CHAPTER ONE

INTRODUCTION

1.1 BACKGROUND OF THE STUDY

Many important results in traditional time series analysis follow from the assumption that the population being sampled or investigated is normally distributed with a common variance. When the relevant theoretical assumptions relating to a selected method of analysis are approximately satisfied, the usual procedures can be applied in order to make inferences about unknown parameters. In situations where the assumptions are seriously violated, several options are available of which transformation of data has recently attracted the most attention. To transform data, one performs a mathematical operation on each observation, then use these transformed data in a statistical analysis of interest. In time series analysis using the multiplicative model, the six most popular transformations are logarithm, inverse, square-root, square, inverse-square-root, and inverse-square transformations. Before now, studies had been carried out on the effect of a particular transformation on the error component of the multiplicative time series model with the overall aim of establishing the conditions for the successful application of a particular transformation. Iwueze(2007), Otuonye et al.(2011), Nwosu et al. (2013), Ohakwe et al. (2013) and Gabriel et al. (2014) had established the conditions for the successful applications of logarithm, square-root, inverse square and inverse square transformations respectively . The details of the results of their findings would be discussed later.

1.2 THE LEFT TRUNCATED NORMAL DISTRIBUTION

The normal or Gaussian distribution is one of the most widely used probability distributions. Bell-shaped distributions such as the normal and t - distributions are usually encountered in a large number of applications. A normally distributed random variable in theory assumes values in the range $-\infty$ to ∞ . However, there are cases where this random variable is constrained to lie only on the negative or positive region of the cartesian coordinates, which is often called the right or left truncation of the random variable. This restriction necessitated the derivation of distributions under the right or left truncation of the random variable. One such example is the error term of the multiplicative time series model which is assumed to be $N(1,\sigma^2)$ whose square root is constrained to the positive real numbers (Otuonye et al (2011)).

Consider the normally distributed random variable X with a probability density function $f(x)$ specified as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-1}{\sigma}\right)^2\right\}, \quad -\infty < x < \infty, \quad \sigma > 0 \tag{1.1}$$

In many applications, the random variable X which has $N(1,\sigma^2)$ distribution does not admit values less than or equal to zero (Iwueze (2007)). Hence, we disregard or truncate all values of $x \leq 0$ to take care of the region of x > 0. If the values of $x \leq 0$ cannot be observed due to censoring or truncation, the resulting distribution is a left-truncated normal distribution whose probability density function denoted by $f^*(x)$ is obtained by (Iwueze,2007)

$$f^*(x)=\begin{cases}\dfrac{\exp\left\{-\frac{1}{2}\left(\frac{x-1}{\sigma}\right)^2\right\}}{\sigma\sqrt{2\pi\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}} & ,\ 0<x<\infty \\ 0, & -\infty<x\leq 0\end{cases} \tag{1.2}$$

with

$$E^*(X)=1+\frac{\sigma e^{-\frac{1}{2\sigma^2}}}{\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]} \tag{1.3}$$

and

$$Var(X)=\frac{\sigma^2}{2\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\left[1+\Pr\left[\chi^2_{(1)}<\frac{1}{\sigma^2}\right]\right]-\frac{\sigma e^{-\frac{1}{2\sigma^2}}}{\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}-\left[\frac{\sigma e^{-\frac{1}{2\sigma^2}}}{\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\right]^2 \tag{1.4}$$

(Iwueze, 2007)

1.3 TIME SERIES ANALYSIS

Time-series as a stochastic process is an ordered sequence of observations (a collection of observations made sequentially in time). Usually, the values are evenly distributed across the time domain (Dunham, 2002).There are two types of time series namely univariate and multivariate. Univariate time series is a time series where only one variable is measured over time, whereas multivariate time series are those, where more than one variable are measured simultaneously. Multivariate time series analysis is a powerful tool for the analysis of multivariate time series data. The application is widespread. Examples are found in the medical field (Crabtree et al. 1990), and engineering (De Vries and Wu, 1978).Time series data provide useful information about the physical, biological, social and economic systems such as share prices, school enrollments, amount of pollutants in the environments, rainfall, number of SARS(Severe Acute Respiratory Syndrom) cases over time, blood pressure measurements over time

The intrinsic nature of a time series is that its observations are dependent or correlated. The order of the observations is therefore important. Hence, statistical procedures and techniques that rely on independence assumptions are no longer applicable and different methods are needed. Future values may be predicted from past observations. If future values of a time series are exactly determined by some mathematical functions, the time series is said to be deterministic. If the future values can be described only in terms of a probability distribution, the time series is said to be non-deterministic or simply a statistical time series. A time series data set is a realization or sample function from a certain stochastic process. We may consider a time series as a sequence of random variables, x_1, x_2, x_3,..., where the random variable x_1 denotes the value taken by the series at the first time point, the variable x_2 denotes the value for the second time period, x_3 denotes the value for the third time period, and so on. In general, a collection of random variables, $X_t, t \in Z$ indexed by t is referred to as a stochastic process. The observed values of a stochastic process are referred to as a realization of the stochastic process.

Time series analysis refers to principles and techniques which deal with analysis of observed data, $X_t, t = 1, 2, ..., n$, usually analysed to find a model that describes the true underlying generating random process, $X_t, t \in Z$; and to obtain the future values.

1.4 COMPONENTS OF TIME SERIES

Typically, a time series comprises four components (Chatfield, 2004), namely

1. **Trend component (T_t) -** Trend is a long term movement in a time series. It is the underlying direction (upward or downward) and rate of change in a time series. The reasons for trend analysis are (i) to obtain the trend values and (ii) to measure local fluctuations.

2. **Seasonal component (S_t) -** It is the component of variation in a time series which is dependent on the time of the year. It describes any regular fluctuations with a period of less than one year. There are several reasons for examining the seasonal effects namely

(i) to remove seasonal effect from the series in order to study its other constituents uncontaminated by seasonal elements.

(ii) to compare a variable at different points of the year as a purely intra-year phenomenon.

(iii) to “correct” the current figure for seasonal effects, for example to state what the unemployment in a month would have been if customary seasonal influences had not increased them.

3. **Cyclic component (C_t) -**This refers to the long term oscillations or swings about the trend. These cycles as they are sometimes called may or may not be periodic. That is, they may or may not have exactly similar patterns after equal intervals of time. An important example of cyclical movement is the so called business cycles representing intervals of prosperity, recession, depression and recovery. Only long period sets of data will show cyclical fluctuations of any appreciable magnitude.

4. **Irregular component (e_t) –** Random or chaotic noisy residuals left over when other components of the series (trend, seasonal and cyclical) have been accounted for.

1.5 DESCRIPTIVE TIME SERIES ANALYSIS

Methods of time series analysis constitute an important area in statistics (Chatfield, 2004)). Time series analysis comprises of methods that attempt to analyse a time series data, often either to explain the underlying context of the data points (Where did they come from? What generated them?) or to make forecasts. Time series analysis involve the use of a model to forecast or predict future events based on known past events.

Methods for time series analyses are often divided into three classes: descriptive methods, time domain methods and frequency domain methods. Frequency domain methods centre on spectral analysis and recently wavelet analysis (Percival and Walden, 2000, Priestly, 1981), which are regarded as model-free analysis. Time domain methods (Box et al., 1994; Wei, 1989) have a distribution-free subset consisting of the autocorrelation and cross-correlation analysis.

Descriptive methods (Chatfield, 2004, Kendal and Ord, 1990) involve the separation of an observed time series into components representing trend, the seasonal, cyclical and irregular components.

Decomposition models as given in Iwueze et al. (2011) are typically additive or multiplicative, but can also take other forms such as pseudo-additive/mixed (combining the elements of both the additive and multiplicative models)

Additive model: $X_t = T_t + S_t + C_t + e_t$ (1.5)

Multiplicative model: $X_t = T_t * S_t * C_t * e_t$ (1.6)

Pseudo-additive/Mixed model: $X_t = T_t * S_t * C_t + e_t$ (1.7)

where X_t is the observed time series for t = 1, 2, . . ., n, T_t is the trend at time t, S_t, the seasonal component at time t, C_t, the cyclical component at time t, and e_t, the irregular/residual component at time t. If short periods of time are involved, the cyclical component is assumed to be superimposed into the trend (Chatfield, 2004) and the observed time series $(X_t, t = 1, 2, ..., n)$ can be decomposed into the trend-cycle component (M_t), seasonal component (S_t) and the irregular/residual component (e_t). Thus models (1.5) through (1.7) become

Additive model: $$X_t = M_t + S_t + e_t \quad (1.8)$$

Multiplicative model: $$X_t = M_t * S_t * e_t \quad (1.9)$$

Pseudo-additive/Mixed model: $$X_t = M_t * S_t + e_t \quad (1.10)$$

Considering that most of the data set we encounter in practice are of short period of time, therefore we shall be interested in models (1.8) through (1.10). It is important to note that the pseudo-additive model is used when the original time series contains very small or zero values (Iwueze et al., 2011). For this reason we shall discuss only the additive and the multiplicative models.

Adopting the traditional method of decomposition, the first step is to estimate and eliminate M_t for each time period from the actual data either by subtraction from model (1.8) or division of model (1.9) by M_t. The de-trended series is obtained as $X_t - \hat{M}_t$ for model (1.8) or $\frac{X_t}{\hat{M}_t}$ for model (1.9). In the second step, the seasonal effect is obtained by estimating the average of the de-trended series at each season.

The de-trended, de-seasonalized series is obtained as $X_t - \hat{M}_t - \hat{S}_t$ for model (1.8) or $\frac{X_t}{\hat{M}_t \hat{S}_t}$ for model (1.9). This gives the residual or irregular component. Having fitted a model to a time series, one often wants to see if the residuals are purely random. For detailed discussions on residual analysis, see Box et al. (1994) and Ljung and Box (1978).

It is always assumed that the seasonal effect when it exists has period, s. That is, it repeats after s time periods.

$$S_{t+s} = S_t, \text{ for all t} \qquad (1.11)$$

For model (1.8), it is convenient to make further assumption that the sum of the seasonal components over a complete period is zero.

$$\sum_{j=1}^{s} S_{t+j} = 0 \qquad (1.12)$$

Similarly, for model (1.9), the convenient variant assumption is that the sum of the seasonal components over a complete period is s.

$$\sum_{j=1}^{s} S_{t+j} = s \qquad (1.13)$$

It is also assumed that the irregular component e_t is a Gaussian $N\left(0, \sigma_1^2\right)$ white noise for model (1.8) while it is a Gaussian $N\left(1, \sigma_2^2\right)$ white noise for model (1.9).

After the estimation of the components of the multiplicative model (1.9), there is need to assess the model adequacy by checking whether the model assumptions are satisfied. The component of the time series used for this assessment is the irregular component or the residual series, e_t. The basic assumption is that e_t is a Gaussian $N\left(1, \sigma_2^2\right)$ white noise. That is e_t's are uncorrelated random shocks with unit-mean and constant variance. For any fitted time series model, the residuals which are the error component are the estimates of these unobserved white noise. To check whether the errors are normally distributed, one can construct a histogram of the standardized residuals and compare it with the standard normal distribution using the chi-square goodness of fit test or any other test of normality such as Turkey, Kolmogorov-Smirnov, Andersen Darling tests of normality and so on. The hypotheses are as follows;

H_0: the residual series follow a normal distribution

against

H_1: the residual series do not follow a normal distribution

The vertical scale on the graph resembles the vertical scale found on normal probability paper. The horizontal axis is a linear scale. The line forms an estimate of the cumulative distribution function for the population from which data are drawn. Numerical estimates of the population parameters, μ and σ, the normality test value, and the associated p-value are displayed with the plot.

To check whether the variance is constant, we examine the residual plot or use the appropriate test of homogeneity of variance such as Bartlett and Levene's test of homogeneity of variance. The tests are used to test if k samples have equal variances. The Bartlett's test is sensitive to departures from normality, that is, if your samples come from non-normal distributions, then Bartlett's test may simply be testing for non-normality. Both tests are designed to test for equality of variances across groups against the alternative that variances are unequal for at least two groups.

The hypothesis for the tests are as follows:

$H_0 : {\sigma_1}^2 = {\sigma_2}^2 = \ldots = {\sigma_k}^2$

H_1: ${\sigma_i}^2 \neq {\sigma_j}^2$ for at least one pair (i , j)

The test statistics for the Bartlett's test is given by:

$$T = \frac{(N-K)\,ln S_p^2 - \sum_{i=1}^{k}(N_i-1)\,ln S_i^2}{1+\frac{1}{3(k-1)}\left[\left(\sum_{i=1}^{k}\frac{1}{N_i-1}\right)-\left(\frac{1}{N-K}\right)\right]}$$

where

S_i^2 is the variance of the yearly group

N is the total sample size

N_i is the sample size of the ith group

K is the number of groups

S_p^2 is the pooled variance

The pooled variance is a weighted average of the group variances and it is defined as:

$$S_p^2 = \sum_{i=1}^{k} \frac{(N_i - 1)S_i^2}{N-K}$$

The variances are judged to be unequal if

$$T > \chi^2_{1-\alpha, k-1}$$

where α is the level of significance.

The Levene's test is an alternative to the Bartlett's test. The Levene's test is less sensitive than the Bartlett's test to departures from normality.

The test statistics for the Levene's test is given by:

$$W = \frac{(N-K)\sum_{i=1}^{k} N_i(\bar{Z}_{i.} - \bar{Z}_{..})^2}{(K-1)\sum_{i=1}^{k}\sum_{j=1}^{N_i}(Z_{ij} - \bar{Z}_{i.})^2}$$

where

$Z_{ij} = |Y_{ij} - \bar{Y}_{i.}|$ and

$\bar{Y}_{i.}$ is the mean of the ith group

$\bar{Z}_{i.}$ are the group means of the Z_{ij}

$\bar{Z}_{..}$ is the overall mean of the Z_{ij}

The Levene's test rejects the hypothesis that the variances are equal if

W > $F_{\alpha,k-1,N-k}$.

Many important results in statistical analysis follow from the assumption that the population being sampled or investigated is normally distributed with a common variance and additive error structure. When the relevant theoretical assumptions relating to a selected method of analysis are approximately satisfied, a statistical method can be applied in order to make inferences about unknown parameters of interest. In situations where the assumptions are violated, several options are available (Sakia, 1992), namely;

(i) Ignore the violation of the assumptions and proceed with the analysis as if all assumptions are satisfied.

(ii) Decide what is the correct assumption in place of the one that is violated and use a valid procedure that takes into account the new assumption.

(iii) Design a new model that has important aspects of the original model and satisfies all the assumptions, e.g. by applying a proper transformation to the data or filtering out some suspect data points which are considered outliers.

(iv) Use a distribution-free procedure that is valid even if various assumptions are violated.

For more details on the above listed options, see Graybill (1976).

Most researchers, however, have opted for (iii) which has attracted much attention to data transformation as documented by Thoeni (1967) and Hoyle (1973). In this study our interest would center on transformation as a remedy for situations where the assumptions for parametric data analysis are violated.

1.6 THE BUYS-BALLOT TABLE

A Buys-Ballot Table (see Table 1.1) summarizes seasonal time series data. Each row is one period (usually a year), and each column is a season of the period/year (4 quarters, 12 months, etc). A cell, (i, j) in Table 1.1 contains the observation made during the period i at the season j.

To analyse the data, it is important to include the period and seasonal totals $\left(T_{i.}\ and\ T_{.j}\right)$, period and seasonal averages $\left(\bar{X}_{i.}\ and\ \bar{X}_{.j}\right)$, period and seasonal standard deviations $\left(\hat{\sigma}_{i.}\ and\ \hat{\sigma}_{.j}\right)$ as part of the terms in Buys-Ballot Table. We also include the grand total $(T_{..})$, grand mean $\left(\bar{X}_{..}\right)$ and pooled standard deviation $(\hat{\sigma}_{..})$ in Table 1.1.

Wold (1938) credited the arrangements in Table 1.1 to Buys-Ballot (1847). The stability of the variance of the time series can easily be assessed by observing the row standard deviations.

Table 1.1 Buys-Ballot Table for Seasonal Time Series

Periods	Seasons						Total	Average	Std Dev
	1	2	...	j	...	s	$T_{i.}$	$\bar{X}_{i.}$	$\hat{\sigma}_{i.}$
1	X_1	X_2	$\cdots$	X_j	$\cdots$	X_s	$T_{1.}$	$\bar{X}_{1.}$	$\hat{\sigma}_{1.}$
2	X_{1+s}	X_{2+s}	$\cdots$	X_{j+s}	$\cdots$	X_{2s}	$T_{2.}$	$\bar{X}_{2.}$	$\hat{\sigma}_{2.}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$	
i	$X_{1+(i-1)s}$	$X_{2+(i-1)s}$	$\cdots$	$X_{j+(i-)s}$	$\cdots$	$X_{s+(i-1)s}$	$T_{i.}$	$\bar{X}_{i.}$	$\hat{\sigma}_{i.}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$	
M	$X_{1+(m-1)s}$		$\cdots$	$X_{j+(m-1)s}$	$\cdots$	$X_{1+(m-1)s}$	$T_{m.}$	$\bar{X}_{m.}$	$\hat{\sigma}_{m.}$
Total	$T_{.1}$	$T_{.2}$	$\cdots$	$T_{.j}$	$\cdots$	$T_{.s}$	$T_{..}$		
Average	$\bar{X}_{.1}$	$\bar{X}_{.2}$	$\cdots$	$\bar{X}_{.j}$	$\cdots$	$\bar{X}_{.s}$		$\bar{X}_{..}$	
Std Dev	$\hat{\sigma}_{.1}$	$\hat{\sigma}_{.2}$	$\cdots$	$\hat{\sigma}_{.j}$	$\cdots$	$\hat{\sigma}_{.s}$			$\hat{\sigma}_{..}$

where

$$T_{i.} = \sum_{j=1}^{s} X_{j+(i-1)s},\ i = 1, 2, ..., m \text{ (i}^{\text{th}}\text{ periodic total)}$$

$$T_{.j} = \sum_{i=1}^{m} X_{j+(i-1)s},\ j = 1, 2, ..., s \text{ (j}^{\text{th}}\text{seasonal total)}$$

m = number of periods

s = number of seasons

n = ms = number of observations

$$\bar{X}_{i.} = \frac{1}{s}\sum_{j=1}^{s} X_{j+(i-1)s} = \frac{T_{i.}}{s},\ i = 1, 2, ..., m \qquad \text{(i}^{\text{th}}\text{ periodic average)}$$

$$\bar{X}_{.j} = \frac{1}{m}\sum_{i=1}^{m} X_{j+(i-1)s} = \frac{T_{.j}}{m},\ j = 1, 2, ..., s \qquad \text{(i}^{\text{th}}\text{ seasonal average)}$$

$$T_{..} = \sum_{i=1}^{m} T_{i.} = \sum_{j=1}^{s} T_{.j} \qquad \text{(Grand total)}$$

$$\bar{X}_{..} = \frac{\sum_{i=1}^{m} T_{i.}}{ms} = \frac{\sum_{j=1}^{s} T_{.j}}{ms} = \frac{T_{..}}{ms} \qquad \text{(Grand mean)}$$

$$\hat{\sigma}_{i.} = \sqrt{\frac{1}{s-1}\sum_{j=1}^{s}\left(X_{j+(i-1)s} - \bar{X}_{i.}\right)^2} \qquad (i^{th} \text{ periodic standard deviation})$$

$$\hat{\sigma}_{.j} = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}\left(X_{j+(i-1)s} - \bar{X}_{.j}\right)^2} \qquad (j^{th} \text{ seasonal standard deviation})$$

1.7. USES OF THE BUYS-BALLOT TABLE IN TIME SERIES ANALYSIS

Some of the main uses of the Buys-Ballot Table as contained in Iwueze et al. (2011) are as follows;

1. **Choice of Appropriate Transformation:** This is simply achieved by obtaining a linear regression model of the logarithm of the periodic standard deviations on the logarithm of the periodic means where the value of the slope coefficient determines the choice of the appropriate transformation as given in Table 1.2.
2. **Assessment of trend:** For this purpose, we plot the periodic/annual means of the Buys-Ballot Table from where we examine and select the appropriate trend-curve rather than plotting the entire series.
3. **Choice of appropriate model:** For this purpose, the relationship between the seasonal averages $\left(\bar{\mathbf{X}}_{.j},\ \mathbf{j = 1, 2, \ldots, s}\right)$ and the seasonal standard deviations $\left(\hat{\boldsymbol{\sigma}}_{.j},\ \mathbf{j = 1, 2, \ldots, s}\right)$ gives an indication of the desired model. An additive model is appropriate when the seasonal standard

deviations show no appreciable/decrease relative to any increase or decrease in the seasonal means. On the other hand a multiplicative model is usually appropriate when the seasonal standard deviations show appreciable increase/decrease relative to any increase/decrease in the seasonal means Iwueze and Nwosu (2014)

1.8. DATA CLASSIFICATION AND TRANSFORMATION IN TIME SERIES ANALYSIS

Data classification is defined by Dunham (2002) as mapping data into predefined classes. According to Tarek (2013), data transformation is synonymous with the feature-based classification. Hence, the feature-based time series classification techniques work on transforming the sequential data/time series data into feature-set before using the classification algorithm (Xing et al, 2010). For a time series data set, there are various literature on the procedures for choice of appropriate transformations among which are: Bartlett (1947), Box and Cox (1964), Akpanta and Iwueze (2009) and Iwueze et al (2011).

Data transformation is the application of a deterministic mathematical function to each observation/number in a data set – that is each data point x_i is replaced with the transformed value $y_i = f(x_i)$, where f is a function . Generally, there are a several reasons for data transformation such as: for easy visualization, improvement in interpretation, variance stabilization, ensuring a normally distributed data, additivity of the seasonal effect, reducing the effect of outliers, making a measurement scale more meaningful, and to linearize a relationship. However, for time series data, the three major reasons for data transformation are as follows;

(a) **Variance Stabilization:** A transformation may be used to produce approximately equal spreads, despite marked variations in level, which again makes data easier to handle and interpret. Each data set or subset having about the same spread or variability is a condition called homoscedasticity; while its opposite is called heteroscedasticity**.** Examples of variance stabilization transformations are the Box-Cox transformation or the Bartlett's Transformation.˙

(b) Ensuring a Normally Distributed Data: In most areas of statistics including time series, the fundamental assumption is that the distribution is Gaussian or normal. The main diagnostic tool which is used in time series (time domain and frequency domain approaches) is the autocorrelation function which is unique for stationary normal processes.

(c) Additivity of the Seasonal Effect: A transformation converts seasonal effects that have a multiplicative pattern to one that has an additive pattern for easy analysis and interpretation**.**

The most popular and common transformations are the power transformations such as $\log_e X_t, \sqrt{X_t}, \frac{1}{X_t}, \frac{1}{\sqrt{X_t}}, X_t^2,$ and $\frac{1}{X_t^2}$.

The logarithm transformation converts the multiplicative model (1.9) to additive model (1.8) while the other listed transformations of the multiplicative model (1.9) are shown in Table 1.2. The logarithm transformation of the multiplicative model (1.9) is given by

$$Y_t = \log_e X_t = \log_e M_t + \log_e S_t + \log_e e_t = M_t^* + S_t^* + e_t^* \qquad (1.14)$$

where

$Y_t \equiv$ Transformed observed time series

$M_t^* \equiv$ Transformed trend-cycle component

$S_t^* \equiv$ Transformed seasonal component

and

$e_t^* \equiv$ Transformed error component

Table 1.2: Transformations of the Purely Multiplicative Model

Y_t	M_t^*	S_t^*	e_t^*	Model for Y_t	Assumption on S_t^*	Assumption on e_t^*
$\log_e X_t$	$\log_e M_t$	$\log_e S_t$	$\log_e e_t$	Additive	$\sum_{j=1}^{s} S^* = 0$	$e_t^* \sim N(0,\sigma_1^2)$, $\sigma_1 \le 0.1$
$\sqrt{X_t}$	$\sqrt{M_t}$	$\sqrt{S_t}$	$\sqrt{e_t}$	Multiplicative	$\sum_{j=1}^{s} S^* = s$	$e_t^* \sim N(1,\sigma_2^2)$, $\sigma_2 \le 0.3$
$1/X_t$	$1/M_t$	$1/S_t$	$1/e_t$	Multiplicative	$\sum_{j=1}^{s} S^* = s$	$e_t^* \sim N(1,\sigma_2^2)$, $\sigma_1 \le 0.1$
X_t^2	M_t^2	S_t^2	e_t^2	Multiplicative	$\sum_{j=1}^{s} S^* = s$	$e_t^* \sim N(1,\sigma_2^2)$, $\sigma_1 \le 0.027$
$1/\sqrt{X_t}$	$1/\sqrt{M_t}$	$1/\sqrt{S_t}$	$1/\sqrt{e_t}$	Multiplicative	$\sum_{j=1}^{s} S^* = s$	$e_t^* \sim N(1,\sigma_2^2)$ (Condition yet to to be obtained)
$1/X_t^2$	$1/M_t^2$	$1/S_t^2$	$1/e_t^2$	Multiplicative	$\sum_{j=1}^{s} S^* = s$	$e_t^* \sim N(1,\sigma_2^2)$, $\sigma_1 \le 0.070$

It is clear from Table 1.2, that only the logarithm transformation alters the assumptions placed on the seasonal and error components of the purely multiplicative model.

