3.9^3 + 6(3.9)^2 + 11 * 3.9 + 6)} \right.$$
$$\left. + \frac{15(3.9)^4 - 66(1.4)^2(3.9)^2 + 3(3.9)^5}{(3.9^3 + 6(3.9)^2 + 11 * 3.9 + 6)} \right) \times \frac{(3.9 + 1)^2}{1.4(3.9 - 1.4)^2}$$

$$= \frac{7946.52283}{199.479} \times \frac{24.01}{8.75}$$

$$\therefore ku_3(1) = 74.9358$$

Hence the kurtosis vector becomes

$$\boldsymbol{ku}(1) = \begin{pmatrix} 97.5055 \\ 95.1079 \\ 74.9358 \end{pmatrix}$$

This implies that the data on Table 4.4 is a $p = 3$-variate leptokurtic distribution.

The results obtained in this chapter shall be presented in a concise form, used to draw conclusions and make recommendations in next chapter.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 SUMMARY OF RESULTS

The results obtained in this work are highlighted as follows: the univariate generalized moment generating function as developed by Oyeka *et al* (2008 and 2010) for continuous random variable, X, with probability density function, $f(x)$, about an arbitrarily chosen, λ, is given as the expected value of the n^{th} power of $X^c + \lambda$, that is; $g_n(c;\lambda) = E(X^c + \lambda)^n = \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} E(x^{cr}) f(x)\, dx$ $= \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} \frac{\Gamma(\alpha+\beta).\Gamma(cr+\alpha)}{\Gamma(\alpha).\Gamma(cr+\alpha+\beta)}$, equation 3.17, for the beta family of distributions; $= \sum_{r=0}^{n} \binom{n}{c} \lambda^{n-r} \frac{\beta^{cr}\Gamma(cr+\alpha)}{\Gamma(\alpha)}$, equation 3.29, for the gamma family of distributions and; $= \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} \sum_{t=0}^{cr} \binom{cr}{t} \mu^{cr-t} (2\sigma^2)^{\frac{t}{2}} \frac{\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)}{\sqrt{\pi}}$, equation 3.26, for the univariate normal distribution; the bivariate generalized moment generating function about zero as developed by Oyeka *et al* (2012) was proved to be obtained as the expected value of the n^{th} power of $X^c Y^d$, given as, $g_n(c,d) = \mu_n(c,d) = E(X^c, Y^d)^n; c \geq 0, d \geq 0, n = 0, 1, 2, \ldots = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (X^c)^n (Y^d)^n f(x,y) dx.dy$; the generalized multivariate moment generating functions about a constant vector or matrix, $\boldsymbol{\lambda}$, $E(\boldsymbol{X^c} + \boldsymbol{\lambda})^n$, were developed and illustrated with the multivariate gamma, the normal and dirrichlet distribution as; $\boldsymbol{G}_n(\boldsymbol{c};\, \boldsymbol{\lambda}) = \sum_{r=0}^{n} \binom{n}{r} \boldsymbol{\lambda}^{n-r} \frac{|\Sigma|^{cr} \beta^{pcr} \Gamma_p(\alpha+cr)}{\Gamma_p(\alpha)}$, equation 3.78, for the multivariate gamma distribution; $\boldsymbol{G}_n(\boldsymbol{c}; \boldsymbol{\lambda}) = \sum_{r=0}^{n} \binom{n}{r} \boldsymbol{\lambda}^{n-r} \frac{|\Sigma|^{cr} 2^{pcr} \Gamma_p\left(\frac{n}{2}+cr\right)}{\Gamma_p\left(\frac{n}{2}\right)}$, equation 3.79, for the wishart distribution; $= \left(\sum_{r=0}^{n} \binom{n}{r} \boldsymbol{\lambda}^{n-r}\right)^p \prod_{i=1}^{p} \beta_i^{cr} \frac{\Gamma(\alpha_i+cr)}{\Gamma(\alpha_i)}$, equation 3.84, for a case of the joint distribution of p independent gamma distributions; $= \prod_{i=1}^{p} \sum_{r_i=1}^{n_i} \binom{n_i}{r_i} \boldsymbol{\lambda}_i^{n_i-r_i} \beta_i^{c_i r_i} \frac{\Gamma(\alpha_i+c_i r_i)}{\Gamma(\alpha_i)}$, equation 3.85, for c, λ, n and r varying among the independent distributions; $= \left[\sum_{i=0}^{n} \binom{n}{r} \boldsymbol{\lambda}^{n-r} \beta^{cr} \frac{\Gamma(\alpha+cr)}{\Gamma(\alpha)}\right]^{p}$, equation 3.86, for a situation where $c, \lambda, \beta, \alpha$ and r are constant among all the independent gamma distributions; $\boldsymbol{G}_n(\boldsymbol{c}; \boldsymbol{\lambda}) = \sum_{i=0}^{n} \binom{n}{r} \boldsymbol{\lambda}^{n-r} \sum_{t=0}^{cr} \binom{cr}{t} \mu^{cr-t} (2^p \Sigma)^{\frac{t}{2}} \frac{\left(\Gamma\left(\frac{t}{2}+\frac{1}{2}\right)\right)^{p}}{\pi^{\frac{p}{2}}}$, equation 3.91 for the multivariate normal distribution and; $\boldsymbol{G}_n(c; \boldsymbol{\lambda}) = \sum_{r=0}^{n} \binom{n}{r} \lambda^{n-r} \frac{\Gamma(\sum_{i=1}^{k} \alpha_i)}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i+cr)}{\Gamma(\sum_{i=1}^{k} \alpha_i+cr)}$, Equation 3.113 for the Dirrichlet Distribution (a multivariate extension of the Beta Distribution).

5.2 CONCLUSIONS

Based on the results obtained in this work, the following conclusions may be drawn: the generalized multivariate moment generating function was successfully developed and found to be easier to apply than the traditional methods of generating moments because no further calculus or any other modification is required for its evaluation; it is more versatile than the traditional methods since it could handle all integral and real powers as well as central and non-central moments of random variables; it exists for all continuous probability distributions unlike its competitors such as factorial moments generating functions and moments generating functions which may not always exist and even if they exist, may be tedious and cumbersome to evaluate in practical applications especially for higher moments; results obtained using the generalized moment generating function are the same as results obtained using other methods like moment generating function where it exists; the kurtosis obtained from Multivariate Generalized Moment Generating Function of Multivariate Normal Distribution (equation 4.100) gives account of the number of variables included in the multivariate distribution and;

5.3 CONTRIBUTION TO KNOWLEDGE

This research has made the following contributions to knowledge: it has developed the generalized moment generating function (GMMGF) of Multivariate Distributions; it developed the function for the Multivariate Gamma Family, Multivariate Normal and the Dirrichlet Distributions; it has shown the use of the function in generating moments of random vectors/matrices and; it showed the advantages of the new method over existing traditional/conventional methods.

5.4 RECOMMENDATIONS

The following recommendations may be made based on the conclusions drawn in section 5.2:

1. application of the generalized moment generating function should be preferred to the traditional methods due to their simplicity and versatility of use in practical applications;
2. specifically, the Multivariate Generalized Moment Generating Function of the Multivariate Normal Distribution should be preferred in the evaluation of kurtosis of the distribution because it gives additional information about the distribution and;
3. future studies should aim at developing the generalized moment generating function for discrete probability distributions.

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