The essence of data transformation as used in time series analysis is to ensure that the data approximately meet the assumptions of a statistical inference procedure that is to be applied.

The basic assumptions of interest are that the error component has unit mean and constant variance, and is normally distributed. Thus for a transformation to be successful, these basic assumptions must be maintained or achieved after any transformation.

Studies on the conditions for achieving successful transformations on the trend-cycle and seasonal components of the purely multiplicative model for the six listed transformations in Table 1.2 had been carried out while for the error component, the logarithm, square root, inverse, square and Inverse Square had been investigated leaving out only inverse square root yet to be studied.

1.9. STATEMENT OF THE PROBLEM

The need for a satisfactory data transformation has been established by some authors such as Turkey (1957), Box and Cox (1964), Osborne (2002), Akpanta and Iwueze (2009), Iwu et al (2009), Fink et al., (2009), Osborne (2010),Vidakovic (2012), Watthanacheewakul (2012), Otuonye et al (2011), Nwosu et al. (2013) and Ohakwe et al. (2013). Every multiplicative time series data is classified into one of the following power transformations, $\log_e X_t, \sqrt{X_t}\, \frac{1}{X_t}, \frac{1}{\sqrt{X_t}}, X_t^2,$ and $\frac{1}{X_t^2}$.

Results for successful transformation of the error component of the multiplicative time series data obtained for the power transformations such as $\log_e X_t, \sqrt{X_t}\, \frac{1}{X_t},\ X_t^2, \textit{and}\ \frac{1}{X_t^2}$ cannot be used for any time series data that requires inverse square root transformation. According to Ruppert (1999), the use of any transformation requires that the effect of such transformation on the error structure should be taken into consideration. Hence, the need for this study is to determine condition for successful inverse square root transformation of the error component of the multiplicative time series model.

This study shall provide a statistical framework for successful inverse square root transformation without the rigor of calculating the periodic means and standard deviations as obtained in Bartlett (1947).

1.10. AIM AND OBJECTIVES OF THE STUDY

The main aim of this study is to investigate the effect of inverse-square-root transformation on the error component of the multiplicative time series model with a view to establish the condition for successful inverse-square-root transformation. The objectives are as follow:

(i) to derive the probability density function of the inverse-square-root transformed error component, $e_t^* = \frac{1}{\sqrt{e_t}}$ of the multiplicative time series model.

(ii) to establish the theoretical values of mean(μ) and variance(σ^2) of the transformed component.

(iii) to determine if there is/are the relationship(s) between the basic parameters **u** and $\boldsymbol{\sigma}^2$ of the untransformed component, e_t and the inverse-square-root transformed component, e_t^* with regard to the model assumptions, and thereby identify the condition for the existence of their relationships if it exists.

CHAPTER TWO

LITERATURE REVIEW

In data transformation; Winer (1968) as cited in Fink (2009), viewed the job of the data analyst, strangely enough, as to find a random error. When all systematic variability has been removed from data, the leftover – or residual or disturbance or error will be random, without pattern; the analyst knows that the analysis is complete when random error has been found. To find random error, data often need to be transformed (Fink, 2009) and this is the subject of this study.

Data transformation has a long history (Box and Cox, 1964) and was already in review essays in the 1960s (Kruskal, 1968). Turkey (1977) worked on Exploratory Data Analysis with emphasis on data transformation and he created new interest in this area. Cohen (1990) stated that Exploratory Data Analysis by Turkey (1977) is an inspiring account of how to effect graphic and numeric analyses of the data at hand so as to understand them.

It is not an overstatement to say that statistics is based on various transformations of data. Basic statistical values such as sample mean, variance, z-scores, etc., are all transformed data (Vidakovic, 2012). According to Vidakovic (2012), transformations in statistics are utilized for several reasons, but unifying arguments are that transformed data

(i) are easier to report, store and analyze;

(ii) comply better with a particular modeling framework; and

(iii) allow for additional insight to the phenomenon not available in the domain of non-transformed data.

Vidakovic (2012) further emphasized that the words transformation and transform are often used interchangeably, however the semantic meaning of the two words seem to be slightly different. For the word transformations, the synonyms are alterations, evolution, change, and reconfiguration. On the other hand the word transform carries the meaning of a more radical change in which the nature and/or structure of the transformed object are altered. Hence it is natural that processes which alter the data leaving them unreduced in the same domain should be called transformations (for example Box-Cox and Batlett's transformations) and the processes that radically change the nature, structure, domain and dimension of data should be called transforms (for example Fourier transforms).

In some parametric tests, the basic assumption is that the data are normally distributed or the sample size is large (Watthanacheewakul; 2012). If the data do not obey normality, then the nonparametric tests are chosen to analyse the data. However the power of nonparametric test is usually less than the power of parametric test (Watthanacheewakul, 2012). Hence, the need to continue with parametric tests after transformation. Turkey (1957) suggested that when analyzing data that violate the assumptions of a conventional method of analysis, there are two choices namely, transform the data to fit the assumptions or develop some new robust methods of analysis. Montgomery (2001) suggested that transformations are used for three purposes namely, stabilizing response variance, making the distribution of the response variable closer to a normal distribution, and improving the fit of the model to the data. Moreover, the relationship between the standard deviation and the mean can be used to determine the appropriate transformation. Furthermore, it is possible to transform the data using a family of transformations already extensively studied over a long period of time e.g. Bartlett, Box and Cox transformations. A well-known family of transformations often used is the power transformations proposed by Box and Cox (1964). Moreover, Akpanta and Iwueze (2009) had also shown that Bartlett's transformation is also a power transformation as would be discussed in the next Section.

Data transformations are not only important in time series analysis but in all areas of statistical modeling where there are some basic assumptions required for the applications of the conventional methods of analysis. For instance, in the words of Ruppert(1999), "data transformations such as replacing a variable by its logarithm or by its square-root are used to simplify the structure of the data so that they follow a convenient statistical model" .Ruppert (1999) further stated that "Transformations have one or more objectives including: (i) inducing a simple systematic relationship between a response and predictor variables in regression; (ii) stabilizing a variance, that is inducing a constant variance in a group of populations or in the residuals after a regression analysis; and (iii) inducing a particular type of distribution, e.g. normal or symmetric distribution. The second and third goals as suggested by Ruppert (1999) are concerned with simplifying the "error structure or random component of the data". However, he also stated that transformation should not be used blindly to linearize a systematic relationship since if the errors of the data set before a transformation are homoscedastic then it is likely to be heteroscedastic after transformation. Ruppert (1999) concluded that any use of transformation requires that the effect of the transformation on the error structure should be understood.

According to Osborne (2002), data transformations are the applications of mathematical modifications to the values of a variable. He expressed that caution should be exercised in the choice of the type of transformation to be adopted so that the fundamental structure of the series is not distorted and thereby rendering the interpretation very difficult or impossible.

Bartlett (1947) used the simple relation between grouped means and standard deviations of a data set to find simple transformations which make the variance independent of the mean, in which case the data are likely to follow a normal distribution with uniform variance. Akpanta and Iwueze

(2009) had shown how to apply Bartlett's transformation technique to time series data using the Buys-Ballot Table (see Table 1.1) without considering the time series model structure. The relationship between the periodic standard deviations and periodic means is what is needed. If we take random samples from a population, the means and standard deviations of these samples will be independent (and thus uncorrelated) if the population has a normal distribution (Hogg and Craig, 1978). Akpanta and Iwueze (2009) showed that for Bartlett's transformation for time series data, if we regress the natural logarithms of the group standard deviations $\left(\hat{\boldsymbol{\sigma}}_{\mathbf{i.}}, \mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{m}\right)$ on the natural logarithms of group means $\left(\bar{\mathbf{X}}_{\mathbf{i.}}, \mathbf{i}=\mathbf{1},\mathbf{2},\ldots,\mathbf{m}\right)$, the slope, $\boldsymbol{\beta}$ of the relationship

$$\mathbf{log_e\,\hat{\sigma}_{i.} = \alpha+\beta\,log_e\,\bar{X}_{i.} + e_i, \; i=1,2,\ldots,m} \tag{2.1}$$

determines the appropriate transformation (see Table 2.1), where e_i's are the error.

For non-seasonal data that require transformation, we split the observed time series $X_t, t=1,2,\ldots,n$ chronologically into m fairly equal different parts and compute $\left(\bar{X}_{i.}, i=1,2,\ldots,m\right)$ and $\left(\hat{\sigma}_{i.}, i=1,2,\ldots,m\right)$ for the parts. For seasonal data with the length of the periodic intervals, s, the Buys-Ballot table naturally partitions the observed data into m periods or rows for easy application. Akpanta and Iwueze (2009) also showed that Bartlett's transformation may also be regarded as the power transformation

$$Y_t = \begin{cases} \log_e X_t, \beta = 1 \\ X_t^{(1-\beta)}, \beta \neq 1 \end{cases} \tag{2.2}$$

The summary of transformations for various values of β and data admissibility are given in Table 2.1

Table 2.1: Bartlett's Transformation for some values of β.

S/N	β	Required Transformation	Data admissibility
1	0	No transformation	$-\infty < X_t < \infty$
2	$\frac{1}{2}$	$\sqrt{X_t}$	$X_t > 0$
3	1	$\log_e X_t$	$X_t > 0$
4	$\frac{3}{2}$	$\frac{1}{\sqrt{X_t}}$	$X_t > 0$
5	2	$\frac{1}{X_t}$	$X_t > 0$
6	3	$\frac{1}{X_t^2}$	$X_t > 0$
7	-1	X_t^2	$-\infty < X_t < \infty$

Source: Akpanta and Iwueze (2009).

(b) The Box-Cox transformation technique

Turkey (1957) as contained in Sakia (1992) introduced a family of power transformations such that the transformed values are monotonic function of the observations over some admissible range, and is given by

$$Y_t = \begin{cases} X_t^{\lambda}, \lambda \neq 1 \\ \log_e X_t, \lambda = 1 \end{cases} \tag{2.3}$$

for $X_t > 0$. However, this family has been modified by Box and Cox (1964) to take account of the discontinuity at $\lambda = 0$, such that

$$Y_t = \begin{cases} \dfrac{\left(X_t^{\lambda} - 1\right)}{\lambda}, \lambda \neq 0 \\ \log_e X_t, \lambda = 0 \end{cases} \tag{2.4}$$

and for unknown λ

$$Y_t^{\lambda} = \left(X_1^{\lambda}, X_2^{\lambda}, ..., X_n^{\lambda}\right)\left(\theta_1, \theta_2 ...., \theta_n\right)^T + \varepsilon = \underline{X}\,\underline{\theta} + \varepsilon \tag{2.5}$$

where $\underline{X}$ is a matrix of values of regressor, $\underline{\boldsymbol{\theta}}$ is a vector of unknown parameters associated with the transformed values and $\underline{\boldsymbol{\varepsilon}}$ is a vector of random errors that is multivariate normal(MVN) with zero mean and variance-covariance matrix, $\boldsymbol{\sigma}^2 \mathbf{I_n}$ $\left(\underline{\boldsymbol{\varepsilon}} \sim \mathbf{MVN}\left(\mathbf{0}, \boldsymbol{\sigma}^2 \mathbf{I_n}\right)\right)$. The transformation (2.4) is valid for only $X_t > 0$ and therefore modifications are needed to be made to accommodate the negative observations. Box and Cox (1964) proposed the shifted power transformation with the form

$$Y_t = \begin{cases} \dfrac{\left(X_t + \lambda_2\right)^{\lambda_1}}{\lambda_1}, \lambda_1 \neq 0 \\ \log_e\left(X_t + \lambda_2\right), \lambda_1 = 0 \end{cases} \tag{2.6}$$

where λ_1 is the transformation parameter and λ_2 is chosen such that $X_t > -\lambda_2$. Box and Cox (1964) further proposed maximum likelihood as well as Bayesian methods for the estimation of λ_1.

However, as a matter of simplicity of applications and considering that the subject of this study is a multiplicative time series model whose observed time series data, $X_t > 0$, for $t = 1, 2, ..., n$, we would explore the Bartlett's technique as stipulated in the literature.

Studies on the effects of transformations on the various components of the multiplicative time series model are not new in the literature. The overall aim of such studies is to establish the conditions for successful transformations. A successful transformation is achieved when the desirable properties of a data set remains un-changed after transformation.

Iwu et al (2009) studied the effect of some transformations namely; logarithm, inverse, inverse-square, square-root, inverse-square-root and square on the trend-cycle components of the multiplicative time series model (1.9) when it is exponential, linear and quadratic and the following findings were established. For the exponential trend curve given by $M_t = ae^{bt}$, the logarithm transformation $\mathbf{\log_e M_t = \alpha + \lambda t \neq a e^{bt}}$ converts the exponential trend-curve into linear and the estimate of α of the transformed series can be obtained by applying the corresponding transformation on the parameter, a**.** However, the estimate of the parameter λ is the same as that of the original series, b. For other transformations $M_t^{\beta} = \left(a e^{bt}\right)^{\beta} = \alpha e^{\lambda t}$, $\beta = \frac{1}{2}, \frac{3}{2}, 2, 3, and\ -1$(Note that $\beta = 1$ is for logarithm transformation), the estimate of the parameter β can be obtained by multiplying the constant b by the power of M_t. That is $\lambda = \beta b$, where $\beta = \frac{1}{2}, \frac{3}{2}, 2, 3, and\ -1$ for the square-root, inverse-square-root, inverse, inverse-square and square transformations respectively. For the linear trend-cycle component given by $M_t = a + bt$, the power transformation is given by $M_t^{\beta} = (a + bt)^{\beta}$. In order to approximately preserve the original form of linear trend, for $R^2 \geq 0.99$, where R^2 is the coefficient of determination,

Iwu et al (2009) established that $-0.006 \leq \Delta(a,b) \leq 0.016$ for the logarithm transformation, and $-0.008 \leq \Delta(a,b) \leq 0.083$, $-0.003 \leq \Delta(a,b) \leq 0.006$, $-0.006 \leq \Delta(a,b) \leq 0.019$, $-0.004 \leq \Delta(a,b) \leq 0.009$ and $-0.002 \leq \Delta(a,b) \leq 0.003$ for square-root, inverse, square, inverse-square-root and inverse-square transformations respectively where $\Delta(a,b) = b/a,\ a \neq 0$, a and b are the intercept and slope of the linear trend, $\mathrm{M_t = a + bt}$. Also in order to approximately preserve the original form of the quadratic trend-cycle component given by $M_t = a + bt + ct^2$ whose power transformation is given by $M_t^{\beta} = \left(a + bt + ct^2\right)^{\beta}$, the restriction $\Delta(a,b) = k\left(b^2/c\right)$ for some real constant k must be adopted (Iwu et al, 2009).

Similarly, Iwueze et al. (2008) had studied the effect of some transformations on the seasonal component of the multiplicative time series model. In the study, they constructed interval for the seasonal indices of the purely multiplicative time series model $\left(S_j, j = 1, 2, \ldots, s\right)$ required for successful data transformation. By successful transformation in this context, they meant the ability to obtain the seasonal indices of the transformed series $\left(S^*_j, j = 1, 2, \ldots, s\right)$ directly from those of the original series by merely taking the equivalent transformation of $S_j, j = 1, 2, \ldots, s$. They investigated this problem for all the distinct indices for all s values and for the equality of some indices when s = 2, 3 and 4. Results obtained were shown to be applicable to given patterns of equality of indices for s = 6, 8, 9, 10 and 12. They went ahead to emphasize that as far as the appropriate transformation is chosen for a particular time series data, the established intervals would be of great help in determining the seasonal indices of the transformed series.

Finally, the effect of logarithm (Iwueze, 2007), square-root (Otuonye et al., 2012), inverse (Nwosu et al., 2013) and square (Ohakwe et al.,2013)) transformations on the error component of the multiplicative time series model had been carried out. The overall aim of such studies is to establish the conditions for successful transformations. A successful transformation in this context is achieved when the desirable properties/assumptions placed on the error component remains unchanged after transformation. The basic assumptions of interest for this study are; (i) Normality (ii) Unit mean and (iii) constant variance (which may or may not be equal to initial variance before transformation). Consequently, Iwueze (2007) investigated the effect of logarithmic transformation on the error component (e_t) of a multiplicative time series model where ($e_t \sim N\left(1,\sigma^2\right)$) and observed that the logarithm transform; Y = Log e_t is normally distributed with mean zero and the same variance σ^2 for $\sigma < 0.1$

Otuonye et al. (2012) observed that the square root transform; $Y = \sqrt{e_t}$ is normally distributed with unit mean and variance, 4 σ^2 for $\sigma \leq 0.3$, where σ^2 is the variance of the original error component before transformation. Nwosu et al. (2013) discovered that the inverse transform $Y = \frac{1}{e_t}$ is normally distributed with mean, one and the same variance provided $\sigma \leq 0.1$

Furthermore, the conditions for successful square transformation in time series modelling, was obtained by Ohakwe et al. (2013) to be for $\sigma \leq 0.027$. The probability density function of the square transformed left-truncated $N\left(1,\sigma_1^2\right)$ error component of the multiplicative time series model and the functional expressions for its mean and variance were established. Also, the mean and variance of the square transformed left-truncated $N\left(1,\sigma_1^2\right)$ error component and those of the untransformed component were compared for the purpose of establishing the interval for σ where the properties of the two distributions are approximately the same in terms of equality of means and normality. Finally, they established that the two distributions are normally distributed with means $\cong$ 1.0 correct to 1 decimal place and variance in the interval $0 < \sigma < 0.027$.

Finally, it must be remarked that studies on the effects of transformation of the components of multiplicative time series model are not new in the literature. The overall aim of such studies is to establish the conditions for successful transformation. A successful transformation except the logarithm transformation is achieved for the trend-cycle component when the original structure of a data set remains unchanged after transformation. For example, in the same way that a linear trend-cycle component is expected to remain linear after transformation, a quadratic trend-cycle component should also remain quadratic after transformation. Furthermore, the seasonal component (indices) of the transformed series is expected to be obtained directly by applying the chosen transformation on the original seasonal indices. Similarly the error component that is initially assumed to be $N\left(1,\sigma_1^2\right)$ should also remain $N\left(1,\sigma_2^2\right)$, even though σ_1^2 may or may not be equal to σ_2^2.

In this chapter, we have reviewed the relevant literature related to this study. In the subsequent Chapters, we would study the distribution and other relevant properties of the left-truncated $\mathrm{N}\left(1,\sigma^2\right)$ error term under inverse-square-root transformation and investigate the conditions for achieving successful transformation on the purely multiplicative time series model.

CHAPTER THREE

METHODOLOGY

In this chapter, the change of variable or transformation technique is used to derive the probability density function of the inverse-square-root transformed error component ($\mathbf{f}\left(\mathbf{e}^*_{\mathbf{t}}\right)$) from which the functional expressions for the mean ($\mathbf{E}\left(\mathbf{e}^*_{\mathbf{t}}\right)$) and variance ($\mathbf{Var}\left(\mathbf{e}^*_{\mathbf{t}}\right)$) are established. We shall also derive the probability density function of the transformed error term $Y = e_t^*$ and investigate its normality. Furthermore, the functional expressions of the mean, median and variance of e_t^* which form the properties of the distribution of e_t^* are obtained.

Graphs of the probability density functions (pdf's) of the untransformed and inverse-square-root transformed error component would be used to investigate for what values of σ are curves of the probability density functions satisfy the bell-shaped characteristic of normal distribution with the property of being symmetrical about the mean, $\mu = 1$.

The Rolle's theorem would be applied to obtain the value of σ for which the value of the probability density function of the inverse-square-root transformed error component is maximum with the aim of finding the mode of the transformed distribution.

Furthermore, computations of the derived functional expressions of mean $(\mathbf{E}(\mathbf{e}^*_t))$ and variance $(\mathbf{Var}(\mathbf{e}^*_t))$ of the inverse-square-root transformed error component and those of the untransformed component ($E(e_t)$ and $Var(e_t)$) for various values of the standard deviation, σ are made with the aim of obtaining the values of σ, where the means of the transformed and untransformed are equal to one. Also simulated data generated from $N(1, \sigma^2)$ for e_t and subsequently transformed to $\mathbf{e}^*_t = \frac{\mathbf{1}}{\sqrt{\mathbf{e}_t}}$ are also used to obtain the region, where the normality conditions are satisfied for the two variables e_t and $\mathbf{e}^*_t$ using the Anderson Darling test.

PROBABILITY DENSITY FUNCTION OF THE INVERSE-SQUARE-ROOT TRANSFORMED ERROR COMPONENT.

Consider the transformed error term

$e_t^* = \frac{1}{\sqrt{X}} = Y$, where $X = e_t$ and the probability density function e_t given by

$$f^*(x) = \frac{e^{-\frac{1}{2}\left(\frac{x-1}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]},\ 0 < x < \infty ------- \quad (3.1)$$

Under the transformation $Y = \frac{1}{\sqrt{X}}$, and using change of variable technique; $\boldsymbol{X} = \frac{1}{\boldsymbol{y^2}}$ and $\frac{dx}{dy} = \frac{-2}{y^3}$, we obtain the probability density function (p.d.f) of Y,

$g(y) = f^*(x)\left|\frac{dx}{dy}\right|$, where $\left|\frac{dx}{dy}\right|$ is the Jacobian of the inverse transformation, That is $g(y) = f^*(x)\left|\frac{dx}{dy}\right|$

$$= \frac{e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left|\frac{-2}{y^3}\right|$$

$$g(y) = \begin{cases} \frac{2\,e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}}{y^3\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}, & 0 < y < \infty \\ 0, & -\infty < y \le 0 \end{cases} \tag{3.2}$$

We shall show that $g(y)$ is a probability density function (pdf):

Now,

$$\int_0^\infty g(y)dy = \int_0^\infty \frac{2}{y^3\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]} e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2} dy$$

$$= \frac{2}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\int_0^\infty \frac{1}{y^3} e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2} dy$$

$$let\ u = \frac{1}{\sigma}\left(\frac{1}{y^2}-1\right), \qquad -\frac{1}{\sigma} < u < \infty.$$

$$\Rightarrow \frac{du}{dy} = \frac{1}{\sigma}\left(-\frac{2}{y^3}\right) = -\frac{2}{\sigma y^3}\,, du = -\frac{2}{\sigma y^3}dy \Rightarrow dy = -\frac{\sigma y^3}{2}du$$

$$= \frac{2}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\int_{-\frac{1}{\sigma}}^\infty \frac{1}{y^3} e^{-\frac{u^2}{2}}\left(-\frac{\sigma y^3}{2}\right)du$$

$$\int_0^{\infty} g(y)dy = \frac{1}{\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\int_{-\frac{1}{\sigma}}^{\infty} e^{-\frac{u^2}{2}}du$$

$$= \frac{1}{\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\int_{-\frac{1}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}du$$

$$= \frac{1}{\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]} P\left(Z > \frac{-1}{\sigma}\right)$$

$$= \frac{1}{\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[1 - P\left(Z < \frac{-1}{\sigma}\right)\right]$$

$$= \frac{1}{\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right] = 1$$

Hence, $g(y)$ is the probability density function (p.d.f) of y, which is the pdf of inverse square root of transformed error, e_t^*.

3.1 INVESTIGATING THE NATURE OF THE FUNCTION $g(y)$ WITH REFERENCE TO $f^*(x)$

Using the pdf of the two variables $e_t = X \; and \; e_t^* = \frac{1}{\sqrt{X}} = Y,$ the graphs $f^*(x)$ and $g(y)$ in Equations (3.1) and (3.2) for values of $\sigma \in [0.06\,,\ 0.50]$ are shown in Figures 3.1 to 3.8 respectively.

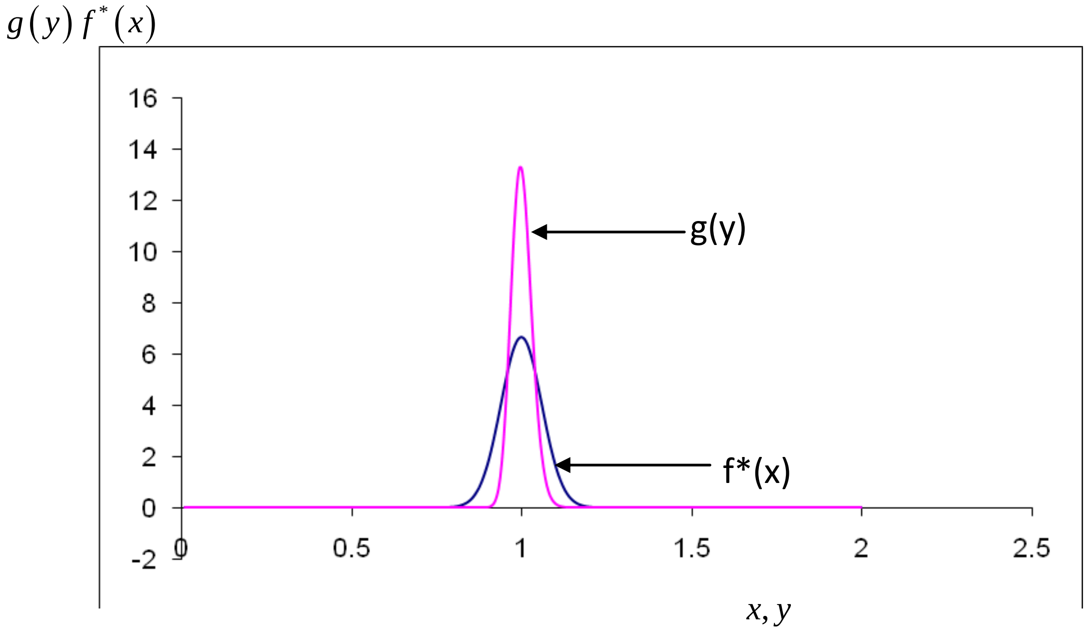


Figure 3.1. Curve Shapes for σ = 0.06

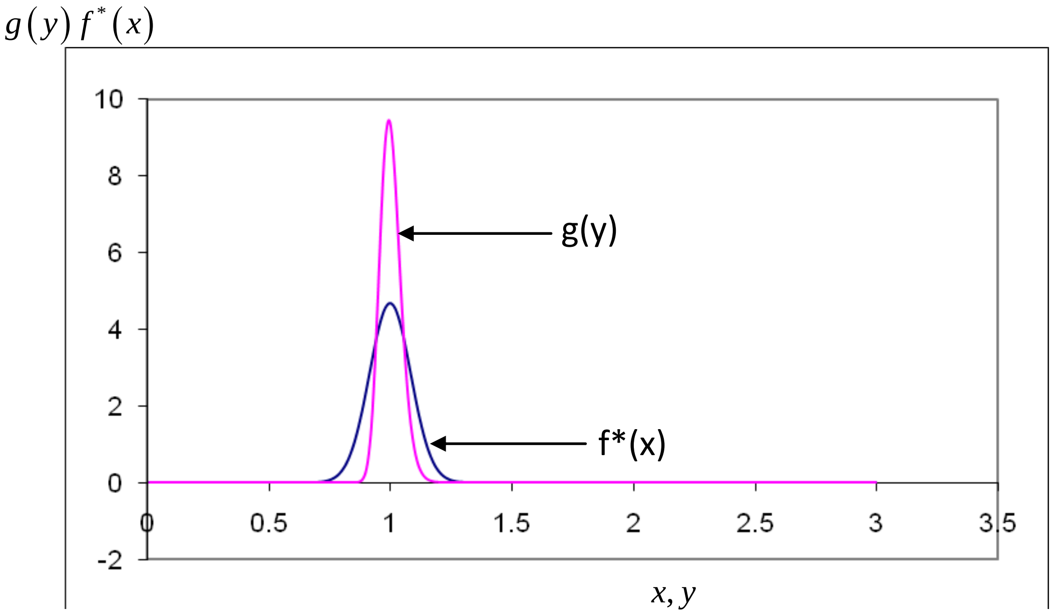


Figure 3.2. Curve Shapes for σ = 0.085

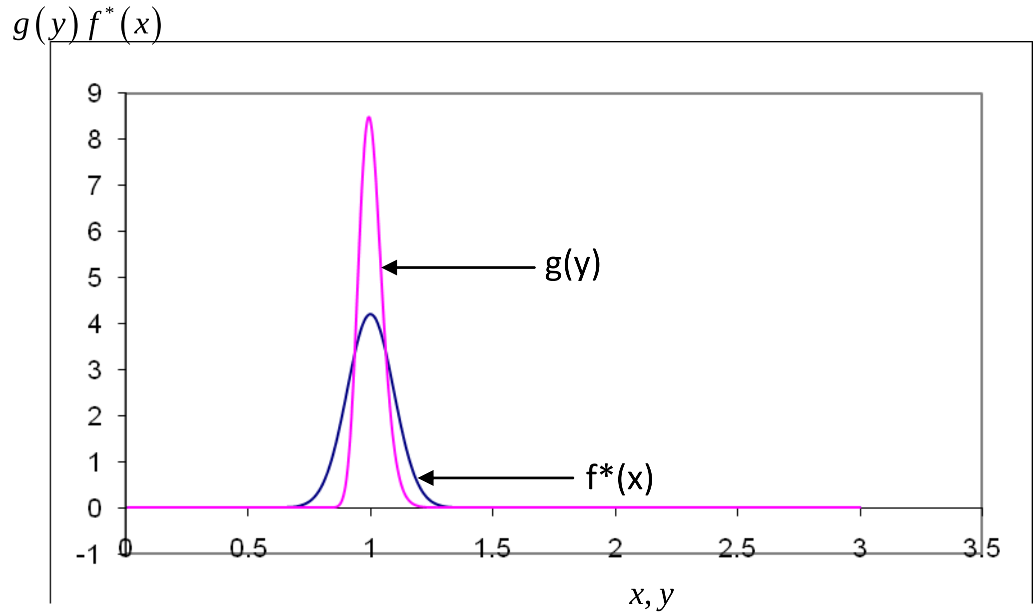


Figure 3.3. Curve Shapes for σ = 0.095

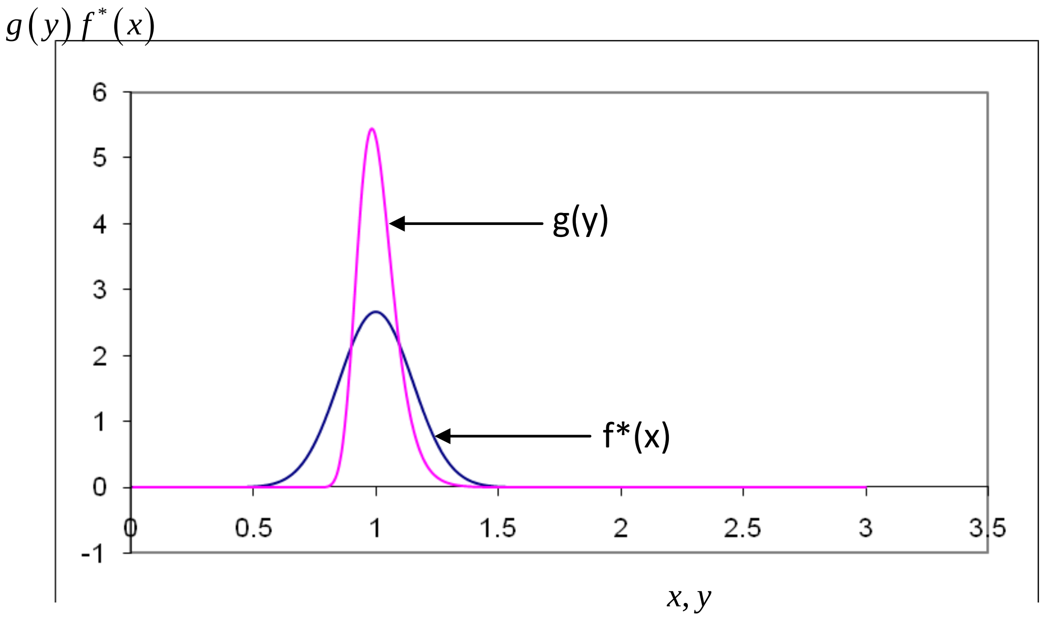


Figure 3.4. Curve Shapes for σ = 0.15

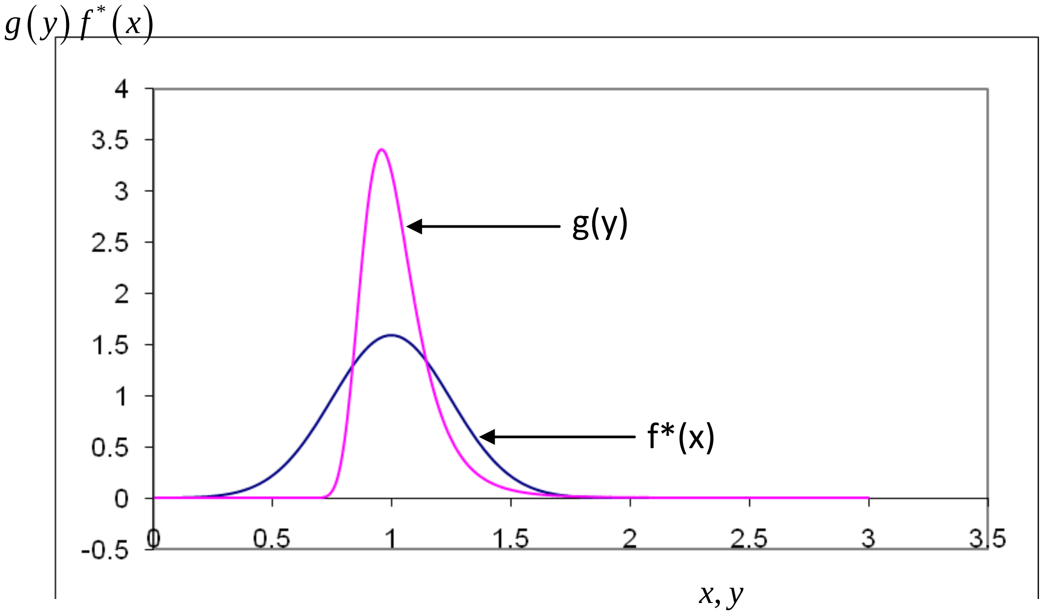


Figure 3.5. Curve Shapes for σ = 0.25

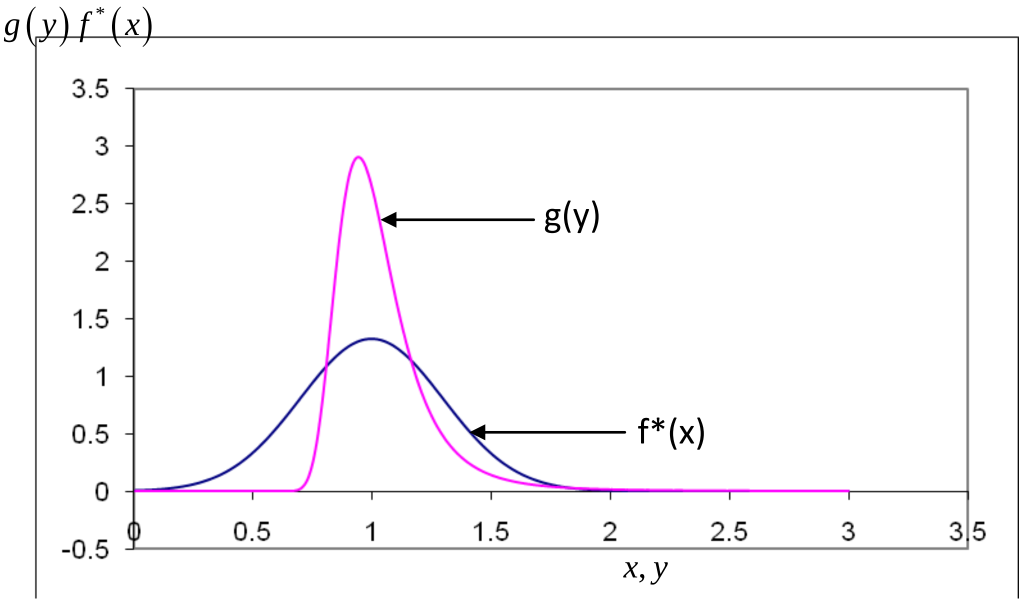


Figure 3.6. Curve Shapes for σ = 0.30

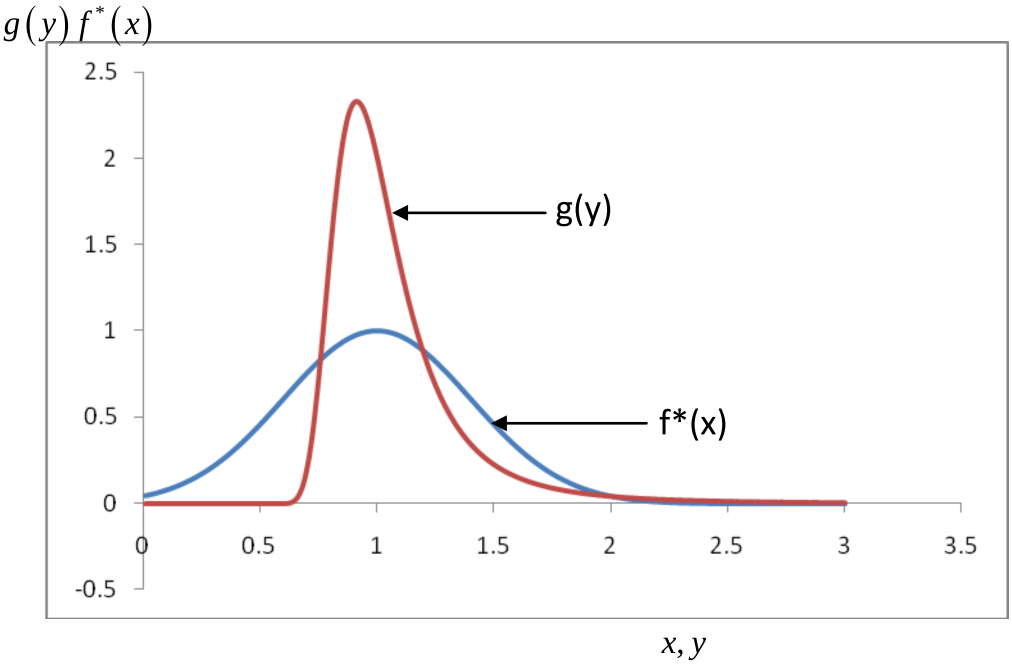


Figure 3.7. Curve Shapes for σ = 0.40

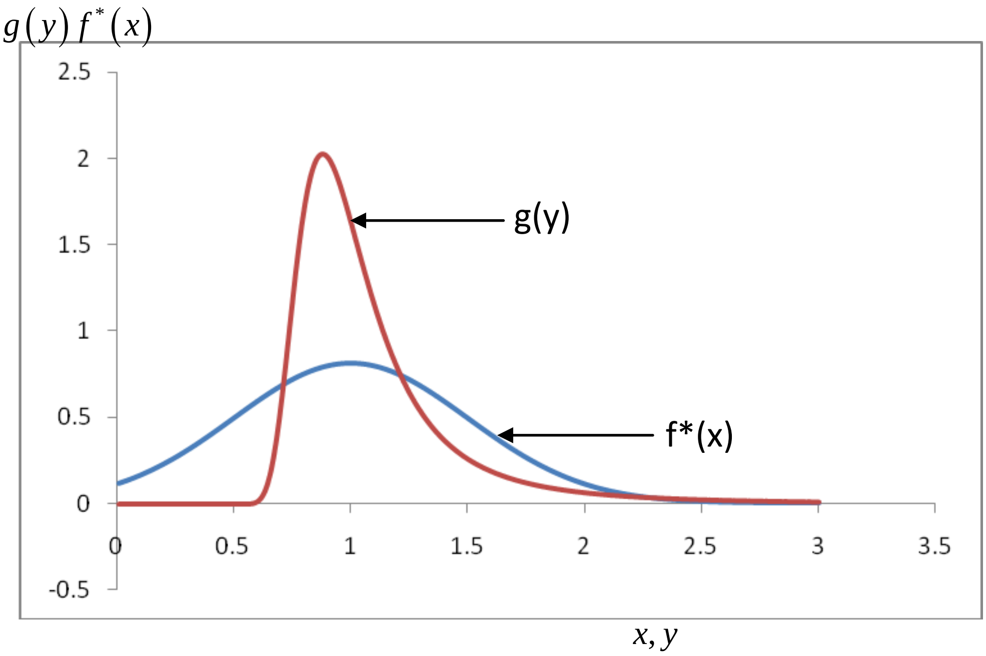


Figure 3.8. Curve Shapes for σ = 0.50

We observe a significant departure from normality to positive skewness of the curve of g(y) for larger values of σ for example for σ = 0.25, 0.3, 0.40 and 0.50 (Figures 3.5, 3.6, 3.7 and 3.8 respectively). However the exact point at which there is a departure from normality of the curves are yet to be investigated in subsequent sections.

From Figures 3.1 through 3.8, it can be observed that the curve g(y) has one maximum point for all values of σ. Hence, we obtain the values of **σ** that satisfy the normality condition using Rolle's Theorem. (Smith and Minton, 2008) which states that, if $f(x)$ is continuous on the interval [a, b] and differentiable on the interval (a, b) with f (a) =f (b), then, there exists a number $c \in (a, b)$ such that $f'(c) = 0$.

Now,

$$g(y) = \frac{2}{y^3 \sigma \sqrt{2\pi}\left[1 - \Phi\left(-\frac{1}{\sigma}\right)\right]} e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}$$

$$g(y) = \frac{2}{y^3 \sigma \sqrt{2\pi}\left[1 - \Phi\left(-\frac{1}{\sigma}\right)\right]} e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2} \quad (3.3)$$

$$= \frac{2}{\sigma \sqrt{2\pi}\left[1 - \Phi\left(-\frac{1}{\sigma}\right)\right]} \left[y^{-3} e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}\right]$$

We shall show that g is continuous as:

(i) $g(y_0)$ is defined for $y_0 \in (0, \infty)$

(ii) $\lim_{y \to y_0} g(y)$ exists

(iii) $\lim_{y \to y_0} g(y) = g(y_0)$

Since g is continuous in $(0, \infty)$. Taking the derivative of g

$$g'(y)=\frac{2}{\sigma\sqrt{2\pi\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}}\left[-3y^{-4}e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}+\frac{2}{\sigma^2y^6}\left(\frac{1}{y^2}-1\right)e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}\right]$$

$$=\frac{2}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\left[\frac{-3}{y^4}e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}+\frac{2}{\sigma^2y^6}\left(\frac{1}{y^2}-1\right)e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}\right] \quad (3.4)$$

$$=\frac{2e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}}{\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\left[\frac{-3}{y^4}+\frac{2}{\sigma^2y^6}\left(\frac{1}{y^2}-1\right)\right]$$

$$g'(y)=\frac{2e^{\frac{-1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[\frac{-3}{y^4}+\frac{2}{\sigma^2y^8}-\frac{2}{\sigma^2y^6}\right]$$

$$=\frac{2e^{\frac{-1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[\frac{2(1-y^2)}{\sigma^2y^8}-\frac{3}{y^4}\right]$$

Thus, $$\boldsymbol{g}'(\boldsymbol{y})=\frac{2e^{\frac{-1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}}{\sigma\sqrt{2\pi}\left[1-\Phi\left(\frac{-1}{\sigma}\right)\right]}\left[\frac{2(1-y^2)}{\sigma^2y^8}-\frac{3}{y^4}\right] \quad (3.5)$$

By Roll's theorem; $\boldsymbol{g}'(\boldsymbol{y})=\mathbf{0}$,

$$\frac{2(1-y^2)}{\sigma^2y^8}-\frac{3}{y^4}=0$$

$$2(1-y^2)-3\sigma^2y^4=0$$

$$2-2y^2-3\sigma^2y^4=0$$

$$3\sigma^2y^4+2y^2-2=0 \quad (3.6)$$

If we let $w = y^2$ in Equation (3.6), we have

$$3\sigma^2 w^2 + 2w - 2 = 0 \tag{3.7}$$

Solving Equation 3.7, we obtain

$$w = \frac{-1 \pm \sqrt{1 + 6\sigma^2}}{3\sigma^2}$$

Hence,

$$y = \pm\sqrt{\frac{-1 + \sqrt{1 + 6\sigma^2}}{3\sigma^2}}$$

and

$$y_{max} = \sqrt{\frac{-1 + \sqrt{1 + 6\sigma^2}}{3\sigma^2}}$$

Therefore, the probability density function g(y) attains its maximum at y_{max}.

The computation of $y_{max.}$ for various values of standard deviation σ is shown in Table 3.1 in the Appendix stipulated $y_{max} \approx 1$.

The summary of the values of $y_{max \cong 1}$ for different values of σ is given in Table 3.2. (see Appendix on page 89).

This shows that g(y) is symmetric about 1 with Mean $\approx$ median $\approx$ Mode ≈ 1 when $0 < \sigma < 0.045$ (for three decimal places) and $0 < \sigma < 0.145$ (for 1 decimal place).

3.5 3-D PLOT OF $f(x)$ AND $g(y)$

Furthermore, in order to ascertain the interval, $\sigma \in (0, 0.15]$, we make a 3-D plot of the untransformed ($f^*(x)$) and the inverse-square-root transformed ($g(y)$) distributions for values of σ = 0.01, 0.02, 0.03, . . . , 0.48, 0.49, 0.50 and for fixed values of $X = Y = 0.00, 0.1, 0.2, \ldots, 1.1$.

The plot is given in Figure 3.9. The aim of the plot is to determine the point at which normality exists among the variables. Based on the property of Gaussian distribution, the point at which bell-shape is spotted on the graph is the point at which normality exists.

Based on the plot, normality was found to exist for $\sigma \leq 0.15$

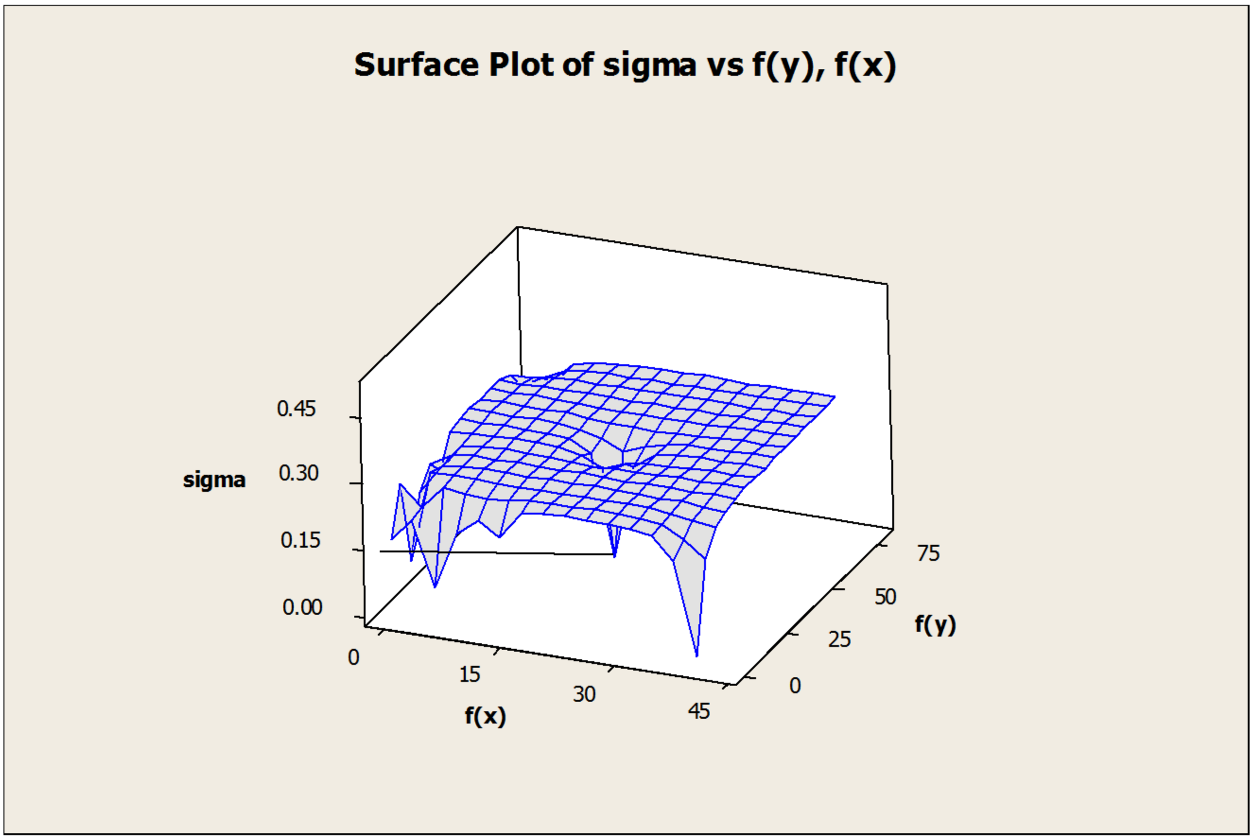


Figure 3.9: 3-D Plots of f*(x) and g(y) for fixed values of σ, X and Y

3.6 USE OF SIMULATED ERROR TERMS

The region where the following conditions;

(i) Mean = Median ≅ 1.0

(ii) Normality is accepted using Anderson Darling test are satisfied,

was obtained using the simulated data generated from $N\left(1,\sigma^2\right)$ for e_t .

Subsequently, transform to obtain $e_t^* = \frac{1}{\sqrt{e_t}}$ for $0.05 \le \sigma \le 0.15$

Values of the required statistical characteristics were obtained for each variable e_t and e_t^* as shown in Tables 3.3 to 3.7 respectively. For each configuration of (n = 100, 0.05 $\le \sigma \le 0.15$), 1000 replications were performed for values σ in steps of 0.005. For want of space the results of the first 25 replications are shown for the configurations, (n = 100, σ = 0.06), (n=100,σ=0.08), (n=100,σ=0.1), (n=100,σ=0.15), (n=100,σ=0.15) and (n=100,σ=0.2). Since the values of skewness $\left(\gamma_1\right)$ and kurtosis $\left(\gamma_2\right)$ are not sufficient to confirm normality; we shall use Anderson Darling's test for normality.

Summary of results:

The following results were obtained from the investigations carried out on $e_t^* = \frac{1}{\sqrt{e_t}}$.

1. The curve shapes are normally distributed about 1 for $\sigma \le 0.15$.

2. By Rolles theorem,

   a. Mode $\approx 1 \approx$ Mean for $\sigma \le 0.045$ to 2 decimal places

   b. Mode $\approx 1 \approx$ Mean for $\sigma \le 0.145$ to 1 decimal place

3. From the simulated random errors, when $\sigma \le 0.2$

   a. Median $\approx$ Mean $\approx$ 1

   b. $Var(e_t^*) \approx \frac{1}{4} Var(e_t)$

4. The p-value of the Anderson Darling's test statistic strongly supports the non-normality of e_t^* at $\sigma < 0.15$.

From the results of the investigations of the distributions of the error term (e_t) of the multiplicative time series model and the inverse square root transformed error term (e_t^*) it is clear that the condition for successful inverse square root transformation is $\sigma < 0.15$. Thus the two error terms e_t and e_t^* are normally distributed with mean 1 but with the variance of inverse square root transformed error term being one quarter of the variance of the untransformed error component when σ < 0.15. That is $Var\left(e_t^*\right) = \frac{1}{4}\ Var\left(e_t\right)$, for σ < 0.15.

This is the same relationship obtained by Otuonye et al. (2011) under square root transformation.

3.7 THE MEAN AND VARIANCE OF TRANSFORMED ERROR TERM, Y

By definition, the mean of Y, E(Y) is given by:

$$E(y) = \int_0^\infty y g(y) dy, \qquad (3.8)$$

$$= \int_0^\infty y \frac{2e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}}{\sigma\sqrt{2\pi}\left[1-\phi\left(\frac{1}{\sigma}\right)\right]y^3} dy$$

$$= \frac{2}{\sigma\sqrt{2\pi}\left[1-\phi\left(\frac{1}{\sigma}\right)\right]} \int_0^\infty \frac{e^{\frac{-1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}}{y^2} dy$$

$$let\ u = \frac{1}{y^2},\ \ then\ y = \frac{1}{u^{1/2}}\ and\ dy = \frac{-du}{2u^{3/2}}, \qquad with\ k = \frac{2}{\sigma\sqrt{2\pi}\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}$$

$$\therefore E(Y) = k\int_0^\infty \frac{e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}}{y^2} dy \ = k\int_\infty^0 u \frac{e^{-\frac{1}{2}\left(\frac{u-1}{\sigma}\right)^2}}{2u^{\frac{3}{2}}} - du$$

$$= \frac{-k}{2}\int_\infty^0 u^{-\frac{1}{2}} e^{-\frac{1}{2}\left(\frac{u-1}{\sigma}\right)^2} du$$

$$let\ z = \frac{u-1}{\sigma}, then\ z\,\sigma + 1 = u\ and\ \frac{du}{dz} = \sigma\ then\ du = \sigma dz\ \ for\ \frac{-1}{\sigma} < z < \infty$$

$$\therefore E(Y) = \frac{k}{2}\int_{\frac{-1}{\sigma}}^{\infty}(1+z\sigma)^{-\frac{1}{2}}e^{-\frac{1}{2}z^2}\,\sigma dz \;\; = \frac{\sigma k}{2}\int_{\frac{-1}{\sigma}}^{\infty}(1+z\sigma)^{-\frac{1}{2}}e^{-\frac{z^2}{2}}\,dz \qquad (3.9)$$

Using negative Binomial expansion theorem to expand the term $(1+z\,\sigma)^{\frac{-1}{2}}$

$$\therefore (1+z\,\sigma)^{\frac{-1}{2}} = 1 + \frac{(\frac{-1}{2})z\,\sigma}{1!} + \frac{\frac{-1}{2}(\frac{-1}{2}-1)(z\,\sigma)^2}{2!} + \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)(\frac{-1}{2}-2)(z\,\sigma)^3}{3!} + \ldots$$

$$= 1 - \frac{z\,\sigma}{2} + \frac{3(z\,\sigma)^2}{8} - \frac{15(z\,\sigma)^3}{48} + \ldots \qquad (3.10)$$

Since $E(Y) = \frac{\sigma k}{2}\int_{\frac{-1}{\sigma}}^{\infty}(1+z\sigma)^{-\frac{1}{2}}e^{-\frac{z^2}{2}}\,dz$

$$= \frac{\sigma k}{2}\int_{\frac{-1}{\sigma}}^{\infty}\left[1 - \frac{z\,\sigma}{2} + \frac{3(z\,\sigma)^2}{8} - \frac{15(z\,\sigma)^3}{48} + \ldots\right]\; e^{\frac{-z^2}{2}}dz \qquad (3.11)$$

$$= \frac{\sigma}{2}\frac{2}{\sigma\sqrt{2\pi}\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}\left[\int_{\frac{-1}{\sigma}}^{\infty} e^{\frac{-z^2}{2}}dz - \int_{\frac{-1}{\sigma}}^{\infty}\frac{z\,\sigma}{2}e^{\frac{-z^2}{2}}dz + \int_{\frac{-1}{\sigma}}^{\infty}\frac{3(z\,\sigma)^2}{8}e^{\frac{-z^2}{2}}dz - \int_{\frac{-1}{\sigma}}^{\infty}\frac{15(z\,\sigma)^3}{48}e^{\frac{-z^2}{2}}dz + ---\right]$$

$$= \frac{1}{\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}\left[\frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} e^{\frac{-z^2}{2}}dz - \frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{z\,\sigma}{2}e^{\frac{-z^2}{2}}dz + \frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{3(z\,\sigma)^2}{8}e^{\frac{-z^2}{2}}dz - \frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{15(z\,\sigma)^3}{48}e^{\frac{-z^2}{2}}dz + ---\right]$$

$$E(Y) = \frac{1}{\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}\left[\frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} e^{\frac{-z^2}{2}}dz - \frac{\sigma}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{z}{2}e^{\frac{-z^2}{2}}dz + \frac{\sigma^2}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{3z^2}{8}e^{\frac{-z^2}{2}}dz - \frac{\sigma^3}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{15z^3}{48}e^{\frac{-z^2}{2}}dz + \ldots\right] \qquad (3.12)$$

$\therefore$ *Evaluating each term in Equation (3.12, we have:*

$$\frac{\sigma}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{z}{2}e^{\frac{-z^2}{2}}dz = \frac{\sigma}{2\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} z\,e^{\frac{-z^2}{2}}dz$$

$$let\ y = z^2\ \ and\ \ dz = \frac{1}{2}y^{\frac{-1}{2}}dy$$

$$\frac{\sigma}{2\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} z\,e^{\frac{-z^2}{2}}dz = \frac{\sigma}{2\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} y^{\frac{1}{2}}e^{\frac{-y}{2}}\frac{1}{2}y^{\frac{-1}{2}}dy = \frac{\sigma}{2\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{e^{\frac{-y}{2}}}{2}dy = \frac{\sigma}{4\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} e^{\frac{-y}{2}}dy$$

$$= \frac{\sigma}{4\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}} \qquad (3.13)$$

$$\frac{\sigma^2}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{3z^2}{8}e^{\frac{-z^2}{2}}dz = \frac{3\sigma^2}{8\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} z^2\,e^{\frac{-z^2}{2}}dz = \frac{3\sigma^2}{8\sqrt{2\pi}}\left[\int_{\frac{-1}{\sigma}}^{0} z^2 e^{\frac{-z^2}{2}}dz + \int_{0}^{\infty} z^2 e^{\frac{-z^2}{2}}dz\right]$$

$$note\ that\ \int_{0}^{\infty} z^2 e^{\frac{-z^2}{2}}dz = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}},\ \ and\ \ \int_{\frac{-1}{\sigma}}^{0} z^2 e^{\frac{-z^2}{2}}dz = \frac{-1}{\sigma}e^{\frac{-1}{2\sigma^2}} + \frac{1}{2}\int_{0}^{\frac{1}{\sigma^2}} y^{\frac{-1}{2}}e^{\frac{-y}{2}}dy$$

Hence,

$$\frac{3\sigma^2}{8\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} z^2\,e^{\frac{-z^2}{2}}dz = \frac{3\sigma^2}{8\sqrt{2\pi}}\left[\frac{-1}{\sigma}\,e^{\frac{-1}{2\sigma^2}} + \frac{1}{2}\int_0^{\frac{1}{\sigma^2}} y^{\frac{-1}{2}}\,e^{\frac{-y}{2}}\,dy + \sqrt{\frac{\pi}{2}}\right] = \frac{3\sigma^2}{8\sqrt{2\pi}}\left[\sqrt{\frac{\pi}{2}} - \frac{e^{\frac{-1}{2\sigma^2}}}{\sigma} + \frac{1}{2}\int_0^{\frac{1}{\sigma^2}} y^{\frac{1}{2}-1}\,e^{\frac{-y}{2}}\,dy\right]$$

$$= \frac{3}{8}\left[\frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}} + \frac{\sigma^2}{\sqrt{2\pi}}.\frac{1}{2}\int_0^{\frac{1}{\sigma^2}} y^{\frac{1}{2}-1}\,e^{\frac{-y}{2}}dy\right] = \frac{3}{8}\left[\frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}} + \frac{\sigma^2}{2}.Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right]$$

$$= \frac{3\sigma^2}{16} - \frac{3\sigma e^{\frac{-1}{2\sigma^2}}}{8\sqrt{2\pi}} + \frac{3\sigma^2}{16}.Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right) \qquad (3.14)$$

Also,

$$\frac{\sigma^3}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{15z^3}{48}e^{\frac{-z^2}{2}}dz = \frac{15\sigma^3}{48\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} z^3\, e^{\frac{-z^2}{2}}dz$$

$$let\ p = z^2,\quad dz = \frac{dp}{2z},\quad \frac{1}{\sigma^2} < p < \infty$$

$$= \int_{\frac{1}{\sigma^2}}^{\infty} pz\, e^{\frac{-p}{2}}\frac{dp}{2z} = \frac{1}{2}\int_{\frac{1}{\sigma^2}}^{\infty} p\, e^{\frac{-p}{2}}dp = -pe^{\frac{-p}{2}} - 2e^{\frac{-p}{2}}\Big|_{\frac{1}{\sigma^2}}^{\infty} = \frac{1}{\sigma^2}e^{\frac{-1}{2\sigma^2}} + 2e^{\frac{-1}{2\sigma^2}}$$

$$\int_{\frac{-1}{\sigma}}^{\infty} z^3\, e^{\frac{-z^2}{2}}dz = \frac{1}{\sigma^2}e^{\frac{-1}{2\sigma^2}} + 2e^{\frac{-1}{2\sigma^2}}$$

$$\therefore \frac{15\sigma^3}{48\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} z^3\, e^{\frac{-z^2}{2}}dz = \frac{15\sigma^3}{48\sqrt{2\pi}}\left[\frac{1}{\sigma^2}e^{\frac{-1}{2\sigma^2}} + 2e^{\frac{-1}{2\sigma^2}}\right] = \frac{15\sigma^3}{48\sqrt{2\pi}}\cdot\frac{1}{\sigma^2}e^{\frac{-1}{2\sigma^2}} + \frac{30\sigma^3}{48\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}}$$

$$= \frac{15\sigma}{48\sqrt{2\pi}}\cdot e^{\frac{-1}{2\sigma^2}} + \frac{30\sigma^3}{48\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}} \qquad (3.15)$$

Thus, using equations (3.13), (3.14) and (3.15) in equation (3.12) we have

$$E(Y) = \frac{1}{\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}\Bigg[\frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} e^{\frac{-z^2}{2}}dz - \frac{\sigma}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{z}{2}e^{\frac{-z^2}{2}}dz + \frac{\sigma^2}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{3z^2}{8}e^{\frac{-z^2}{2}}dz$$
$$- \frac{\sigma^3}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty}\frac{15z^3}{48}e^{\frac{-z^2}{2}}dz + ---\Bigg]$$

$$= \frac{1}{\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}\Bigg[Pr\left(z > \frac{-1}{\sigma}\right) - \frac{\sigma}{4\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}} + \frac{3\sigma^2}{16} - \frac{3\sigma e^{\frac{-1}{2\sigma^2}}}{8\sqrt{2\pi}} + \frac{3\sigma^2}{16}.Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)$$
$$- \frac{15\sigma}{48\sqrt{2\pi}}\cdot e^{\frac{-1}{2\sigma^2}} - \frac{30\sigma^3}{48\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}} + \ldots\Bigg]$$

$$= \frac{1}{\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}\left[\left[1-\phi\left(\frac{-1}{\sigma}\right)\right] - \frac{15\sigma}{16\sqrt{2\pi}}\,.\,e^{\frac{-1}{2\sigma^2}} + \frac{3\sigma^2}{16}\left[1+Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right] - \frac{5\sigma^3}{8\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}} + \ldots\right]$$

$$\therefore\ E(Y) = 1 - \frac{15\sigma e^{\frac{-1}{2\sigma^2}}}{16\sqrt{2\pi}\left[1-\phi\left(\frac{1}{\sigma}\right)\right]} + \frac{3\sigma^2}{16\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}\left[1+Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right]$$

$$-\frac{5\sigma^3 e^{-\frac{1}{2\sigma^2}}}{8\sqrt{2\pi}\left[1-\phi\left(\frac{1}{\sigma}\right)\right]} + \ldots \tag{3.16}$$

Now,

$$Var\ (Y) = E(Y^2) - [E(Y)]^2 \tag{3.17}$$

where

$$E(Y^2) = \int y^2 g(y)dy \tag{3.18}$$

$$= \int_0^\infty y^2 . \frac{2}{\sigma\sqrt{2\pi}\left[1-\phi\left(\frac{1}{\sigma}\right)\right]y^3} e^{\frac{-1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2} dy$$

$$= \int_0^\infty \frac{2e^{\frac{-1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}}{\sigma\sqrt{2\pi}\left[1-\phi\left(\frac{1}{\sigma}\right)\right]y} dy = \frac{2}{\sigma\sqrt{2\pi}\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}\int_0^\infty \frac{e^{\frac{-1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}}{y} dy$$

$$let\ u = \frac{1}{y^2}\ then\ du = \frac{-2}{y^3}dy, \quad and \quad dy = \frac{-du}{2u^{\frac{3}{2}}}\quad with\ \ k = \frac{2}{\sigma\sqrt{2\pi}\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}$$

$$\therefore\ E(Y^2) = k\int_0^\infty \frac{1}{u^{\frac{1}{2}}} e^{\frac{-1}{2\sigma^2}(u-1)^2} \frac{-du}{2u^{\frac{3}{2}}}$$

$$= k\int_0^{\infty} -u^{\frac{1}{2}}.\frac{u^{\frac{-3}{2}}}{2}\, e^{\frac{-1}{2}\left(\frac{u-1}{\sigma}\right)^2}\, du \; = \frac{k}{2}\int_{\infty}^{0} u^{-2}.\,e^{\frac{-1}{2}\left(\frac{u-1}{\sigma}\right)^2}\, du$$

$$let\ z = \frac{u-1}{\sigma}\ and\ u = z\sigma + 1\ then\ du = \ \sigma dz, \quad and \quad \frac{-1}{\sigma} < z < \infty$$

$$E(Y^2) = \frac{k}{2}\int_{\infty}^{0} u^{-2}.\,e^{\frac{-1}{2}\left(\frac{u-1}{\sigma}\right)^2}\, du$$

$$= \frac{k}{2}\int_{\frac{-1}{\sigma}}^{\infty} (1+z\sigma)^{-2}.\,e^{\frac{-z^2}{2}}\,\sigma dz \; = \frac{\sigma k}{2}\int_{\frac{-1}{\sigma}}^{\infty} (1+z\sigma)^{-2}.\,e^{\frac{-z^2}{2}}\, dz \qquad (3.19)$$

Using negative Binomial expansion on $(1+z\,\sigma)^{-2}$, defined as

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ \ldots$$

$$\therefore\ (1+z\,\sigma)^{-2} \; = 1 + \frac{(-2)z\,\sigma}{1!} + \frac{-2(-2-1)(z\,\sigma)^2}{2!} + \frac{-2(-2-1)(-2-2)(z\,\sigma)^3}{3!} + \ \ldots$$

$$= 1 - 2z\,\sigma + 3(z\,\sigma)^2 - \ 4(z\,\sigma)^3 + . \; . \; .$$

$$since\ E(Y^2) = \frac{\sigma k}{2}\int_{\frac{-1}{\sigma}}^{\infty} (1+z\,\sigma)^{-2}\, e^{\frac{-z^2}{2}} dz\, . \; . \; .$$

$$= \frac{\sigma k}{2}\int_{\frac{-1}{\sigma}}^{\infty} [1 - 2z\,\sigma + 3(z\,\sigma)^2 - \ 4(z\,\sigma)^3 + \ \ldots] \quad e^{\frac{-z^2}{2}} dz$$

$$= \frac{\sigma}{2}\frac{2}{\sigma\sqrt{2\pi}\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}\left[\int_{\frac{-1}{\sigma}}^{\infty} e^{\frac{-z^2}{2}}dz - \int_{\frac{-1}{\sigma}}^{\infty} 2z\,\sigma\, e^{\frac{-z^2}{2}}dz + \int_{\frac{-1}{\sigma}}^{\infty} 3(z\,\sigma)^2\, e^{\frac{-z^2}{2}}dz - \int_{\frac{-1}{\sigma}}^{\infty} 4(z\,\sigma)^3\, e^{\frac{-z^2}{2}}dz + \ \ldots\right]$$

$$E(Y^2) = \frac{1}{\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}\left[\frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} e^{\frac{-z^2}{2}}dz - \frac{2\,\sigma}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} z\, e^{\frac{-z^2}{2}}dz + \frac{3\sigma^2}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} z^2\, e^{\frac{-z^2}{2}}dz - \frac{4\sigma^3}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} z^3\, e^{\frac{-z^2}{2}}dz + \ \ldots\right] \quad (3.20)$$

Evaluating the terms in Equation (3.20) we have

$$\frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{0} e^{\frac{-z^2}{2}} = \ Pr\left(z > \frac{-1}{\sigma}\right) \quad (3.21)$$

$$\frac{2\,\sigma}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{0} z\, e^{\frac{-z^2}{2}}dz = \frac{2\,\sigma}{\sqrt{2\pi}}\left(\frac{e^{\frac{-1}{2\sigma^2}}}{2}\right) = \frac{\sigma}{\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}} \quad (3.22)$$

$$\frac{3\sigma^2}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} z^2\, e^{\frac{-z^2}{2}}dz \ = \frac{3\sigma^2}{8\sqrt{2\pi}}\left[\sqrt{\frac{\pi}{2}} - \frac{e^{\frac{-1}{2\sigma^2}}}{\sigma} + \frac{1}{2}\int_0^{\frac{1}{\sigma^2}} y^{\frac{1}{2}-1}\, e^{\frac{-y}{2}}dy\right]$$

$$= \frac{3\sigma^2}{2} - \frac{3\sigma e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}} + \frac{3\sigma^2}{\sqrt{2\pi}}\,.\frac{1}{2}\int_0^{\frac{1}{\sigma^2}} y^{\frac{1}{2}-1}\, e^{\frac{-y}{2}}dy$$

$$= \frac{3\sigma^2}{2} - \frac{3\sigma e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}} + \frac{3\sigma^2}{4}\ Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right) \quad (3.23)$$

and

$$\frac{4\sigma^3}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} z^3 e^{\frac{-z^2}{2}} dz = \frac{4\sigma^3}{\sqrt{2\pi}}\left[\frac{1}{\sigma^2}e^{\frac{-1}{2\sigma^2}} + 2e^{\frac{-1}{2\sigma^2}}\right] = \frac{4\sigma e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}} + \frac{8\sigma^3 e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}} \qquad (3.24)$$

Using Equations (3.21) (3.22), (3.23) and (3.24) in Equation (3.20) we have:

$$E(Y^2) = \frac{1}{\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}\left[\frac{1}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} e^{\frac{-z^2}{2}} dz - \frac{2\,\sigma}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} z\, e^{\frac{-z^2}{2}} dz + \frac{3\sigma^2}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} z^2 e^{\frac{-z^2}{2}} dz \right.$$
$$\left. - \frac{4\sigma^3}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma}}^{\infty} z^3 e^{\frac{-z^2}{2}} dz + \ldots\right]$$

$$= \frac{1}{\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}\left[Pr\left(z > \frac{-1}{\sigma}\right) - \frac{\sigma}{\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}} + \frac{3\sigma^2}{2} - \frac{3\sigma e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}} + \frac{3\sigma^2}{4}\, Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right.$$
$$\left. - \frac{4\sigma e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}} - \frac{8\sigma^3 e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}} + \ldots\right]$$

$$= \frac{1}{\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}\left[\left(1-\phi\left(\frac{1}{\sigma}\right)\right) - \frac{8\sigma}{\sqrt{2\pi}}e^{\frac{-1}{2\sigma^2}} + \frac{3\sigma^2}{4}\left[1 + Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right] - \frac{8\sigma^3 e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}} + \ldots\right]$$

$\therefore$

$$E(Y^2) = 1 - \frac{8\sigma}{\sqrt{2\pi}\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}e^{\frac{-1}{2\sigma^2}} + \frac{3\sigma^2}{4\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}\left[1 + Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right] - \frac{8\sigma^3 e^{\frac{-1}{2\sigma^2}}}{\sqrt{2\pi}\left[1-\phi\left(\frac{1}{\sigma}\right)\right]} + \cdots \qquad (3.25)$$

We observe that

(i) Subsequent terms in the series for $E(Y)$ *and* $E(Y^2)$ respectively all have $e^{\frac{-1}{2\sigma^2}}$ as a factor.

(ii) $e^{\frac{-1}{2\sigma^2}} = 0$ for $\sigma \leq 0.22$ correct to 4 decimal places. (See Table 3.7. column 3 in the Appendix).

(iii) Conditions (i) and (ii) above imply that all subsequent terms for $E(Y)$ *and* $E(Y^2)$ are all zeros for $\sigma \leq 0.22$.

Thus,

$$E(Y) = 1 + \frac{3\sigma^2}{16\left[1-\phi\left(\frac{-1}{\sigma}\right)\right]}\left[1 + Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right] \quad for \quad \sigma \le 0.22 \tag{3.26}$$

and

$$E(Y^2) = 1 + \frac{\sigma^2}{2\left[1-\phi\left(\frac{-1}{\sigma}\right)\right]}\left[1 + Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right] \quad for \quad \sigma \le 0.22$$

Thus

$$Var(Y) = 1 + \frac{\sigma^2}{2\left[1-\phi\left(\frac{1}{\sigma}\right)\right]}\left[1 + Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right] - \left[1 + \frac{3\sigma^2}{16\left[1-\phi\left(\frac{-1}{\sigma}\right)\right]}\left[1 + Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right]\right]^2$$

$$= \mathbf{Var(Y)} = \frac{9\sigma^2\left(1+\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right)}{8[1-\Phi\left(-\frac{1}{\sigma}\right)]}\left\{1 - \frac{\sigma^2\left(1+\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right)}{32[1-\Phi\left(-\frac{1}{\sigma}\right)]}\right\} \tag{3.27}$$

3.8 THE NUMERICAL COMPUTATIONS OF THE MEAN AND VARIANCE OF THE PROBABILITY DENSITY FUNCTION G(Y) OF THE INVERSE SQUARE ROOT TRANSFORMED ERROR TERM.

Using the expressions for $E(Y)$and $Var\ (Y)$ in Equations 3.26 and 3.27 respectively, we now compute values of $E(Y)$ and $Var(Y)$ for $\sigma \in [0.01,\ 0.22]$.

Table 3.7 (Appendix) shows the values of $E(Y)$, and Var(Y). For this computation we write

$$E(Y) = 1 + \frac{3\sigma^2 B}{16A}$$

and

$$\mathbf{Var(Y)} = \frac{\mathbf{9\sigma^2}}{\mathbf{8A}}\left(\mathbf{1} - \frac{\mathbf{\sigma^2 B}}{\mathbf{32A}}\right)$$

where $A = \left[1 - \phi\left(\frac{1}{\sigma}\right)\right]$

$$B = \left[1 + Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right]$$

From Table 3.7(appendix), columns 4 and 5: A = 1 and B = 2 for σ< 0.22, therefore the mean E(Y) and the Var (Y) can equivalently be obtained using the expressions

$$\therefore \quad E(Y) = 1 + \frac{3\sigma^2}{8}, \quad \sigma < 0.22 \tag{3.28}$$

and $$\mathbf{Var}(\mathbf{Y}) = \frac{\mathbf{9\sigma^2}}{\mathbf{8}} - \frac{\mathbf{1}}{\mathbf{4}}\left(\frac{\mathbf{3\sigma^2}}{\mathbf{8}}\right)^{\mathbf{2}} \sigma < 0.22. \tag{3.29}$$

as confirmed by the simulated data in Tables 3.3 to 3.6 (Appendix)

From Table 3.7 in the Appendix, we observe the following within feasible region of investigation, σ ≤ 0.22

i. $E\left(e_t^*\right) = 1$ correct to 1 decimal place (d.p) when σ ≤ 0.22

ii. $Var\left(e_t^*\right) = \frac{1}{4} Var\left(e_t\right)$ Correct to the nearest whole number within the feasible region of study, σ ≤ 0.22.

3.9 MEDIAN OF THE INVERSE- SQUARE-ROOT TRANSFORMED ERROR COMPONENT

Having shown in Section 3.4 using the Rolles Theorem that the mode ≈ mean ≈ 1.0 (to 1 dp) for $0 < \sigma \leq 0.15$, hence we wish to find the region where the value of median is also one since the measures of central tendency, mean, median and mode are equal for a normally distributed random variable.

The median of a random variable Y of the discrete or continuous type is a value m such that $p_r(Y < m) \leq \frac{1}{2}$. Recall that for a continuous random variable y, we have the relation. $p_r(Y < m) = p_r(y \leq m)$, since $p_r(Y = m) = 0$. This implies that if y is continuous and $p_r(Y < m) = p_r(y \leq m) = \frac{1}{2}$, then m is the median of Y.

Thus, we say that m is the median of Y if and only if

i. $\sum_{y<m} p(y) \leq \frac{1}{2}$ when Y is discrete,

ii. $F(m) = P(Y \leq m) = \int_{-\infty}^{m} P(y)dy = \frac{1}{2}$ when Y is continuous.

$$g(y) = \frac{2e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}}{y^3\sigma\sqrt{2\pi}\left(1-\phi\left(-\frac{1}{\sigma}\right)\right)}$$

$$\therefore F(m) = P(y \leq m) = \int_{0}^{m} g(y)dy = \frac{1}{2}$$

$$\int_0^m \frac{2}{\sigma\sqrt{2\pi}\left(1-\phi\left(-\frac{1}{\sigma}\right)\right)y^3} e^{-\frac{1}{2}\left(\frac{y^{-2}-1}{\sigma}\right)^2} dy = \frac{1}{2}$$

$$\Rightarrow \qquad \frac{2}{\sigma\sqrt{2\pi}\left(1-\phi\left(-\frac{1}{\sigma}\right)\right)} \int_0^m y^{-3} e^{-\frac{1}{2}\left(\frac{y^{-2}-1}{\sigma}\right)^2} dy = \frac{1}{2} \tag{3.30}$$

Let

$$u = \frac{y^{-2}-1}{\sigma} = \frac{\frac{1}{y^2}-1}{\sigma} \tag{3.31}$$

therefore, when

$$y = 0,\ u = \infty \text{ and when } y = m,\ u = \frac{m^{-2}-1}{\sigma} \tag{3.32}$$

Thus,

$$y = (\sigma u + 1)^{-1/2} \tag{3.33}$$

and

$$dy = -\frac{1}{2}\sigma(\sigma u + 1)^{-3/2} du \tag{3.34}$$

Now,

$$F(m) = \frac{2}{\sigma\sqrt{2\pi}\left(1-\phi\left(-\frac{1}{\sigma}\right)\right)}\left(-\frac{1}{2}\sigma\right) \int_{\infty}^{\frac{m^{-2}-1}{\sigma}} (\sigma u + 1)^{-3/2} e^{-u^2/2} (\sigma u + 1)^{3/2} du = \frac{1}{2}$$

$$F(m) = \frac{1}{\sqrt{2\pi}\left(1-\phi\left(-\frac{1}{\sigma}\right)\right)} \int_{\frac{m^{-2}-1}{\sigma}}^{\infty} e^{-u^2/2}\, du = \frac{1}{2} \tag{3.35}$$

which implies that

$$\frac{1}{\left(1-\phi\left(-\frac{1}{\sigma}\right)\right)} P\left(u > \frac{m^{-2}-1}{\sigma}\right) = \frac{1}{2}$$

$$= \frac{1}{\left(1-\Phi\left(-\frac{1}{\sigma}\right)\right)} \left(1-P\left(u < \frac{m^{-2}-1}{\sigma}\right)\right) = \frac{1}{2}$$

$$= \frac{1-\Phi\left(\frac{m^{-2}-1}{\sigma}\right)}{1-\Phi\left(-\frac{1}{\sigma}\right)} = \frac{1}{2} \tag{3.36}$$

where $= P\left(u < \frac{m^{-2}-1}{\sigma}\right) = \Phi\left(\frac{m^{-2}-1}{\sigma}\right)$, therefore from (3.36) we obtain

$$\phi\left(\frac{m^{-2}-1}{\sigma}\right) = \frac{1}{2}\left(1+\phi\left(\frac{-1}{\sigma}\right)\right) \tag{3.37}$$

In (3.37), we are interested in the values of σ for which $\boldsymbol{\Phi}\left(\boldsymbol{-\frac{1}{\sigma}}\right) = \mathbf{0}$. For this purpose we obtain the values of $\boldsymbol{\Phi}\left(\boldsymbol{-\frac{1}{\sigma}}\right)$ $for, \sigma = 0.001, 0.002, \ldots, 0.499, 0.500$ and discovered that

$\boldsymbol{\Phi\left(-\frac{1}{\sigma}\right) = 0}$ to 4 decimal places for $\sigma \leq 0.256$. Within the interval $\sigma \in (0,\ 0.15)$

$$\Phi\left(\frac{m^{-2}-1}{\sigma}\right) = \frac{1}{2}$$

From the standard normal table,

$$\Phi\left(\frac{m^{-2}-1}{\sigma}\right) = 0.5 \quad \text{when} \quad \frac{m^{-2}-1}{\sigma} = 0$$

$$\therefore \quad m^{-2} - 1 = 0$$

thus

$$m = 1$$

$\therefore$ For $\sigma \leq 0.256$, the distribution of the inverse square of the error term of the multiplicative time series model is normal with median equal to one. Finally it is important to note that $0 < \sigma \leq 0.15 \subset \sigma \leq 0.256$, thus the median of the inverse-square-root transformed error component is equal to one within the region, $0 < \sigma \leq 0.15$ of its successful application.

CHAPTER FOUR

ANALYSIS OF DATA AND DISCUSSIONS

4.1: INTRODUCTION

In this chapter, a real life data shall be used to validate the properties of the Inverse-square-root transformed error component of the multiplicative time series model. The results to be validated in this chapter are the major findings of this study listed as follows;

(i) The interval for the successful application of inverse-square root transformation is $0 < \sigma \leq 0.15$, where σ is standard deviation of the untransformed error component.

(ii) The variance-ratio of the transformed error component to that of the untransformed error component is approximately four and

(iii) The mean of the transformed error $\approx$ the median of the transformed error component ≈ 1.0

The steps of analysis of the study data, X_t are to:

(i) confirm the applicability of the multiplicative time series model on X_t

(ii) justify that the inverse-square-root transformation is appropriate for X_t.

(iii) decompose the time series data, X_t to obtain e_t

(iv) obtain the mean, median and variance of (e_t) and verify its normality using Anderson-Darling test.

(v) use the inverse-square-root transformation of X_t to obtain Y_t and decomposition of Y_t to obtain the residual series (error component), e_t^*

(vi) obtain the mean, median and standard deviation of e_t^* respectively, and verify its normality using Anderson-Darling test.

(vii) compare the properties of e_t and e_t^*.

The appropriate data transformation would be assessed using the Bartlett's transformation technique as applied by Akpanta and Iwueze (2009). This method is achieved by obtaining a linear relationship between the natural logarithm of the periodic (yearly) standard deviations and the natural logarithm of the periodic mean. The slope of the linear relationship obtained is used to determine the appropriate transformation to be adopted. For the inverse-square-root transformation, a slope of 1.5 or approximately 1.5 is required. The appropriateness of data transformation and the type of transformation to be made will provide the medium for choice of transformation.

4.2 DESCRIPTION OF DATA

Data on the Monthly interest rates Government Bond Yield 2-year securities, Reserve Bank of Australia, Jan 1976 – Dec 1993 is used for this study (see Column 2 of Table 4.1in the Appendix). The data were obtained from the Time Series Data Library exported from datamarket.com. The time plot is shown in Figure 4.1**.**

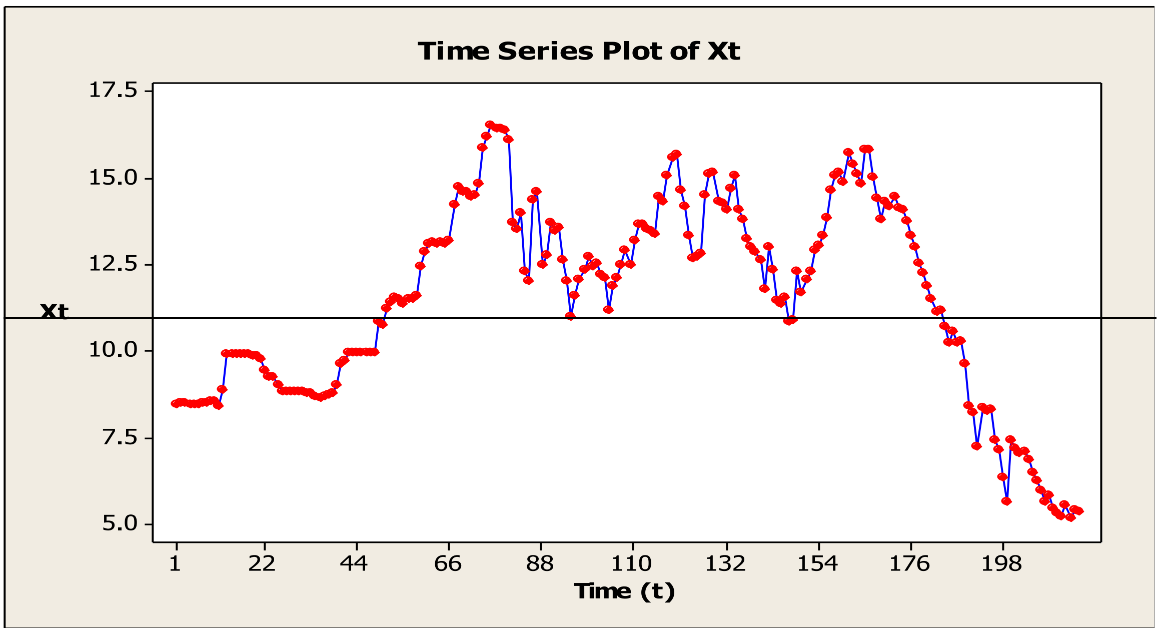


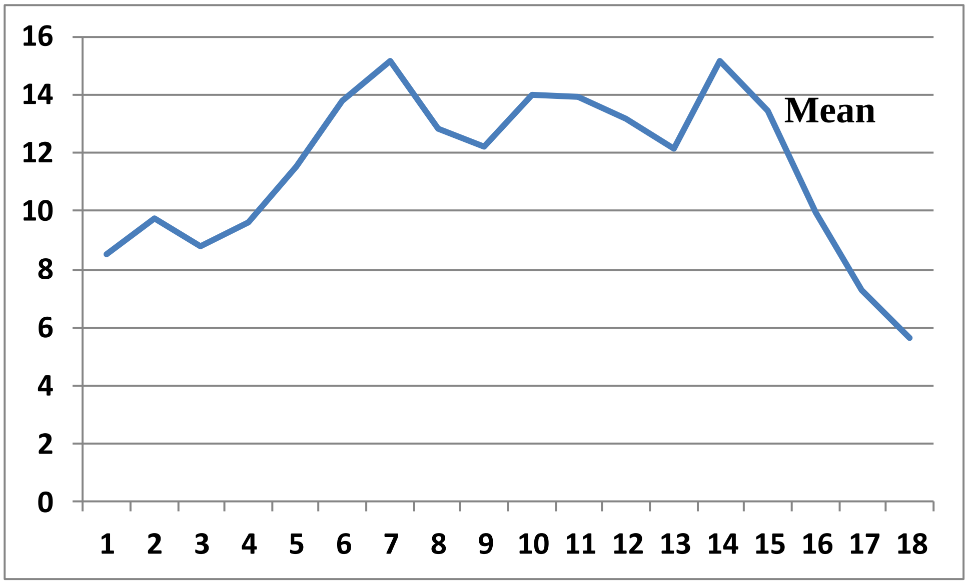
Figure 4.1: Time series plot of the Data, Monthly interest rates Government Bond Yield 2-year securities, Reserve Bank of Australia. Jan 1976 – Dec 1993

4.2.1 CHOICE OF MODEL

The choice of a multiplicative model for this series is clear. As given in Table 1.2, except the logarithm transformation, all the other common power transformations still leaves the model multiplicative. To put it simply, multiplicative model is appropriate for all series that requires the application of all other forms of transformations listed in Table 1.2 except the logarithm transformation that automatically converts a multiplicative model to an additive model, thus a multiplicative model is appropriate for the data of this study. Furthermore, using the plots of the periodic means and standard deviations (see Table 4.2, Figures 4.2 and 4.3) it is clear that the graph of the standard deviations is mimicking that movement of the means which also suggests the use of multiplicative model.

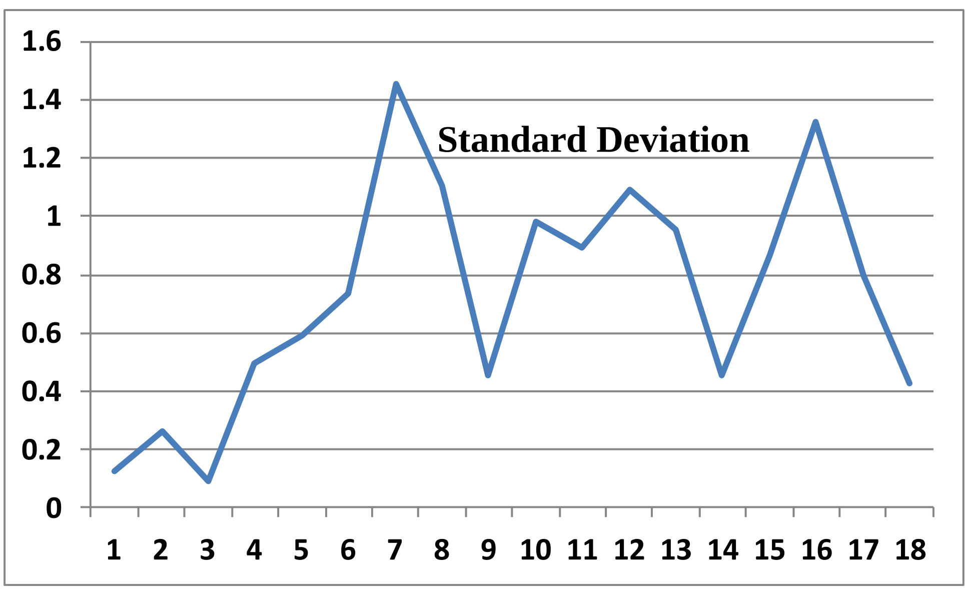
Table 4.2: Periodic means and standard deviations and their natural logarithms.

Year	$\bar{X}_{i.}$ (Periodic Mean)	$\hat{\sigma}_{i.}$ (Periodic Std. dev.)
1	8.5175	0.12462
2	9.7342	0.25875
3	8.7967	0.09029
4	9.6200	0.49360
5	11.5342	0.58775
6	13.8225	0.73632
7	15.1750	1.45438
8	12.8417	1.10820
9	12.2417	0.45569
10	14.0250	0.98269
11	13.9667	0.89502
12	13.1583	1.08896
13	12.1750	0.95762
14	15.1417	0.45419
15	13.4625	0.86553
16	9.9375	1.32341
17	7.2542	0.79557
18	5.6333	0.42498



Period

Figure 4.2: Time Series Plot of the Periodic Means of the Study Data



Period

Figure 4.3: Time Series Plot of the Periodic Standard Deviations of the Study Data

4.2.2 DETERMINATION OF THE APPROPRIATE TRANSFORMATION

For the choice of appropriate transformation, we obtain the natural logarithm of the periodic (yearly) means and standard deviations of X_t (see Table 4.3). Thereafter, the natural log of the periodic standard deviations are plotted against that of periodic means (see Figure 4.4) Table 4.3: Natural Log of the Periodic Means and Standard Deviations.

Table 4.3: Natural Log of the Periodic Means and Standard Deviations

Year	$\text{Log}_e \bar{X}_{i.}$	$\text{Log}_e \hat{\sigma}_{i.}$
1	2.14212	-2.08251
2	2.27564	-1.35188
3	2.17437	-2.40478
4	2.26384	-0.70604
5	2.44531	-0.53146
6	2.62630	-0.30609
7	2.71965	0.37458
8	2.55270	0.10274
9	2.50485	-0.78595
10	2.64084	-0.01746
11	2.63667	-0.11091
12	2.57706	0.08522
13	2.49938	-0.04330
14	2.71745	-0.78924
15	2.59991	-0.14441

16	2.29632	0.28021
17	1.98158	-0.22869
18	1.72870	-0.85572

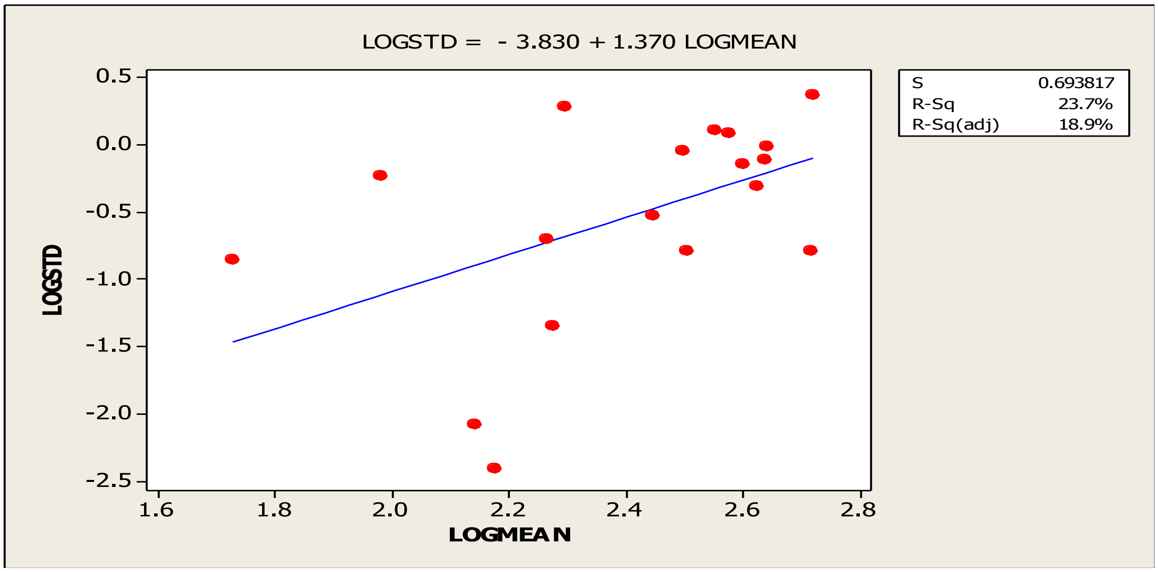


Figure 4.4: Fitted Regression Line plot of the Natural Logarithms of periodic Standard Deviation against that of the Periodic Means.

We obtained the regression line using minitab as

$$\mathbf{Log_e\,\hat{\sigma}_{i.} = \underset{(1.4920)}{-3.830} + \underset{(0.6152)}{1.37}\,\bar{X}_{i.},\ i = 1, 2, \ldots, 18}$$

where the values in brackets under the coefficients are their respective standard deviations. Here, the slope $\hat{\beta} = 1.37$ which is not exactly 1.5 as required by the Bartlett's transformation technique for the inverse-square-root transformation. Hence, we shall test whether the slope, $\hat{\beta} = 1.5$

4.2.3 : Test for the significance of the theoretical value of the slope coefficient $\hat{\beta}$

To test whether the estimated value of the slope is significantly different from the Bartlett's value of the slope for the inverse-square-root transformation ($\beta = 1.5$), we test the null hypothesis

$H_0 : \beta = 1.5$, against the alternative $H_1 : \beta \neq 1.5$

using the test statistic given by

$t = \frac{\hat{\beta} - 1.5}{\text{Std Error}\left(\hat{\beta}\right)} \; t_{n-2}$, where $\hat{\beta} = 1.37$, $\text{Std Dev}\left(\hat{\beta}\right) = 0.6152$. Thus t = - 0.2113 and the statistical table value is $\mathbf{t_{0.025,16} = \pm 2.10}$. Since |0.2113| < 2.10, we therefore accept that $\hat{\beta} \approx 1.5$, which shows that the inverse-square-root transformation is appropriate for the data.

4.2.4 Decomposition of the Data on Monthly interest rates Government Bond Yield 2-year securities, Reserve Bank of Australia. Jan 1976 – Dec 1993.

Having justified the use of inverse-square-root transformation and the appropriateness of the multiplicative model in Section 4.2.1 and 4.2.2, we now decompose the data on the Monthly interest rates Government Bond Yield 2-year securities, Reserve Bank of Australia into its components namely: trend, seasonal and residual/irregular components.

We shall determine the appropriate trend-curve to adopt among the two likely fits, linear and quadratic trend-curves using three accuracy measurements namely, Mean Absolute percentage Error (MAPE), Mean Absolute Deviation (MAD) and Mean Squared Deviation (MSD).

$$\mathbf{MAPE} = \frac{1}{n}\sum_{t=1}^{n}\frac{\left|X_t - \hat{X}_t\right|}{X_t} \times 100$$

$$\mathbf{MAD} = \frac{1}{n}\sum_{t=1}^{n}\left|X_t - \hat{X}_t\right|$$

$$\mathbf{MSD} = \frac{1}{n}\sum_{t=1}^{n}\left(X_t - \hat{X}_t\right)^2$$

Where X_t, $\hat{X}_t$ and n are the actual data, fitted trend-curve and number of data points respectively.

The fitted linear and quadratic trend curves are given in Figures 4.5 and 4.6 respectively while their accuracy measures are given in Table 4.3.

Table 4.4: Accuracy Measures for the Fitted Linear and Quadratic Trend Curves

Accuracy Measure	Linear Trend	Quadratic Trend
MAPE	24.5064	12.4829
MAD	2.3891	1.3495
MSD	7.9343	2.5042

It is important to mention that Minitab version 18 package was used for model fitting and generation of the values of MAPE, MAD and MSD. Based on the results of Table 4.4, it is clear that the quadratic trend-curve is the appropriate fit and its curve is given in Figure 4.6. The quadratic trend-curve equation is given by

$$X_t = \mathbf{6.3510 + 0.1442 * t - 0.0007 * t^2} \tag{4.1}$$

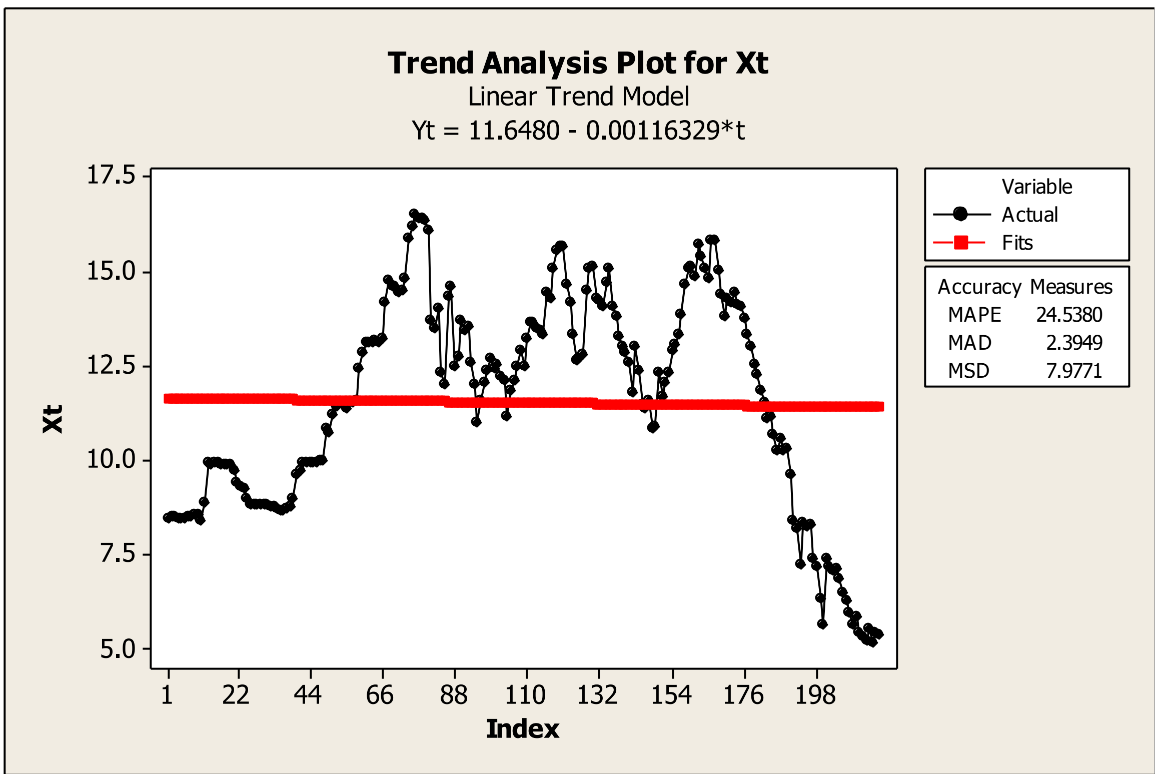


Figure 4.5: Linear trend fit for the Data on Monthly interest rates

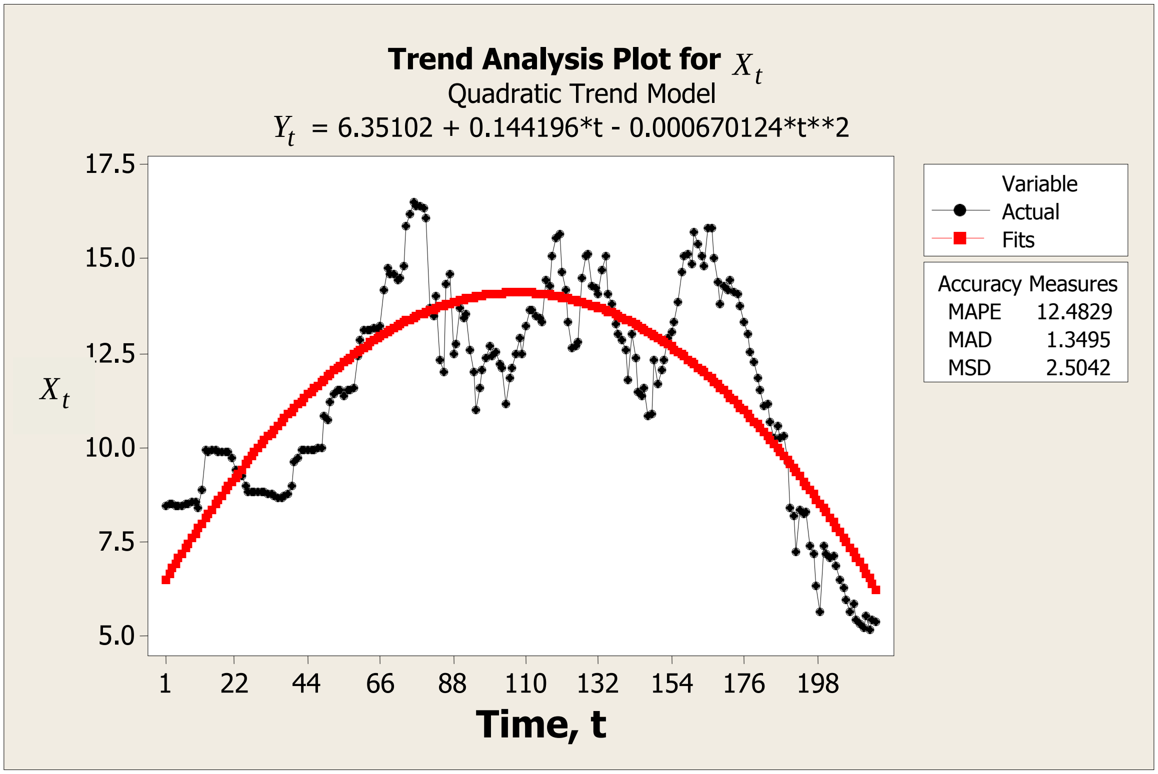


Figure 4.6: Quadratic trend fit for the Data on Monthly interest rates

The results of the decomposition of the time series data on the monthly interest rate into the trend, seasonal and irregular components are shown in Table 4.1(see columns 3, 4 and 5) for the trend, seasonal and irregular components respectively for the untransformed data.

The Mean ($\mathbf{E}(\hat{\mathbf{e}}_t)$), Median, Standard Deviation and Variance of the Irregular component/Residual series ($\hat{\mathbf{e}}_t$) are; $E(\hat{e}_t) = 0.9992$, median = 0.9719, σ = 0.1444 and σ^2 = 0.0209 as given in Table 4.5.

The irregular/error component was assessed for normality using the Kolmogorov - Smirnov test of normality. The result of the test is given in Figure 4.7 and it suggests that the residual/irregular component is not normally distributed at 10% level of significance which suggests lack of fit. This may be evidence that the data were also not normally distributed. The non-normality of the original data was confirmed using Anderson Darling test (value of the test statistic 2.827) and p-value < 0.005). That is the null hypothesis that the data were normally distributed is rejected, hence the need for a transformation is evident.

The summary of the descriptive statistics of e_t and e_t^* is given in Table 4.5

Table 4.5: Summary of the descriptive statistics of e_t and e_t^* from the real life - data and from the functional expresion obtained in Chapter 3.

Error component	Mean	Median	Standard Deviation	KS test Statistic	p-value of the KS test	Decision	$\frac{\mathbf{Var}\left(\hat{\mathbf{e}}_\mathbf{t}\right)}{\mathbf{Var}\left(\hat{\mathbf{e}}_\mathbf{t}^*\right)}$
e_t	0.9992	0.9719	0.1444	0.099	< 0.01	Reject Normality At 1% level of significance	$3.72 \approx 4.0$
e_t^*	1.0004	1.0092	0.0749	0.065	0.034	Do not reject normality at 1% level of significance	

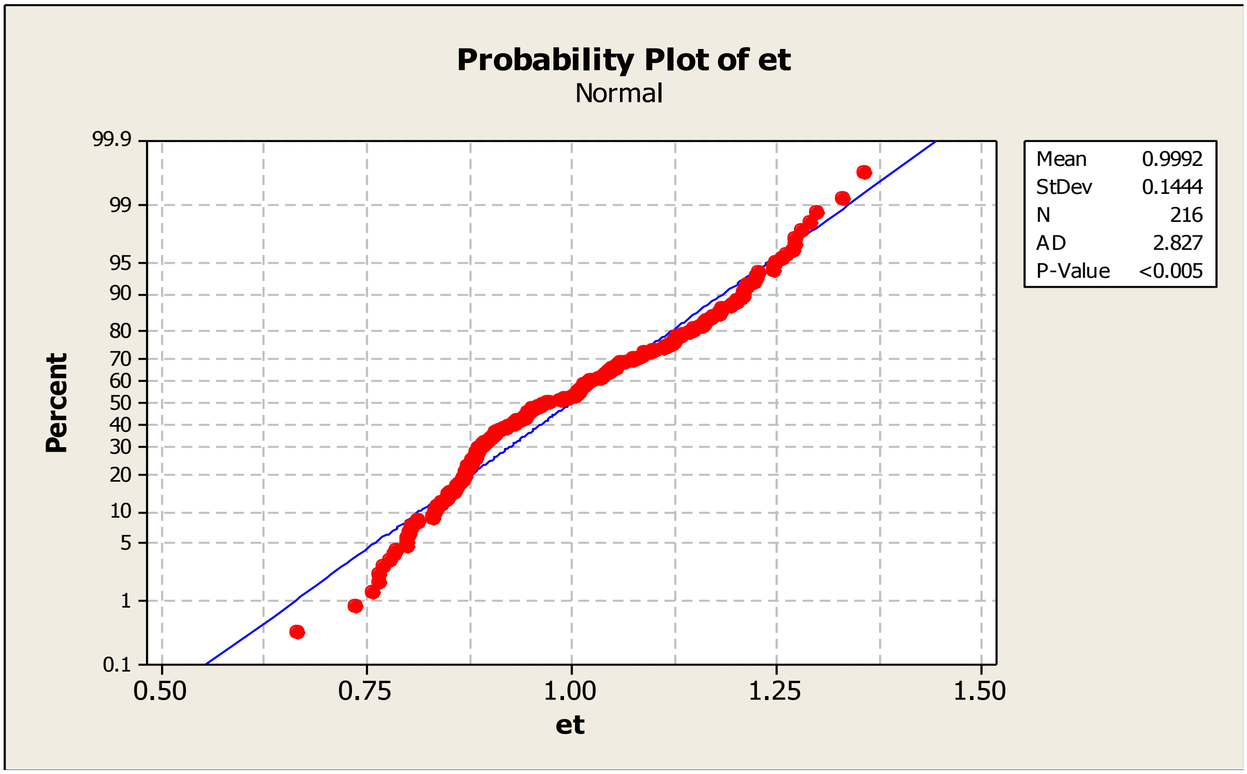


Figure 4.7: Normality Test for the Irregular Component of the Untransformed Interest Rate Data

For want of space, the inverse-square-root transformed data denoted as Y_t is

4.2.5: Time Series Analysis of the Inverse-Square-Root Transformed Data of Monthly interest rates Government Bond Yield 2-year securities, Reserve Bank of Australia. Jan 1976 – Dec 1993.

The inverse-square root transformed data are also presented in Table 4.1 (Column 6), while its plot is given in Figure 4.8. The decomposed components of the transformed data, Y_t are also presented in Table 4.1 (columns 7, 8 and 9) in the Appendix.

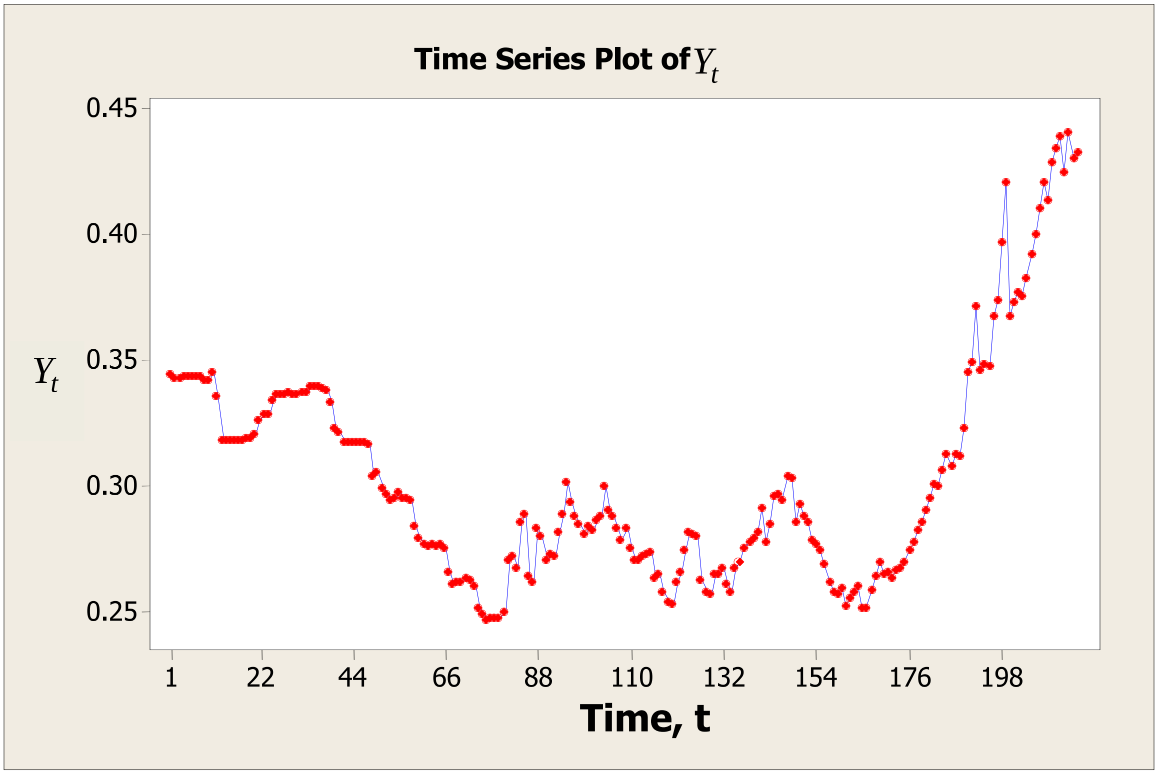


Figure 4.8: Time Series Plot of the Inverse-Square-Root Transformed Monthly Interest rateData

Similarly as was done for the untransformed data, we determined the trend-curve of best fit for the transformed data using the accuracy measures- MAPE, MAD and MSD as given in Table 4.6.

Table 4.6: Accuracy Measures for the Fitted Linear and Quadratic Trend Curves of the Inverse-Square-Root Transformed Data.

Accuracy Measure	Linear Trend	Quadratic Trend
MAPE	11.8085	6.3500
MAD	0.0365	0.0192
MSD	0.0019	0.0006

From Figure 4.8, the trend curve of the transformed series can either be linear or quadratic, therefore the two curves were fitted (see Figures 4.9 and 4.10) and the fit with smaller accuracy measures in Table 4.6 is selected. Based on the results in Table 4.6, the appropriate trend-curve is quadratic given by

$$\mathbf{M_t^* = 0.3759 - 0.0022 * t + 0.00001 * t^2} \tag{4.2}$$

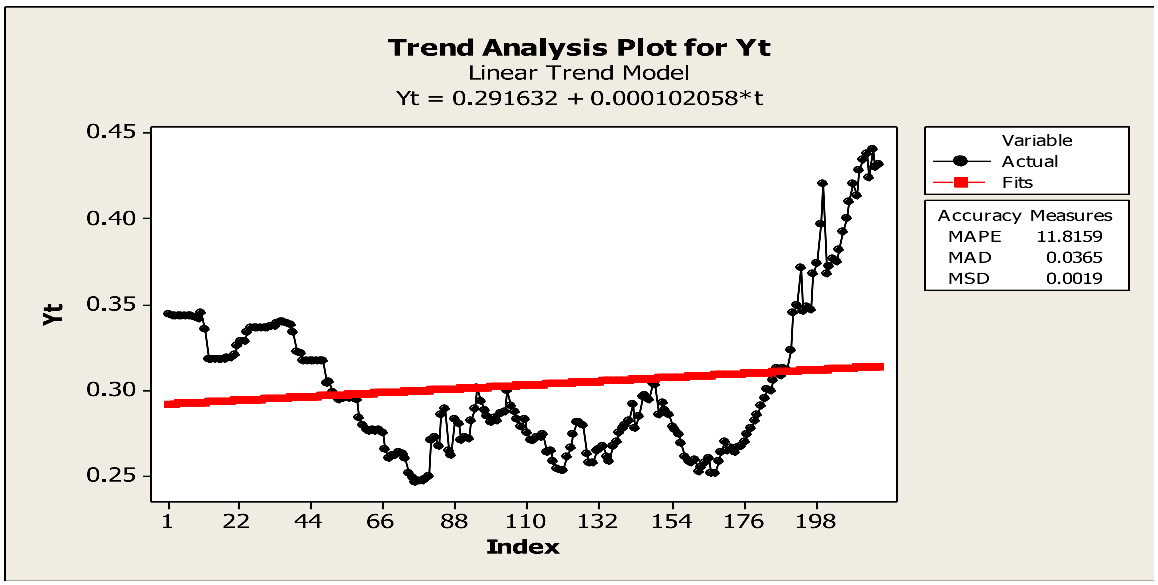


Figure 4.9: Linear Trend Curve of the Inverse-square-Root Transfromed Data on the monthly Interest Rate

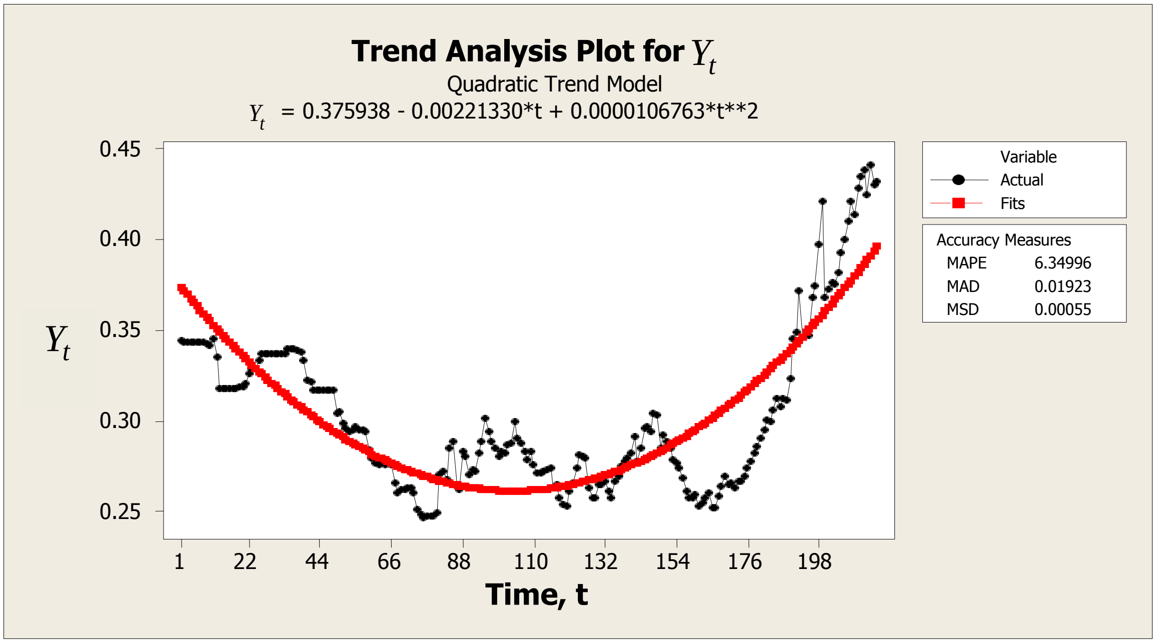


Figure 4.10: Quadratic Trend Curve of the Inverse-square-Root Transfromed Data on the monthly Interest Rate

The mean $\left(\mathbf{E}\left(\hat{\mathbf{e}}_{\mathbf{t}}^{*}\right)\right)$, median, standard deviation, σ and variance, σ^2 of the Irregular component/Residual series for inverse-square-root transformed series are respectively given by $\mathbf{E}\left(\hat{\mathbf{e}}_{\mathbf{t}}^{*}\right) = 1.0004$, median = 1.0092, σ = 0.0749, and $\sigma^2 = 0.0056$.

The Kolmogorov-Smirnov normality test was also carried out on the error/irregular component of the inverse square root transformed data. The results obtained using Minitab 18 software is shown in Figure 4.11. From the normal probability plot, it is observed that the inverse-square root transformation normalized the original error component of the data which was not initially normal (Normality of the error component is true at 1%, 2% and 3% levels of significance). Thus the error component of the transformed data has mean ≈ 1.0 and variance 0.0056. Hence, $\mathbf{e}_t^* \sim \mathbf{N}(\mathbf{1, 0.0056})$.

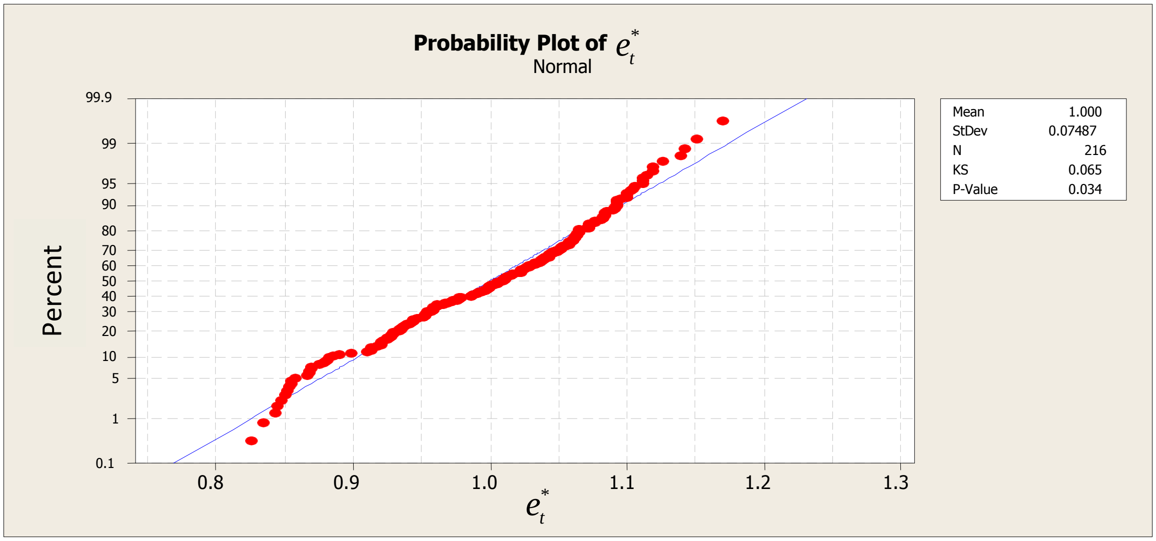


Figure 4.11: Kolmogorov-Smirnov Normality Test for the Error Component Inverse-Square-Root Transformed Data

4.2.6 VALIDATION OF RESULTS

In this section, the functional expressions obtained in Chapter 3 for the mean, mode and variance of inverse square root transformed error term in equations (3.28), (3.7) and (3.29) respectively are now used to obtain the descriptive statistics of the inverse-square root transformed error component.

From the equivalent results established for the mean and variance of the transformed error component (Equations 3.28 and 3.29), we obtain

$$\textbf{Mean of } \hat{\mathbf{e}}_t^* = 1 + \frac{3\sigma^2}{8} = 1 + \frac{3(0.1444)^2}{8} = 1 + 0.0026 = 1.0026 \approx 1.0$$

where $1-\Phi\left(-\frac{1}{\sigma}\right)=1$ and $1+\Pr\left(\chi^2_{(1)}<\frac{1}{\sigma^2}\right)=2$ for $\sigma \le 0.15$

Variance of $\mathbf{\hat{e}_t^*}$ $\quad \mathbf{V}\left(\mathbf{\hat{e}_t^*}\right) \;\; = \;\; \frac{\sigma^2}{4}-\left(\frac{9\sigma^4}{64}\right)$

$$= \frac{\left(0.1444\right)^2}{4} - \left(\frac{9*0.1444^4}{64}\right)$$

$$= \quad 0.0052 - 0.000061 \quad = \quad 0.0052$$

which compares favourably with the sample estimate of 0.0056 obtained for the error component using the study data.

Relationship between the variance of transformed and untransformed for the study data is given as

$$\frac{V\left(\hat{e}_t\right)}{V\left(\hat{e}_t^*\right)} = \frac{0.0209}{0.0056} \mathbf{= 3.72 \approx 4.0}$$

$$\Rightarrow V\left(e_t^*\right) \approx \frac{1}{4} V\left(e_t\right)$$

Hence, the theoretical relationship obtained in Chapter 3, compares favourably with the descriptive statistics obtained using the real life data (correct to 1 d.p.)

From the analysis of the real life data, it is evident that the application of inverse-square-root transformation normalized the untransformed error component $\hat{e}_t$ that was initially non-normal. Furthermore, the mean of the inverse-square-root transformed error component is 1, and the standard deviation is $\boldsymbol{\sigma} \mathbf{= 0.0749}$ which is in line with the result obtained from the simulated data (see Table 3.5 in the Appendix). Furthermore the mean and median of the transformed error component were both found to be approximately equal to 1.0, thus a successful transformation is achieved when $0 < \sigma < 0.15$ and the $\mathbf{Var}\left(\hat{\mathbf{e}}_{\mathbf{t}}^{*}\right) \approx \frac{\mathbf{1}}{\mathbf{4}}\mathbf{Var}\left(\hat{\mathbf{e}}_{\mathbf{t}}\right)$.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 INTRODUCTION

In this Chapter, the summary of the findings of this study, contribution to knowledge, conclusion, recommendations and suggestions for further research would be given.

5.2 SUMMARY OF THE FINDINGS

The findings of this study are;

(i) the distribution of the inverse-square-root transformed error component of the multiplicative time series model was established and is given by

$$\mathbf{g(y)} = \begin{cases} \dfrac{2}{y^3\sigma\sqrt{2\pi}\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]} e^{-\frac{1}{2\sigma^2}\left(\frac{1}{y^2}-1\right)^2}, & 0 < y < \infty \\ 0, & -\infty < y \le 0 \end{cases} \qquad (5.1)$$

with mean $E\left(e_t^*\right) = 1 + \dfrac{3\sigma^2}{16\left[1-\Phi\left(-\frac{1}{\sigma}\right)\right]}\left[1+\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right]$, when $\sigma \le 0.22$ and

$$\mathrm{Var}\left(e_t^*\right) = \mathrm{Var}(Y) = \frac{\sigma^2}{8[1-\Phi(\frac{1}{\sigma})}\left[1+\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right]\left\{1 - \frac{9\sigma^2}{32[1-\Phi(\frac{1}{\sigma})}\left[1+\Pr\left(\chi^2_{(1)} < \frac{1}{\sigma^2}\right)\right]\right\} \qquad (5.2)$$

for $\sigma \le 0.22$

(ii) The distribution function g(y) is symmetric about a unit mean for values of standard deviation, $\sigma \in (0, 0.15)$.

(iii) It was observed that the functional form of E ($\hat{e}_t^*$) and Var ($\hat{e}_t^*$) confirmed the mean of $\hat{e}_t^*$ to be 1 and the variance, Var ($\hat{e}_t^*$) is approximately equal to $\frac{1}{4}\mathrm{Var}(\hat{e}_t)$.

(iv) For inverse square root transformation to be applied on the error term of the multiplicative time series model without any violation of the basic assumptions, the standard deviation must be less than 0.15.

5.3 CONCLUSION

We investigated the distribution and properties of the left-truncated $\mathbf{N}\left(\mathbf{1},\boldsymbol{\sigma}^{\mathbf{2}}\right)$ error term, $\mathbf{e_t}$, of the multiplicative time series model under inverse-square-root transformation with a view to establish the condition for the transformed error term, $e_t^* = \frac{1}{\sqrt{e_t}}$, to be normally distributed with mean, 1. It was found that the normality of $\mathbf{e_t^*}$ is attained for $\boldsymbol{\sigma} < \mathbf{0.15}$ the functional forms of $\mathrm{E}\left(e_t^*\right)$ and $\mathrm{Var}\left(e_t^*\right)$ confirmed the mean of $\mathbf{e_t^*}$ to be 1 with $\mathrm{Var}\left(e_t^*\right) \approx \frac{1}{4}\mathrm{Var}\left(e_t\right)$, whenever $\boldsymbol{\sigma} < \mathbf{0.15}$. Hence $\boldsymbol{\sigma} < \mathbf{0.15}$ is the recommended condition for successful inverse square root transformation.

5.4 RECOMMENDATION

We recommend that for successful and valid inverse square root transformation in time series data the standard deviation of the error component must lie in the interval (0, 0.15).

5.5 CONTRIBUTION TO KNOWLEDGE

In this study, we have established the conditions for a successful inverse square root transformation of the error term of the multiplicative time series model which include:

(i) the establishment of the distribution and properties of the Left-truncated $N(1, \sigma^2)$ error term under inverse square root transformation.

and

(ii) the establishment of the interval $\boldsymbol{\sigma} \in (\mathbf{0}, \mathbf{0.15})$ for a successful and valid inverse-square-root transformation of the multiplicative time series.

5.6 SUGGESTIONS FOR FURTHER RESEARCH

The following are suggestions for further research;

(i) The effect of the inverse-square-root transformation on the error component of the multiplicative error model whose distribution is assumed to be non-normal such as Gamma, Weibull, etc are suggested for investigations to establish if the results of this study will also be valid.

(ii) Since we only used simulations to establish that the Variance of the untransformed error component to that of the transformed is 4, we also suggest its establishment analytically.

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Table 3.1: COMPUTATION OF $y_{max} = \sqrt{\frac{-1+\sqrt{1+6\sigma^2}}{3\sigma^2}}, for \sigma \in [0.01, \ 0.3]$

σ	$y_{max} = \sqrt{\frac{-1+\sqrt{1+6\sigma^2}}{3\sigma^2}}$	$1 - y_{max}$	σ	$y_{max} = \sqrt{\frac{-1+\sqrt{1+6\sigma^2}}{3\sigma^2}}$	$1 - y_{max}$
0.010	0.99992502	0.00007500	0.155	0.94470721	0.05529300
0.015	**0.99970031**	0.00030000	0.160	0.94163225	0.05836800
0.020	0.99932659	0.00067300	0.165	0.93852446	0.06147600
0.025	0.99880501	0.00119500	0.170	0.93538739	0.06461300
0.030	0.99813720	0.00186300	0.175	0.93222440	0.06777600
0.035	0.99732519	0.00267500	0.180	0.92903869	0.07096100
0.040	0.99637147	0.00362900	0.185	0.92583333	0.07416700
0.045	**0.99527886**	0.00472100	0.190	0.92261120	0.07738900
0.050	0.99405059	0.00594900	0.195	0.91937505	0.08062500
0.055	0.99269018	0.00731000	0.200	0.91612748	0.08387300
0.060	0.99120149	0.00879900	0.205	0.91287093	0.08712900
0.065	0.98958860	0.01041100	0.210	0.90960772	0.09039200
0.070	0.98785584	0.01214400	0.215	0.90634001	0.09366000
0.075	0.98600775	0.01399200	0.220	0.90306986	0.09693000
0.080	0.98404899	0.01595100	0.225	0.89979918	0.10020100
0.085	0.98198438	0.01801600	0.230	0.89652976	0.10347000
0.090	0.97981881	0.02018100	0.235	0.89326328	0.10673700
0.095	0.97755725	0.02244300	0.240	0.89000132	0.10999900
0.100	0.97520469	0.02479500	0.245	0.88674534	0.11325500
0.105	0.97276613	0.02723400	0.250	0.88349669	0.11650300
0.110	0.97024653	0.02975300	0.255	0.88025665	0.11974300
0.115	0.96765082	0.03234900	0.260	0.87702640	0.12297400
0.120	0.96498387	0.03501600	0.265	0.87380702	0.12619300
0.125	0.96225045	0.03775000	0.270	0.87059952	0.12940000
0.130	0.95945523	0.04054500	0.275	0.86740484	0.13259500
0.135	0.95660279	0.04339700	0.280	0.86422383	0.13577600
0.140	0.95369754	0.04630200	0.285	0.86105729	0.13894300
0.145	**0.95074378**	0.04925600	0.290	0.85790594	0.14209400
0.150	0.94774567	0.05225400	0.295	0.85477043	0.14523000
			0.300	0.85165139	0.14834900

Table 3.2: Conditions For $mode \approx 1 \approx mean$, where $y_{max} \approx 1$

Decimal places	$mode \approx 1 \approx mean$
3	$0 < \sigma < 0.015$
2	$0 < \sigma < 0.045$
1	$0 < \sigma < 0.145$

Table 3.3: Simulation Results When σ = 0.06

X= $e_t \sim N(1,\sigma^2), \sigma = 0.06$						Y= $e_t^* = \frac{1}{\sqrt{e_t}}$, $e_t \sim N(1,\sigma^2), \sigma = 0.06$						
Mean	StD	Variance	Median	AD	p-value	Mean	StDev	Variance	Median	AD	p-value	$\frac{Var(e_t)}{Var(e_t^*)}$
1	0.06	0.0036	0.9927	.235	.788	1.0013	0.0303	0.000918	1.0037	.206	.867	4
1	0.06	0.0036	1.0009	.183	.908	1.0013	0.0302	0.000914	0.9995	.298	.580	4
1	0.06	0.0036	1.0002	.195	.889	1.0013	0.0303	0.000916	0.9999	.275	.654	4
1	0.06	0.0036	1.0029	.234	.790	1.0013	0.0303	0.000917	0.9985	.334	.505	4
1	0.06	0.0036	1.0037	.178	.918	1.0013	0.0302	0.000915	0.9982	.312	.546	4
1	0.06	0.0036	1.0045	.435	.294	1.0013	0.0301	0.000908	0.9978	.364	.433	4
1	0.06	0.0036	1.0037	.178	.918	1.0013	0.0302	0.000915	0.9982	.312	.546	4
1	0.06	0.0036	1.0013	.137	.976	1.0013	0.0302	0.00091	0.9993	.213	.851	4
1	0.06	0.0036	0.9941	.196	.888	1.0013	0.0302	0.000911	1.003	.302	.569	4
1	0.06	0.0036	1.0017	.250	.739	1.0014	0.0304	0.000924	0.9991	.453	.266	4
1	0.06	0.0036	1.0004	.200	.880	1.0013	0.0302	0.000915	0.9998	.314	.540	4
1	0.06	0.0036	1.0045	.435	.294	1.0013	0.0301	0.000908	0.9978	.364	.433	4
1	0.06	0.0036	0.9991	.183	.908	1.0013	0.0303	0.000916	1.0005	.214	.846	4
1	0.06	0.0036	0.9983	.250	.739	1.0013	0.0301	0.000908	1.0009	.206	.866	4
1	0.06	0.0036	1.001	.209	.859	1.0013	0.03	0.000901	0.9995	.241	.767	4
1	0.06	0.0036	1.0028	.195	.889	1.0013	0.0302	0.000913	0.9986	.284	.625	4
1	0.06	0.0036	1.0031	.141	.972	1.0013	0.0302	0.000911	0.9985	.208	.862	4
1	0.06	0.0036	0.9975	.310	.550	1.0013	0.0299	0.000894	1.0012	.232	.795	4
1	0.06	0.0036	1.0006	.262	.699	1.0014	0.0304	0.000924	0.9997	.385	.387	4
1	0.06	0.0036	0.9983	.182	.911	1.0013	0.0302	0.000913	1.0009	.318	.531	4
1	0.06	0.0036	0.9958	.150	.962	1.0013	0.0303	0.000916	1.0021	.218	.835	4
1	0.06	0.0036	0.9938	.290	.606	1.0013	0.0299	0.000896	1.0031	.185	.906	4
1	0.06	0.0036	0.9931	.450	.270	1.0013	0.03	0.000903	1.0035	.336	.503	4
1	0.06	0.0036	0.995	.199	.882	1.0013	0.0301	0.000907	1.0025	.390	.376	4
1	0.06	0.0036	0.9987	.216	.841	1.0013	0.0302	0.000914	1.0006	.315	.538	4
1	0.06	0.0036	0.9942	.311	.546	1.0013	0.03	0.000899	1.0029	.165	.940	4

NB i. $\frac{3}{8}\sigma^2 = (0.375\sigma^2) = 0.00135$ ii. $E(e_t^*) \approx 1 + \frac{3}{8}\sigma^2$.

Table 3.4: Simulation Results when σ = 0.1

X= $e_t \sim N\left(1,\sigma^2\right), \sigma = 0.1$						Y= $e_t^* = \frac{1}{\sqrt{e_t}}$, $e_t \sim N\left(1,\sigma^2\right), \sigma = 0.1$						
Mean	StD	Variance	Median	AD	p-value	Mean	StD	Variance	Median	AD	p-value	$\frac{Var\left(e_t\right)}{Var\left(e_t^*\right)}$
1	0.1	0.01	0.9878	.235	.788	1.0038	0.0514	0.00265	1.0061	.298	.582	4
1	0.1	0.01	1.0016	.183	.908	1.0038	0.0511	0.00262	0.9992	.457	.260	4
1	0.1	0.01	1.0003	.195	.889	1.0038	0.0513	0.00263	0.9998	.428	.306	4
1	0.1	0.01	1.0049	.234	.790	1.0038	0.0513	0.00264	0.9976	.502	.201	4
1	0.1	0.01	1.0062	.178	.918	1.0038	0.0512	0.00262	0.9969	.495	.211	4
1	0.1	0.01	1.0074	.435	.294	1.0038	0.0509	0.00259	0.9963	.424	313	4
1	0.1	0.01	1.0062	.178	.918	1.0038	0.0512	0.00262	0.9969	.495	.211	4
1	0.1	0.01	1.0022	.137	.976	1.0038	0.0509	0.00259	0.9989	.357	.450	4
1	0.1	0.01	0.9902	.196	.888	1.0038	0.051	0.0026	1.005	.464	.251	4
1	0.1	0.01	1.0029	.250	.739	1.0038	0.0516	0.00267	0.9986	.685	.071	4
1	0.1	0.01	1.0007	.200	.880	1.0038	0.0512	0.00262	0.9997	.495	.210	4
1	0.1	0.01	1.0074	.435	.294	1.0038	0.0509	0.00259	0.9963	.424	.313	4
1	0.1	0.01	0.9984	.183	.908	1.0038	0.0513	0.00263	1.0008	.326	.516	4
1	0.1	0.01	0.9971	.250	.739	1.0038	0.0509	0.00259	1.0014	.272	.664	4
1	0.1	0.01	1.0016	.209	.859	1.0037	0.0505	0.00255	0.9992	.359	.445	4
1	0.1	0.01	1.0047	.195	.889	1.0038	0.0511	0.00261	0.9977	.446	.277	4
1	0.1	0.01	1.0052	.141	.972	1.0038	0.051	0.0026	0.9974	346	.477	4
1	0.1	0.01	0.9959	.310	.550	1.0037	0.0502	0.00252	1.0021	.278	.642	4
1	0.1	0.01	1.0011	.262	.699	1.0038	0.0516	0.00266	0.9995	.554	.150	4
1	0.1	0.01	0.9971	.182	.911	1.0038	0.0511	0.00261	1.0014	.499	.205	4
1	0.1	0.01	0.9931	.150	.962	1.0038	0.0513	0.00263	1.0035	.368	.424	4
1	0.1	0.01	0.9897	.290	.606	1.0037	0.0503	0.00253	1.0052	.221	.827	4
1	0.1	0.01	0.9884	.450	.270	1.0037	0.0506	0.00256	1.0058	.366	.428	4
1	0.1	0.01	0.9917	.306	.559	1.0038	0.0508	0.00258	1.0042	.547	.156	4
1	0.1	0.01	0.9979	.199	.882	1.0038	0.0511	0.00261	1.0011	.497	.207	4
1	0.1	0.01	0.9904	.216	.841	1.0037	0.0504	0.00254	1.0048	.226	.815	4

NB i. $\frac{3}{8}\sigma^2 = 0.375\,(.01) = 0.00375$ ii. $E(e_t^*) \approx 1 + \frac{3}{8}\sigma^2$.

Table 3.5: Simulation Results when σ = 0.15

X= $e_t \sim N\left(1,\sigma^2\right), \sigma = 0.15$						Y= $e_t^* = \frac{1}{\sqrt{e_t}}$, $e_t \sim N\left(1,\sigma^2\right), \sigma = 0.15$						
Mean	StD	Variance	Median	AD	p-value	Mean	StDev	Variance	Median	AD	p-value	$\frac{Var\left(e_t\right)}{Var\left(e_t^*\right)}$
1	0.15	0.0225	0.9818	.235	.788	1.0089	0.0803	0.00645	1.0092	.582	.126	3
1	0.15	0.0225	1.0024	.183	.908	1.0088	0.0791	0.00626	0.9988	.761	.046	4
1	0.15	0.0225	1.0005	.195	.889	1.0088	0.0798	0.00637	0.9997	.756	.047	4
1	0.15	0.0225	1.0073	.234	.790	1.0088	0.0798	0.00636	0.9964	.857	.027	4
1	0.15	0.0225	1.0093	.178	.918	1.0088	0.0792	0.00628	0.9954	.842	.029	4
1	0.15	0.0225	1.0111	.435	.294	1.0087	0.0788	0.0062	0.9945	.646	.089	4
1	0.15	0.0225	1.0093	.178	.918	1.0088	0.0792	0.00628	0.9954	.842	.029	4
1	0.15	0.0225	1.0034	.137	.976	1.0087	0.0786	0.00618	0.9983	.656	.085	4
1	0.15	0.0225	0.9853	.196	.888	1.0087	0.0788	0.00621	1.0075	.785	.040	3
1	0.15	0.0225	1.0043	.250	.739	1.0089	0.0804	0.00646	0.9979	1.109	.005	4
1	0.15	0.0225	1.001	.200	.880	1.0088	0.0793	0.00628	0.9995	.860	.026	4
1	0.15	0.0225	1.0111	.435	.294	1.0087	0.0788	0.0062	0.9945	.646	.089	4
1	0.15	0.0225	0.9976	.183	.908	1.0088	0.0796	0.00633	1.0012	.596	.119	4
1	0.15	0.0225	0.9957	.250	.739	1.0087	0.0788	0.00621	1.0022	.486	.221	4
1	0.15	0.0225	1.0025	.209	.859	1.0086	0.0775	0.00601	0.9988	.620	.104	4
1	0.15	0.0225	1.007	195	889	1.0088	0.0791	0.00626	0.9965	.779	.042	4
1	0.15	0.0225	1.0077	141	. 972	1.0087	0.0787	0.00619	0.9962	.635	.095	4
1	0.15	0.0225	0.9938	.310	.550	1.0085	0.077	0.00593	1.0031	.450	.271	4
1	0.15	0.0225	1.0016	.262	.699	1.0089	0.0799	0.00639	0.9992	.880	.023	4
1	0.15	0.0225	0.9957	.182	.911	1.0087	0.0789	0.00622	1.0022	.838	.030	4
1	0.15	0.0225	0.9896	.500	.962	1.0088	0.0798	0.00636	1.0052	.701	.065	4
1	0.15	0.0225	0.9846	.290	.606	1.0085	0.077	0.00593	1.0078	.398	.361	4
1	0.15	0.0225	0.9826	.450	.270	1.0086	0.0781	0.00609	1.0088	.545	.157	4
1	0.15	0.0225	0.9876	.306	.559	1.0087	0.0782	0.00611	1.0063	.868	.025	4
1	0.15	0.0225	0.9968	.199	.882	1.0088	0.079	0.00624	1.0016	.860	.026	4
1	0.15	0.0225	0.9856	.216	.841	1.0085	0.0772	0.00596	1.0073	.419	.322	4

NB: (i) The p-value is less than 0.05 in 50% of the cases

(ii) The ratio of the two variances is less than 4 in 8% of the cases.

iii. $\frac{3}{8}\sigma^2 = (0.375)\,(0.0225) = 0.0084$

iv. $E(e_t^*) \approx 1 + \frac{3}{8}\sigma^2$..

Table 3.6: Simulation Results when σ = 0.2

$e_t \sim N(1,\sigma^2), \sigma = 0.2$						$e_t^* = \frac{1}{\sqrt{e_t}}, e_t \sim N(1,\sigma^2), \sigma = 0.2$						
Mean	StD	Variance	Median	AD	p-value	Mean	StDev	Variance	Median	AD	p-value	$\frac{Var(e_t)}{Var(e_t^*)}$
1	0.2	0.04	0.9757	.235	.788	1.0167	0.1147	0.0132	1.0124	1.176	<0.005	3
1	0.2	0.04	1.0032	.183	.908	1.0162	0.1107	0.0123	0.9984	1.220	<0.005	3
1	0.2	0.04	1.0007	.195	.889	1.0165	0.1127	0.0127	0.9997	1.315	<0.005	3
1	0.2	0.04	1.0097	.234	.790	1.0164	0.1124	0.0126	0.9952	1.435	<0.005	3
1	0.2	0.04	1.0124	.178	.918	1.0163	0.1109	0.0123	0.9939	1.353	<0.005	3
1	0.2	0.04	1.0148	.435	.294	1.0161	0.1105	0.0122	0.9927	1.097	.007	3
1	0.2	0.04	1.0124	.178	.918	1.0163	0.1109	0.0123	0.9939	1.353	<0.005	3
1	0.2	0.04	1.0045	.137	.976	1.0161	0.1095	0.012	0.9978	1.117	.006	3
1	0.2	0.04	0.9803	.196	.888	1.0161	0.11	0.0121	1.01	1.276	<0.005	3
1	0.2	0.04	1.0057	.250	..739	1.0166	0.1133	0.0128	0.9971	1.734	<0.005	3
1	0.2	0.04	1.0013	.200	.880	1.0163	0.111	0.0123	0.9994	1.418	<0.005	3
1	0.2	0.04	1.0149	.435	.294	1.0161	0.1105	0.0122	0.9927	1.097	.007	3
1	0.2	0.04	0.9968	.183	.908	1.0164	0.112	0.0125	1.0016	1.072	.008	3
1	0.2	0.04	0.9943	.250	.739	1.0162	0.1107	0.0123	1.0029	.915	0.019	3
1	0.2	0.04	1.0033	.209	.859	1.0157	0.1072	0.0115	0.9984	1.026	0.010	3
1	0.2	0.04	1.0094	.195	.889	1.0162	0.1109	0.0123	0.9953	1.293	<0.005	3
1	0.2	0.04	1.0103	.141	.972	1.0161	0.1097	0.012	0.9949	1.084	0.007	3
1	0.2	0.04	0.9917	.310	.550	1.0156	0.1066	0.0114	1.0042	.768	0.045	3
1	0.2	0.04	1.0021	.260	.699	1.0165	0.1119	0.0125	0.9989	1.371	<0.005	3
1	0.2	0.04	0.9942	.182	.911	1.0162	0.11	0.0121	1.0029	1.331	<0.005	3
1	0.2	0.04	0.9862	.150	.962	1.0165	0.1128	0.0127	1.007	1.267	<0.005	3
1	0.2	0.04	0.9795	.290	.606	1.0156	0.1064	0.0113	1.0104	.745	0.051	3
1	0.2	0.04	0.9768	.450	.270	1.0159	0.109	0.0119	1.0118	.933	.017	3
1	0.2	0.04	0.9835	.306	.559	1.0159	0.1084	0.0118	1.0084	1.348	<0.005	3
1	0.2	0.04	0.9958	.199	.882	1.0162	0.1101	0.0121	1.0021	1.402	<0.005	3
1	0.2	0.04	0.9808	.216	.841	1.0156	0.1066	0.0114	1.0097	.766	.045	3

NB: (i) The p-value is less than 0.05 in all the cases iii. $\frac{3}{8}\sigma^2 = 0.015$

(ii) The ratio of the two variances is less than 4 in all the cases. iv. $E(e_t^*) > 1 + \frac{3}{8}\sigma^2$.

Table 3.7: Values of E(Y) and Var(Y) for σ∈[0.01,0.5]

σ	σ^2	$e^{-\frac{1}{2\sigma^2}}$	A	B	E(Y)	Var(Y)	Var(X)	VarY/VarX
0.01	0.0001	0.0000000	1.00000	2.00000	1.00004	0.0000250	0.000100	4.00023
0.02	0.0004	0.0000000	1.00000	2.00000	1.00015	0.0001000	0.000400	4.00090
0.03	0.0009	0.0000000	1.00000	2.00000	1.00034	0.0002249	0.000900	4.00203
0.04	0.0016	0.0000000	1.00000	2.00000	1.00060	0.0003996	0.001600	4.00360
0.05	0.0025	0.0000000	1.00000	2.00000	1.00094	0.0006241	0.002500	4.00563
0.06	0.0036	0.0000000	1.00000	2.00000	1.00135	0.0008982	0.003600	4.00812
0.07	0.0049	0.0000000	1.00000	2.00000	1.00184	0.0012216	0.004900	4.01106
0.08	0.0064	0.0000000	1.00000	2.00000	1.00240	0.0015942	0.006400	4.01445
0.09	0.0081	0.0000000	1.00000	2.00000	1.00304	0.0020158	0.008100	4.01831
0.10	0.0100	0.0000000	1.00000	2.00000	1.00375	0.0024859	0.010000	4.02263
0.11	0.0121	0.0000000	1.00000	2.00000	1.00454	0.0030044	0.012100	4.02741
0.12	0.0144	0.0000000	1.00000	2.00000	1.00540	0.0035708	0.014400	4.03266
0.13	0.0169	0.0000000	1.00000	2.00000	1.00634	0.0041848	0.016900	4.03839
0.14	0.0196	0.0000000	1.00000	2.00000	1.00735	0.0048460	0.019600	4.04459
0.15	0.0225	0.0000000	1.00000	2.00000	1.00844	0.0055538	0.022500	4.05127
0.16	0.0256	0.0000000	1.00000	2.00000	1.00960	0.0063078	0.025600	4.05844
0.17	0.0289	0.0000000	1.00000	2.00000	1.01084	0.0071075	0.028900	4.06610
0.18	0.0324	0.0000002	1.00000	2.00000	1.01215	0.0079524	0.032400	4.07425
0.19	0.0361	0.0000010	1.00000	2.00000	1.01354	0.0088417	0.036100	4.08290
0.20	0.0400	0.0000037	1.00000	2.00000	1.01500	0.0097750	0.040000	4.09204
0.21	0.0441	0.0000119	1.00000	2.00000	1.01654	0.0107515	0.044099	4.10166
0.22	0.0484	0.0000326	1.00000	1.99999	1.01815	0.0117706	0.048397	4.11170
0.23	0.0529	0.0000785	0.99999	1.99999	1.01984	0.0128315	0.052893	4.12211
0.24	0.0576	0.0001699	0.99998	1.99997	1.02160	0.0139334	0.057584	4.13277
0.25	0.0625	0.0003355	0.99997	1.99994	1.02344	0.0150757	0.062467	4.14353
0.26	0.0676	0.0006134	0.99994	1.99988	1.02535	0.0162574	0.067536	4.15420
0.27	0.0729	0.0010503	0.99989	1.99979	1.02734	0.0174777	0.072787	4.16456
0.28	0.0784	0.0016993	0.99982	1.99964	1.02940	0.0187356	0.078210	4.17440
0.29	0.0841	0.0026181	0.99972	1.99944	1.03154	0.0200304	0.083797	4.18349
0.30	0.0900	0.0038659	0.99957	1.99914	1.03375	0.0213609	0.089537	4.19162
0.31	0.0961	0.005501	0.90769	1.99988	1.03604	0.0227263	0.095419	4.19861
0.32	0.1024	0.007576	0.91959	1.99979	1.03840	0.0241254	0.101431	4.20432
0.33	0.1089	0.010139	0.93148	1.99964	1.04084	0.0255573	0.107562	4.20865
0.34	0.1156	0.013230	0.94335	1.99944	1.04335	0.0270208	0.113799	4.21155
0.35	0.1225	0.016880	0.95521	1.99914	1.04594	0.0285147	0.120132	4.21299
0.36	0.1296	0.021110	0.96706	1.99874	1.04860	0.0300380	0.126551	4.21301
0.37	0.1369	0.025931	0.97889	1.99822	1.05134	0.0315895	0.133044	4.21167
0.38	0.1444	0.031348	0.99071	1.99756	1.05415	0.0331678	0.139605	4.20904
0.39	0.1521	0.037354	1.00252	1.99673	1.05704	0.0347717	0.146224	4.20525
0.40	0.1600	0.043937	1.01431	1.99573	1.06000	0.0364000	0.152895	4.20041
0.41	0.1681	0.051077	0.90769	1.99453	1.06304	0.0380513	0.159613	4.19467
0.42	0.1764	0.058750	0.91959	1.99312	1.06615	0.0397242	0.166372	4.18817

Table 3.7 Continues								
0.43	0.1849	0.066926	0.95521	1.98758	1.06934	0.0414173	0.173168	4.18106
0.44	0.1936	0.075574	0.96706	1.98527	1.07260	0.0431292	0.179999	4.17349
0.45	0.2025	0.084658	0.97889	1.98273	1.07594	0.0448585	0.186862	4.16560
0.46	0.2116	0.094142	0.99071	1.97996	1.07935	0.0466036	0.193756	4.15753
0.47	0.2209	0.103989	1.00252	1.97696	1.08284	0.0483629	0.200678	4.14941
0.48	0.2304	0.114162	1.01431	1.99988	1.08640	0.0501350	0.207628	4.14138
0.49	0.2401	0.124623	0.90769	1.99979	1.09004	0.0519182	0.214607	4.13355
0.50	0.2500	0.135335	0.91959	1.99964	1.09375	0.0537109	0.221613	4.12603

Table 4.1: **Time Series Decomposition Table for The Monthly Interest Rates Government Bond Yield 2-Year Securities, Reserve Bank of Australia. Jan 1976 – Dec 1993.**

T	X_t	$\hat{M}_t^*$	$\hat{S}_t^*$	$\hat{e}_t^*$	Y_t	$\hat{M}_t^*$	$\hat{S}_t^*$	$\hat{e}_t^*$
1	8.46	6.4945	1.00904	1.29096	0.343807	0.373735	0.99673	0.92294
2	8.50	6.6367	1.02472	1.24985	0.342997	0.371554	0.98857	0.93381
3	8.50	6.7776	1.02025	1.22925	0.342997	0.369394	0.99047	0.93747
4	8.47	6.9171	1.00158	1.22257	0.343604	0.367255	1.00016	0.93545
5	8.47	7.0552	1.00564	1.19379	0.343604	0.365138	0.99702	0.94384
6	8.47	7.1921	0.99741	1.18074	0.343604	0.363042	1.00214	0.94443
7	8.48	7.3276	0.99608	1.16183	0.343401	0.360968	1.00365	0.94787
8	8.48	7.4617	1.00873	1.12663	0.343401	0.358915	0.99498	0.96160
9	8.54	7.5945	0.99702	1.12786	0.342193	0.356883	1.00001	0.95883
10	8.56	7.7260	0.98500	1.12482	0.341793	0.354872	1.00637	0.95705
11	8.39	7.8561	0.98262	1.08685	0.345238	0.352883	1.00695	0.97158
12	8.89	7.9849	0.97193	1.14551	0.335389	0.350916	1.01299	0.94350
13	9.91	8.1123	1.00904	1.21065	0.317660	0.348969	0.99673	0.91327
14	9.89	8.2384	1.02472	1.17151	0.317982	0.347044	0.98857	0.92685
15	9.91	8.3632	1.02025	1.16144	0.317660	0.345140	0.99047	0.92923
16	9.91	8.4866	1.00158	1.16588	0.317660	0.343258	1.00016	0.92528
17	9.90	8.6087	1.00564	1.14355	0.317821	0.341397	0.99702	0.93373
18	9.88	8.7294	0.99741	1.13474	0.318142	0.339557	1.00214	0.93493
19	9.86	8.8488	0.99608	1.11866	0.318465	0.337739	1.00365	0.93950
20	9.86	8.9669	1.00873	1.09008	0.318465	0.335942	0.99498	0.95276
21	9.74	9.0836	0.99702	1.07547	0.320421	0.334167	1.00001	0.95886
22	9.42	9.1990	0.98500	1.03962	0.325818	0.332412	1.00637	0.97396
23	9.27	9.3130	0.98262	1.01298	0.328443	0.330680	1.00695	0.98638
24	9.26	9.4257	0.97193	1.01079	0.328620	0.328968	1.01299	0.98613
25	8.99	9.5371	1.00904	0.93419	0.333519	0.327278	0.99673	1.02242
26	8.83	9.6471	1.02472	0.89322	0.336527	0.325609	0.98857	1.04548
27	8.83	9.7558	1.02025	0.88714	0.336527	0.323962	0.99047	1.04878
28	8.83	9.8631	1.00158	0.89384	0.336527	0.322336	1.00016	1.04386
29	8.82	9.9691	1.00564	0.87977	0.336718	0.320731	0.99702	1.05299
30	8.83	10.0738	0.99741	0.87880	0.336527	0.319147	1.00214	1.05220
31	8.83	10.1771	0.99608	0.87105	0.336527	0.317585	1.00365	1.05579
32	8.79	10.2791	1.00873	0.84773	0.337292	0.316045	0.99498	1.07261
33	8.79	10.3797	0.99702	0.84937	0.337292	0.314525	1.00001	1.07237
34	8.69	10.4790	0.98500	0.84190	0.339227	0.313027	1.00637	1.07684
35	8.66	10.5770	0.98262	0.83324	0.339814	0.311551	1.00695	1.08319

Table 4.1 Continues								
T	X_t	$\hat{M}_t^x$	$\hat{S}_t^x$	$\hat{e}_t^x$	Y_t	$\hat{M}_t^y$	$\hat{S}_t^y$	$\hat{e}_t^y$
36	8.67	10.6736	0.97193	0.83574	0.339618	0.310095	1.01299	1.08116
37	8.72	10.7689	1.00904	0.80248	0.338643	0.308661	0.99673	1.10074
38	8.77	10.8628	1.02472	0.78786	0.337676	0.307249	0.98857	1.11173
39	9.00	10.9554	1.02025	0.80521	0.333333	0.305858	0.99047	1.10032
40	9.61	11.0467	1.00158	0.86857	0.322581	0.304488	1.00016	1.05925
41	9.70	11.1366	1.00564	0.86612	0.321081	0.303139	0.99702	1.06235
42	9.94	11.2252	0.99741	0.88781	0.317181	0.301812	1.00214	1.04868
43	9.94	11.3124	0.99608	0.88214	0.317181	0.300506	1.00365	1.05165
44	9.94	11.3983	1.00873	0.86451	0.317181	0.299222	0.99498	1.06536
45	9.95	11.4829	0.99702	0.86910	0.317021	0.297959	1.00001	1.06397
46	9.94	11.5661	0.98500	0.87250	0.317181	0.296717	1.00637	1.06220
47	9.96	11.6479	0.98262	0.87021	0.316862	0.295497	1.00695	1.06490
48	9.97	11.7285	0.97193	0.87462	0.316703	0.294297	1.01299	1.06233
49	10.83	11.8077	1.00904	0.90898	0.303869	0.293120	0.99673	1.04007
50	10.75	11.8855	1.02472	0.88264	0.304997	0.291963	0.98857	1.05672
51	11.20	11.9620	1.02025	0.91772	0.298807	0.290828	0.99047	1.03732
52	11.40	12.0372	1.00158	0.94557	0.296174	0.289715	1.00016	1.02213
53	11.54	12.1111	1.00564	0.94750	0.294372	0.288622	0.99702	1.02297
54	11.50	12.1835	0.99741	0.94634	0.294884	0.287552	1.00214	1.02331
55	11.34	12.2547	0.99608	0.92900	0.296957	0.286502	1.00365	1.03272
56	11.50	12.3245	1.00873	0.92502	0.294884	0.285474	0.99498	1.03817
57	11.50	12.3930	0.99702	0.93072	0.294884	0.284467	1.00001	1.03661
58	11.58	12.4601	0.98500	0.94352	0.293864	0.283481	1.00637	1.03006
59	12.42	12.5259	0.98262	1.00908	0.283752	0.282517	1.00695	0.99744
60	12.85	12.5904	0.97193	1.05010	0.278964	0.281574	1.01299	0.97802
61	13.10	12.6535	1.00904	1.02601	0.276289	0.280653	0.99673	0.98769
62	13.12	12.7152	1.02472	1.00694	0.276079	0.279753	0.98857	0.99827
63	13.10	12.7757	1.02025	1.00504	0.276289	0.278874	0.99047	1.00026
64	13.15	12.8348	1.00158	1.02294	0.275764	0.278017	1.00016	0.99174
65	13.10	12.8925	1.00564	1.01039	0.276289	0.277180	0.99702	0.99977
66	13.20	12.9489	0.99741	1.02203	0.275241	0.276366	1.00214	0.99380
67	14.20	13.0040	0.99608	1.09627	0.265372	0.275572	1.00365	0.95948
68	14.75	13.0577	1.00873	1.11982	0.260378	0.274800	0.99498	0.95229
69	14.60	13.1101	0.99702	1.11697	0.261712	0.274050	1.00001	0.95497
70	14.60	13.1612	0.98500	1.12621	0.261712	0.273320	1.00637	0.95147
71	14.45	13.2109	0.98262	1.11314	0.263067	0.272613	1.00695	0.95833
72	14.50	13.2592	0.97193	1.12516	0.262613	0.271926	1.01299	0.95336
73	14.80	13.3063	1.00904	1.10229	0.259938	0.271261	0.99673	0.96140
74	15.85	13.3520	1.02472	1.15845	0.251180	0.270617	0.98857	0.93890
75	16.20	13.3963	1.02025	1.18529	0.248452	0.269994	0.99047	0.92906
76	16.50	13.4393	1.00158	1.22580	0.246183	0.269393	1.00016	0.91370
77	16.40	13.4810	1.00564	1.20970	0.246932	0.268813	0.99702	0.92135

Table 4.1 Continues								
T	X_t	$\hat{\mathbf{M}}_t^x$	$\hat{\mathbf{S}}_t^x$	$\hat{\mathbf{e}}_t^x$	Y_t	$\hat{\mathbf{M}}_t^y$	$\hat{\mathbf{S}}_t^y$	$\hat{\mathbf{e}}_t^y$
78	16.40	13.5213	0.99741	1.21604	0.246932	0.268255	1.00214	0.91855
79	16.35	13.5603	0.99608	1.21047	0.247310	0.267718	1.00365	0.92041
80	16.10	13.5979	1.00873	1.17375	0.249222	0.267202	0.99498	0.93742
81	13.70	13.6342	0.99702	1.00783	0.270172	0.266708	1.00001	1.01298
82	13.50	13.6692	0.98500	1.00266	0.272166	0.266234	1.00637	1.01581
83	14.00	13.7028	0.98262	1.03976	0.267261	0.265783	1.00695	0.99862
84	12.30	13.7351	0.97193	0.92138	0.285133	0.265352	1.01299	1.06076
85	12.00	13.7661	1.00904	0.86389	0.288675	0.264943	0.99673	1.09315
86	14.35	13.7957	1.02472	1.01508	0.263982	0.264556	0.98857	1.00936
87	14.60	13.8239	1.02025	1.03518	0.261712	0.264189	0.99047	1.00015
88	12.50	13.8509	1.00158	0.90105	0.282843	0.263844	1.00016	1.07183
89	12.75	13.8765	1.00564	0.91367	0.280056	0.263521	0.99702	1.06593
90	13.70	13.9007	0.99741	0.98812	0.270172	0.263219	1.00214	1.02422
91	13.45	13.9236	0.99608	0.96979	0.272671	0.262938	1.00365	1.03324
92	13.55	13.9452	1.00873	0.96325	0.271663	0.262678	0.99498	1.03942
93	12.60	13.9654	0.99702	0.90493	0.281718	0.262440	1.00001	1.07345
94	12.00	13.9843	0.98500	0.87117	0.288675	0.262223	1.00637	1.09391
95	11.00	14.0018	0.98262	0.79951	0.301511	0.262028	1.00695	1.14274
96	11.60	14.0180	0.97193	0.85140	0.293610	0.261854	1.01299	1.10689
97	12.05	14.0329	1.00904	0.85100	0.288076	0.261701	0.99673	1.10440
98	12.35	14.0464	1.02472	0.85802	0.284555	0.261569	0.98857	1.10045
99	12.70	14.0586	1.02025	0.88544	0.280607	0.261459	0.99047	1.08356
100	12.45	14.0694	1.00158	0.88350	0.283410	0.261371	1.00016	1.08415
101	12.55	14.0789	1.00564	0.88640	0.282279	0.261303	0.99702	1.08350
102	12.20	14.0871	0.99741	0.86829	0.286299	0.261257	1.00214	1.09351
103	12.10	14.0939	0.99608	0.86191	0.287480	0.261233	1.00365	1.09647
104	11.15	14.0994	1.00873	0.78397	0.299476	0.261229	0.99498	1.15219
105	11.85	14.1035	0.99702	0.84273	0.290496	0.261247	1.00001	1.11195
106	12.10	14.1063	0.98500	0.87083	0.287480	0.261287	1.00637	1.09328
107	12.50	14.1078	0.98262	0.90171	0.282843	0.261347	1.00695	1.07478
108	12.90	14.1079	0.97193	0.94079	0.278423	0.261430	1.01299	1.05134
109	12.50	14.1067	1.00904	0.87816	0.282843	0.261533	0.99673	1.08503
110	13.20	14.1041	1.02472	0.91332	0.275241	0.261658	0.98857	1.06407
111	13.65	14.1002	1.02025	0.94886	0.270666	0.261804	0.99047	1.04380
112	13.65	14.0950	1.00158	0.96690	0.270666	0.261971	1.00016	1.03302
113	13.50	14.0884	1.00564	0.95286	0.272166	0.262160	0.99702	1.04127
114	13.45	14.0805	0.99741	0.95770	0.272671	0.262371	1.00214	1.03704
115	13.35	14.0712	0.99608	0.95248	0.273690	0.262602	1.00365	1.03843
116	14.45	14.0606	1.00873	1.01880	0.263067	0.262855	0.99498	1.00585
117	14.30	14.0487	0.99702	1.02093	0.264443	0.263129	1.00001	1.00498
118	15.05	14.0354	0.98500	1.08862	0.257770	0.263425	1.00637	0.97234
119	15.55	14.0208	0.98262	1.12868	0.253592	0.263742	1.00695	0.95488

Table 4.1 Continues								
T	X_t	$\hat{M}_t^x$	$\hat{S}_t^x$	$\hat{e}_t^x$	Y_t	$\hat{M}_t^y$	$\hat{S}_t^y$	$\hat{e}_t^y$
120	15.65	14.0048	0.97193	1.14974	0.252780	0.264080	1.01299	0.94493
121	14.65	13.9875	1.00904	1.03798	0.261265	0.264440	0.99673	0.99124
122	14.15	13.9689	1.02472	0.98853	0.265841	0.264821	0.98857	1.01545
123	13.30	13.9489	1.02025	0.93456	0.274204	0.265223	0.99047	1.04381
124	12.65	13.9276	1.00158	0.90684	0.281161	0.265647	1.00016	1.05823
125	12.70	13.9049	1.00564	0.90822	0.280607	0.266092	0.99702	1.05770
126	12.80	13.8809	0.99741	0.92452	0.279508	0.266559	1.00214	1.04634
127	14.50	13.8555	0.99608	1.05063	0.262613	0.267047	1.00365	0.97982
128	15.10	13.8289	1.00873	1.08247	0.257343	0.267556	0.99498	0.96668
129	15.15	13.8008	0.99702	1.10104	0.256917	0.268086	1.00001	0.95833
130	14.30	13.7715	0.98500	1.05419	0.264443	0.268638	1.00637	0.97815
131	14.25	13.7408	0.98262	1.05540	0.264906	0.269211	1.00695	0.97722
132	14.05	13.7087	0.97193	1.05449	0.266785	0.269806	1.01299	0.97612
133	14.70	13.6753	1.00904	1.06529	0.260820	0.270422	0.99673	0.96766
134	15.05	13.6406	1.02472	1.07670	0.257770	0.271059	0.98857	0.96196
135	14.05	13.6045	1.02025	1.01225	0.266785	0.271718	0.99047	0.99129
136	13.80	13.5671	1.00158	1.01556	0.269191	0.272398	1.00016	0.98807
137	13.25	13.5284	1.00564	0.97393	0.274721	0.273099	0.99702	1.00895
138	13.00	13.4883	0.99741	0.96630	0.277350	0.273822	1.00214	1.01072
139	12.85	13.4469	0.99608	0.95937	0.278964	0.274566	1.00365	1.01232
140	12.60	13.4041	1.00873	0.93187	0.281718	0.275331	0.99498	1.02836
141	11.80	13.3600	0.99702	0.88587	0.291111	0.276118	1.00001	1.05429
142	13.00	13.3145	0.98500	0.99124	0.277350	0.276926	1.00637	0.99519
143	12.35	13.2678	0.98262	0.94729	0.284555	0.277755	1.00695	1.01741
144	11.45	13.2196	0.97193	0.89115	0.295527	0.278606	1.01299	1.04713
145	11.35	13.1702	1.00904	0.85407	0.296826	0.279478	0.99673	1.06556
146	11.55	13.1193	1.02472	0.85914	0.294245	0.280372	0.98857	1.06161
147	10.85	13.0672	1.02025	0.81385	0.303588	0.281287	0.99047	1.08967
148	10.90	13.0137	1.00158	0.83626	0.302891	0.282223	1.00016	1.07306
149	12.30	12.9589	1.00564	0.94383	0.285133	0.283180	0.99702	1.00991
150	11.70	12.9027	0.99741	0.90914	0.292353	0.284159	1.00214	1.02663
151	12.05	12.8452	0.99608	0.94179	0.288076	0.285160	1.00365	1.00655
152	12.30	12.7863	1.00873	0.95364	0.285133	0.286181	0.99498	1.00136
153	12.90	12.7261	0.99702	1.01669	0.278423	0.287224	1.00001	0.96935
154	13.05	12.6646	0.98500	1.04612	0.276818	0.288289	1.00637	0.95413
155	13.30	12.6017	0.98262	1.07408	0.274204	0.289374	1.00695	0.94104
156	13.85	12.5375	0.97193	1.13658	0.268705	0.290481	1.01299	0.91317
157	14.65	12.4720	1.00904	1.16410	0.261265	0.291610	0.99673	0.89888
158	15.05	12.4051	1.02472	1.18394	0.257770	0.292759	0.98857	0.89066
159	15.15	12.3369	1.02025	1.20366	0.256917	0.293930	0.99047	0.88248
160	14.85	12.2673	1.00158	1.20863	0.259500	0.295123	1.00016	0.87915
161	15.70	12.1964	1.00564	1.28005	0.252377	0.296337	0.99702	0.85420

Table 4.1 Continues								
T	X_t	$\hat{M}_t^x$	$\hat{S}_t^x$	$\hat{e}_t^x$	Y_t	$\hat{M}_t^y$	$\hat{S}_t^y$	$\hat{e}_t^y$
162	15.40	12.1241	0.99741	1.27349	0.254824	0.297572	1.00214	0.85451
163	15.10	12.0505	0.99608	1.25799	0.257343	0.298828	1.00365	0.85804
164	14.80	11.9756	1.00873	1.22515	0.259938	0.300106	0.99498	0.87052
165	15.80	11.8993	0.99702	1.33178	0.251577	0.301405	1.00001	0.83467
166	15.80	11.8217	0.98500	1.35687	0.251577	0.302726	1.00637	0.82578
167	15.00	11.7427	0.98262	1.29998	0.258199	0.304068	1.00695	0.84329
168	14.40	11.6624	0.97193	1.27039	0.263523	0.305431	1.01299	0.85172
169	13.80	11.5808	1.00904	1.18095	0.269191	0.306816	0.99673	0.88025
170	14.30	11.4978	1.02472	1.21370	0.264443	0.308222	0.98857	0.86788
171	14.15	11.4135	1.02025	1.21516	0.265841	0.309649	0.99047	0.86678
172	14.45	11.3279	1.00158	1.27360	0.263067	0.311098	1.00016	0.84547
173	14.10	11.2409	1.00564	1.24731	0.266312	0.312568	0.99702	0.85456
174	14.05	11.1525	0.99741	1.26307	0.266785	0.314059	1.00214	0.84766
175	13.75	11.0629	0.99608	1.24779	0.269680	0.315572	1.00365	0.85147
176	13.30	10.9718	1.00873	1.20170	0.274204	0.317106	0.99498	0.86907
177	13.00	10.8795	0.99702	1.19848	0.277350	0.318661	1.00001	0.87035
178	12.55	10.7858	0.98500	1.18128	0.282279	0.320238	1.00637	0.87589
179	12.25	10.6907	0.98262	1.16612	0.285714	0.321836	1.00695	0.88164
180	11.85	10.5944	0.97193	1.15082	0.290496	0.323456	1.01299	0.88658
181	11.50	10.4967	1.00904	1.08577	0.294884	0.325097	0.99673	0.91004
182	11.10	10.3976	1.02472	1.04180	0.300150	0.326759	0.98857	0.92918
183	11.15	10.2972	1.02025	1.06133	0.299476	0.328442	0.99047	0.92058
184	10.70	10.1955	1.00158	1.04783	0.305709	0.330147	1.00016	0.92583
185	10.25	10.0924	1.00564	1.00992	0.312348	0.331873	0.99702	0.94398
186	10.55	9.9880	0.99741	1.05901	0.307875	0.333621	1.00214	0.92085
187	10.25	9.8822	0.99608	1.04130	0.312348	0.335390	1.00365	0.92791
188	10.30	9.7751	1.00873	1.04458	0.311588	0.337180	0.99498	0.92876
189	9.60	9.6667	0.99702	0.99607	0.322749	0.338992	1.00001	0.95208
190	8.40	9.5569	0.98500	0.89233	0.345033	0.340825	1.00637	1.00594
191	8.20	9.4458	0.98262	0.88347	0.349215	0.342679	1.00695	1.01204
192	7.25	9.3333	0.97193	0.79922	0.371391	0.344555	1.01299	1.06406
193	8.35	9.2195	1.00904	0.89757	0.346064	0.346452	0.99673	1.00216
194	8.25	9.1044	1.02472	0.88430	0.348155	0.348371	0.98857	1.01093
195	8.30	8.9879	1.02025	0.90514	0.347105	0.350310	0.99047	1.00038
196	7.40	8.8700	1.00158	0.83295	0.367607	0.352272	1.00016	1.04337
197	7.15	8.7509	1.00564	0.81248	0.373979	0.354254	0.99702	1.05884
198	6.35	8.6304	0.99741	0.73768	0.396838	0.356258	1.00214	1.11153
199	5.65	8.5085	0.99608	0.66665	0.420703	0.358283	1.00365	1.16995
200	7.40	8.3854	1.00873	0.87485	0.367607	0.360330	0.99498	1.02534
201	7.20	8.2608	0.99702	0.87419	0.372678	0.362398	1.00001	1.02836
202	7.05	8.1350	0.98500	0.87982	0.376622	0.364487	1.00637	1.02675
203	7.10	8.0078	0.98262	0.90232	0.375293	0.366597	1.00695	1.01666

Table 4.1 Continues								
T	X_t	$\hat{M}_t^x$	$\hat{S}_t^x$	$\hat{e}_t^x$	Y_t	$\hat{M}_t^y$	$\hat{S}_t^y$	$\hat{e}_t^y$
204	6.85	7.8792	0.97193	0.89448	0.382080	0.368729	1.01299	1.02292
205	6.50	7.7493	1.00904	0.83126	0.392232	0.370883	0.99673	1.06104
206	6.25	7.6181	1.02472	0.80062	0.400000	0.373057	0.98857	1.08461
207	5.95	7.4856	1.02025	0.77909	0.409960	0.375253	0.99047	1.10300
208	5.65	7.3516	1.00158	0.76732	0.420703	0.377471	1.00016	1.11435
209	5.85	7.2164	1.00564	0.80611	0.413449	0.379709	0.99702	1.09211
210	5.45	7.0798	0.99741	0.77179	0.428353	0.381970	1.00214	1.11904
211	5.30	6.9419	0.99608	0.76649	0.434372	0.384251	1.00365	1.12633
212	5.20	6.8026	1.00873	0.75779	0.438529	0.386554	0.99498	1.14018
213	5.55	6.6620	0.99702	0.83557	0.424476	0.388878	1.00001	1.09153
214	5.15	6.5201	0.98500	0.80189	0.440653	0.391223	1.00637	1.11922
215	5.40	6.3768	0.98262	0.86180	0.430331	0.393590	1.00695	1.08580
216	5.35	6.2322	0.97193	0.88324	0.432338	0.395978	1.01299	1.07782
Note: The superscripts x and y is used to denote the component of the original series X and that of the inverse-square-root transformed series, Y								